WP 61_13

Yin-Hei (Michael) Cheng
University of Waterloo, Canada

Tony S. Wirjanto
University of Waterloo, Canada
The Rimini Centre for Economic Analysis (RCEA), Italy

PRICING FINANCIAL DERIVATIVES BY GRAM-CHARLIER EXPANSIONS

Pricing Financial Derivatives by Gram-Charlier Expansions

Yin-Hei (Michael) Cheng and Tony S. Wirjanto

Department of Statistics & Actuarial Science

School of Accounting & Finance

University of Waterloo

Current Version: May 2013

Abstract. In this paper we provide several applications of Gram-Charlier expansions in financial derivative pricing. We first give an exposition on how to calculate swaption prices under a two-factor Cox-Ingersoll-Ross (CIR2) model. Then we apply this method to an extended version of the model (CIR2++). We also develop a procedure to calculate European call options under Heston's model of stochastic volatility by the Gram-Charlier Expansions.

We thank Adam Kolkiewicz and Chengguo Weng for their comments and suggestions for the original and long version of this paper, which is a Master Thesis of the first author under the supervision of the second author. Any remaining errors and omissions are ours.

1 Introduction

A Gram-Charlier expansion, which was introduced by Jogen Pedersen Gram and Carl Charlier, is an infinite series that approximates a probability distribution in terms of its cumulants (or moments). The idea underlying the Gram-Charlier expansion is relatively straightforward. Suppose that the first four moments of a random variable are given. Then we can calculate its mean, variance, skewness and kurtosis. Heuristically speaking, the shape of the density function of a random variable can be well characterized by these four moments. As a result, the distribution function of a random variable is fixed if its moments are known.

At the same time, the arbitrage-free price of a European-type pay-off function of a financial asset is an expectation with respect to an equivalent martingale measure if the model is arbitrage free. If the moments of the underlying asset with respect to the numeraire measure are known, then we are able to calculate its probability density (or cumulative distribution) function. It follows that the price of the derivative can be obtained in a relatively straightforward manner. This is the appeal of the Gram-Charlier expansion when applied to pricing financial derivatives. The objective of this paper is to explore the feasibility of such applications.

The remaining parts of this paper are organized as follows. In the first three subsections of section 2, we provide a detailed discussion on the study by Tanaka, Yamada, Watanabe (2010) in [12], which presents a way to calculate the swaption prices under the CIR2 model by using the Gram-Charlier expansions. Most of the original works presented in our paper, which starts with subsection 3.3, are inspired by this paper. In the last subsection of section 2, we disuss a method to calculate the the swaption prices under an extended CIR2 (known as CIR2++) model by using the Gram-Charlier expansions. This is achieved by a modification of the formula for the bond moment, which is an important concept mentioned in [12]. We discuss the applications of the Gram-Charlier approach to general diffusion processes in section 3. To make the discussion concrete, we consider the Black-Scholes model and a simplified version of the Brennan-Schwarz model for interest rates. The Black-Scholes model is chosen as a representation of the class of diffusion processes of which the moments are easily obtained. In subsection 3.1, we use the Gram-Charlier approach to price European call options for this class of models. A simplified version of the Brennan-Schwarz model is chosen for a study since the process does not admit a simple closed-form solution. In subsection 3.2, we show that the moments are solutions to a system of ordinary differential equations. The solutions can be obtained by any existing symbolic calculation software. In general, this can be obtained by using numerical methods of ordinary differential equations. In section 4, we apply the Gram-Charlier expansion to Heston's Model of stochastic volatility. As the characteristic functions of the discounted log-price is known, the moments of the log-price are readily obtained by taking derivatives. Then we provide a formula to calculate the truncated moment-generating function in subsection 4.2. This is the key step in obtaining the approximation formula of the price of the European call options. We also suggest a way to simulate the Heston model to avoid negative measures of volatility. Some numerical results and discussions are given at the end of this section. Section 5 contains the conclusion of this paper in which we also give a summary of the results and methods presented in this paper and discuss limitations of our approach. In Appendix A, we provide background information on Gram-Charlier expansions and an important class of ordinary differential equations. This is because the Gram-Charlier expansion is used throughout this paper and the solution of the class of ordinary differential equations mentioned above is used prominently in section 2 of this paper. In Appendix B, we review the concepts of and discuss the properties of T-forward measures, swaps, swaptions and Black's implied volatilities of swaptions.

2 Pricing swaptions using Gram-Charlier expansions

2.1 Introduction to the CIR2 model

It is well-known that the term structure as well as the prices of any interest-rate derivatives are completely determined by the short-rate dynamics under the risk-neutral measure Q which is assumed to be known. The procedure of specifying the Q-dynamics is known as a *martingale modeling*. We begin the discussion of swaptions with the following definition.

Definition 2.1. If the term structure $\{P(t,T) : 0 \leq t \leq T, T > 0\}$ has the form

$$P(t,T) = F(t, r(t), T),$$

where F has the form

$$F(t, r(t), T) = e^{A(t,T)+(t,T)\cdot X(t)},$$

and where A and B are deterministic functions, then the model is said to possess an affine term structure.

An assumption often made in a term-structure model is that r under the Q-measure has dynamics given by

$$r(t) = \delta_0 + \delta_X \cdot X(t)$$

where $X(t)$ satisfies the following system of Stochastic Differential Equations (SDEs):

$$dX(t) = K(\theta - X(t))dt + \Sigma D(X(t))dW(t),$$

$$\alpha_i, \beta_i \in \mathbb{R}, \theta \in \mathbb{R}^n, K \in M_{n\times n}(\mathbb{R})$$

and

$$D(x) = \text{diag}[\sqrt{\alpha_1 + \beta_1 \cdot x}, ..., \sqrt{\alpha_n + \beta_n \cdot x}],\ x \in \mathbb{R}^n.$$

Proposition 2.2. *The model of the form assumed above has an affine term structure.*

There are a number short-rate models introduced and studied in the literature. One of them is proposed by [9], known as the Cox-Ingersoll-Ross (CIR) model. It is specified as

$$dr = a(b - r) + \sigma\sqrt{r}dW.$$

See Cox, Ingersoll and Ross (1985) in [9] for more details.

This model ensures mean reversion of the interest rate towards its long-run mean value b, with a speed of adjustment governed by a strictly positive parameter a. We can show that r is non-negative in this model. In addition, r is strictly positive whenever $2ab \leq \sigma^2$.

In order to accommodate a more complicated shape of typical yield curves, Brigo and Mercurio (2006) in [6, Chapter 4] discussed an extension to the CIR model, known as as a two-factor CIR (CIR2) model. It is given by

$$r(t) = X_1(t) + X_2(t) + \delta_0.$$

where the Q-dynamics of $X(t) = (X_1(t), X_2(t))$ are given by the following SDEs:

$$dX_i(t) = K_i(\theta_i - X_i(t))dt + \sigma_i\sqrt{X_i(t)}dW_i(t)\ ,\ i = 1, 2.$$

In the CIR2 model, the short rate is given by

$$r(t) = X_1(t) + X_2(t) + \delta_0.$$

where the Q-dynamics of $X(t) = (X_1(t), X_2(t))$ are given by the following SDEs:

$$dX_1(t) = K_1(\theta_1 - X_1(t))dt + \sigma_1\sqrt{X_1(t)}dW_1(t);$$

$$dX_2(t) = K_2(\theta_1 - X_2(t))dt + \sigma_2\sqrt{X_2(t)}dW_2(t)$$

and the initial conditions are given by

$$X(0) = (X_1(0), X_2(0)) \text{ are given},$$

where W_1, W_2 are independent standard Q-Brownian motions.

Lemma 2.3. *Suppose that the discounted price of a derivative is a Q-martingale and admits an affine structure*

$$V(t,T) = e^{A(t,T)+B(t,T)\cdot X(t)}$$

where A and B are deterministic quantities. Then A and B satisfy the following system of Ordinary Differential Equations (ODEs)

$$\frac{\partial A}{\partial t} = \delta_0 - B_1 K_1 \theta_1 - B_2 K_2 \theta_2, \tag{1}$$
$$\frac{\partial B_1}{\partial t} = 1 + B_1 K_1 - \frac{1}{2} B_1^2 \sigma_1^2, \tag{2}$$
$$\frac{\partial B_2}{\partial t} = 1 + B_2 K_2 - \frac{1}{2} B_2^2 \sigma_2^2, \tag{3}$$

Proof. Assume an affine structure for V

$$V(t,T) = F(t, X_1(t), X_2(t)) = e^{A(t,T)+B_1(t,T)X_1(t,T)+B_2(t,T)X_2(t,T)}.$$

Then we have

$$F_t = (A_t + B_{1,t} X_1 + B_{2,t} X_2) F, \tag{4}$$
$$F_{X_1} = B_1 F, \tag{5}$$
$$F_{X_2} = B_2 F, \tag{6}$$
$$F_{X_1 X_1} = B_1^2 F, \tag{7}$$
$$F_{X_1 X_2} = B_1 B_2 F, \tag{8}$$
$$F_{X_2 X_2} = B_2^2 F, \tag{9}$$

Let $D(t) = e^{-\int_t^T r(s)ds}$ be the discount factor. Then we have

$$\begin{aligned}
& d(DF) \\
= & -rDFdt + DdF \\
= & -rDFdt + D[F_t dt + \\
& F_{X_1} dX_1 + F_{X_2} dX_2 + \frac{1}{2} F_{X_1 X_1} dX_1 dX_1 + F_{X_1 X_2} dX_1 dX_2 + \frac{1}{2} F_{X_2 X_2} dX_2 dX_2] \\
= & D[-rFdt + F_t dt + F_{X_1}(K_1(\theta_1 - X_1)dt + \sigma_1 \sqrt{X_1} dW_1) + \\
& F_{X_2}(K_2(\theta_2 - X_2)dt + \sigma_1 \sqrt{X_2} dW_2) + \frac{1}{2} F_{X_1 X_1} \sigma_1^2 X_1 dt + \frac{1}{2} F_{X_2 X_2} \sigma_2^2 X_2 dt] \\
= & D\{[-rF + F_t + F_{X_1} K_1(\theta_1 - X_1) + F_{X_2} K_2(\theta_2 - X_2) + \\
& \frac{1}{2} F_{X_1 X_1} \sigma_1^2 X_1 + \frac{1}{2} F_{X_2 X_2} \sigma_2^2 X_2]dt + \sigma_1 \sqrt{X_1} F_{X_1} dW_1 + \sigma_2 \sqrt{X_2} F_{X_2} dW_2\}
\end{aligned}$$

Since DF is a Q-martingale, the drift-term is equal to 0. Thus, we have

$$-rF + F_t + F_{X_1} K_1(\theta_1 - X_1) + F_{X_2} K_2(\theta_2 - X_2) + \frac{1}{2} F_{X_1 X_1} \sigma_1^2 X_1 + \frac{1}{2} F_{X_2 X_2} \sigma_2^2 X_2 = 0$$

Therefore,

$$\begin{aligned}
& -(X_1 + X_2 + \delta_0) + (A_t + B_{1,t} X_1 + B_{2,t} X_2) + B_1 K_1(\theta_1 - X_1) + B_2 K_2(\theta_2 - X_2) \\
& \quad + \frac{1}{2} B_1^2 \sigma_1^2 X_1 + \frac{1}{2} B_2^2 \sigma_2^2 X_2 = 0 \\
& \Rightarrow (-\delta_0 + A_t + B_1 K_1 \theta_1 + B_2 K_2 \theta_2) + \\
& \quad (-1 + B_{1,t} - B_1 K_1 + \frac{1}{2} B_1^2 \sigma_1^2) X_1 + (-1 + B_{2,t} - B_2 K_2 + \frac{1}{2} B_2^2 \sigma_2^2) X_2 = 0
\end{aligned}$$

Next we obtain the following system

$$\frac{\partial A}{\partial t} = \delta_0 - B_1 K_1 \theta_1 - B_2 K_2 \theta_2, \tag{10}$$

$$\frac{\partial B_1}{\partial t} = 1 + B_1 K_1 - \frac{1}{2} B_1^2 \sigma_1^2, \tag{11}$$

$$\frac{\partial B_2}{\partial t} = 1 + B_2 K_2 - \frac{1}{2} B_2^2 \sigma_2^2, \tag{12}$$

□

Theorem 2.4. *The price at time t of a zero-coupon bond maturing at time T and with a unit face value given by*

$$P^{CIR}(t,T) = e^{A(t,T)+B(t,T)\cdot X(t)}$$

where

$$\gamma_j = \sqrt{K_j^2 + 2\sigma_j^2}, j = 1,2$$

$$A(t,T) = -\delta_0(T-t) - \sum_{j=1}^{2} K_j\theta_j \, [\frac{2}{\sigma_j^2} \ln \frac{(K_j+\gamma_j)(e^{\gamma_j(T-t)}-1)}{2\gamma_j} + \frac{2}{K_j-\gamma_j}(T-t)],$$

$$B_j(t,T) = \frac{-2(e^{\gamma_j(T-t)}-1)}{(K_j+\gamma_j)(e^{\gamma_j(T-t)}-1)+2\gamma_j}, j = 1,2.$$

Proof. Consider an affine term structure

$$P^{CIR}(t,T) = F(t, X_1(t), X_2(t)) = e^{A(t,T)+B_1(t,T)X_1(t,T)+B_2(t,T)X_2(t,T)}.$$

Then we obtain the following system of ODEs by Lemma 2.3

$$\frac{\partial A}{\partial t} = \delta_0 - B_1K_1\theta_1 - B_2K_2\theta_2, \quad (13)$$

$$\frac{\partial B_1}{\partial t} = 1 + B_1K_1 - \frac{1}{2}B_1^2\sigma_1^2, \quad (14)$$

$$\frac{\partial B_2}{\partial t} = 1 + B_2K_2 - \frac{1}{2}B_2^2\sigma_2^2, \quad (15)$$

$$A(T,T) = B_1(T,T) = B_2(T,T) = 0 \quad (16)$$

The solution can be derived straightforwardly usng the results discussed in subsection B.2 of the Appendix. □

Finally, we calculate the bond moments under the T_0-forward measure, which is defined by

$$\mu^{T_0}(t, T_0, \{T_{i_1}, T_{i_2}, ..., T_{i_m}\}) := \mathbb{E}^{T_0}\left[\prod_{k=1}^{m} P(T_0, T_{i_k})\Big|\mathcal{F}_t\right].$$

This is the key to the pricing formula given in next section.

The formula given below can be found in Collin-Dufresne and Goldstein (2002) in [8] and [12] and we provide a detailed proof of this formula.

Theorem 2.5. *The bond moments under the T_0-forward measure is given by*

$$\mu^{T_0}(t, T_0, \{T_{i_1}, T_{i_2}, ..., T_{i_m}\}) := \frac{e^{M(t)+N(t)\cdot X(t)}}{P(t, T_0)}.$$

where

$$M(t) = F_0 - \delta_0 \tau - \sum_{j=1}^{n} K_j \theta_j \left[\frac{2}{\sigma_j^2} \ln \frac{(K_j + \gamma_j - \sigma_j^2 F_j)(e^{\gamma_j \tau} - 1) + 2\gamma_j}{2\gamma_j} + \frac{(K_j + \gamma_j)F_j + 2}{K_j - \gamma_j - \sigma_j^2 F_j} \tau \right],$$

$$N_j(t) = \frac{-[(K_j - \gamma_j)F_j + 2](e^{\gamma_j \tau} - 1) + 2\gamma_j F_j}{(K_j + \gamma_j - \sigma_j^2 F_j)(e^{\gamma_j \tau} - 1) + 2\gamma_j},$$

$\gamma_j = \sqrt{K_j^2 + 2\sigma_j^2}$, $\tau = T_0 - t$, $F_0 = \sum_{i=1}^{m} A(T_0, T_i) + A(T_0, T)$, $F_j = \sum_{i=1}^{m} B_j(T_0, T_i) + B_j(T_0, T)$ *and the formulas of A and B_j's are given in the previous theorem , $j = 1, 2$.*

Proof. Since

$$P(T_0, T_{i_k}) = e^{A(T_0, T_{i_k}) + B(T_0, T_{i_k})\cdot X(T_0)},$$

we have

$$\prod_{k=1}^{m} P(T_0, T_{i_k}) = e^{F_0(t, T_0, \{T_{i_1}, T_{i_2}, ..., T_{i_m}\}) + F(t, T_0, \{T_{i_1}, T_{i_2}, ..., T_{i_m}\})\cdot X(T_0)}$$

with

$$F_0 := F_0(t, T_0, \{T_{i_1}, T_{i_2}, ..., T_{i_m}\}) = \sum_{k=1}^{m} A(T_0, T_{i_k})$$

and

$$F := F(t, T_0, \{T_{i_1}, T_{i_2}, ..., T_{i_m}\}) = \sum_{k=1}^{m} B(T_0, T_{i_k}).$$

By the definition of a forward measure, we have

$$\mathbb{E}^{T_0}[e^{F_0 + F\cdot X(T_0)} | \mathcal{F}_t] = \frac{1}{P(t, T_0)} \mathbb{E}[e^{-\int_t^{T_0} r_s ds} e^{F_0 + F\cdot X(T_0)} | \mathcal{F}_t]$$

Let $F = F(t, X)$ be the solution of

$$\frac{\partial F}{\partial t}(t, X) + \sum_{i=1}^{n} \mu_i(t, X) \frac{\partial F}{\partial x_i}(t, X) + \frac{1}{2} \sum_{i,j=1}^{n} C_{ij}(t, X) \frac{\partial^2 F}{\partial x_i \partial x_j}(t, X) - rF(t, X) = 0,$$

$$F(T_0, X(T_0)) = \Phi(X(T_0)).$$

Assume that F has the form $F(t,X) = e^{M(t)+N(t)\cdot X(t)}$. Then it is easy to show that

$$M(T_0) = \sum_{i=1}^{m} A(T_0, T_{i_m}),\ N(T_0) = \sum_{i=1}^{m} B(T_0, T_{i_m}).$$

Furthermore, the Partial Differential Equation (PDE) above implies that

$$\begin{aligned}
&-(X_1 + X_2 + \delta_0) + (M_t + N_{1,t}X_1 + N_{2,t}X_2) + N_1K_1(\theta_1 - X_1) + N_2K_2(\theta_2 - X_2)\\
&\quad + \frac{1}{2}N_1^2\sigma_1^2X_1 + \frac{1}{2}N_2^2\sigma_2^2X_2 = 0\\
&\Rightarrow (-\delta_0 + M_t + N_1K_1\theta_1 + N_2K_2\theta_2) +\\
&\quad (-1 + N_{1,t} - N_1K_1 + \frac{1}{2}N_1^2\sigma_1^2)X_1 + (-1 + N_{2,t} - N_2K_2 + \frac{1}{2}N_2^2\sigma_2^2)X_2 = 0
\end{aligned}$$

This leads to the following system

$$\frac{\partial M}{\partial t} = \delta_0 - N_1K_1\theta_1 - N_2K_2\theta_2, \tag{17}$$

$$\frac{\partial N_1}{\partial t} = 1 + N_1K_1 - \frac{1}{2}N_1^2\sigma_1^2, \tag{18}$$

$$\frac{\partial N_2}{\partial t} = 1 + N_2K_2 - \frac{1}{2}N_2^2\sigma_2^2, \tag{19}$$

$$M(T_0) = F_0\ ,\ N_1(T_0) = F_1 \text{ and } N_2(T_0) = F_2. \tag{20}$$

The solution can be derived straightforwardly from the results discussed in subsection B.2 of the Appendix. □

2.2 Pricing Swaptions under the CIR2 model

The discussion in this subsection relies heavily on [12]. Consider a swaption with the expiry T_0 and the fixed rate K during a period $[T_0, T_N]$. The price of the underlying swap is given by

$$SV(t) = \sum_{i=0}^{N} a_i P(t, T_i)$$

where

$$a_0 = -1\,;\ a_i = \frac{P(t,T_0) - P(t,T_N)}{\sum_{i=1}^{N} P(t,T_N)}\ (i = 1, ..., N-1) \text{ and } a_N = 1 + \frac{P(t,T_0) - P(t,T_N)}{\sum_{i=1}^{N} P(t,T_N)}.$$

The m^{th} swap moment under the T_0-forward measure conditioned on $\mathcal{F}_t$ is given by

$$M_m(t) = \mathbb{E}^{T_0}\left[\left(\sum_{i=0}^{N} a_i P(t,T_i)\right)^m \Big| \mathcal{F}_t\right]$$

Note that

$$\left[\sum_{i=0}^{N} a_i P(t,T_i)\right]^m = \sum_{0\le i_1,...,i_m\le N} a_{i_1}...a_{i_m}\left[\prod_{k=1}^{m} P(T_0,T_{i_k})\right].$$

So,

$$M_m(t) = \sum_{0\le i_1,...,i_m\le N} a_{i_1}...a_{i_m}\mathbb{E}^{T_0}\left[\prod_{k=1}^{m} P(T_0,T_{i_k})\Big|\mathcal{F}_t\right]$$

Remark. Observe that

$$M_m(t) = \sum_{\substack{0\le k_0,...k_M\le N,\\ k_0+...+k_N=m,}} a_0^{k_0}...a_N^{k_N}\mathbb{E}^{T_0}\left[\prod_{j=0}^{N} P(T_0,T_j)^{k_j}\Big|\mathcal{F}_t\right]$$

By simple combinatorics steps, we obtain

$$M_m(t) = \sum_{\substack{0\le k_0\le...\le k_N\le N,\\ k_0+...+k_N=m,}} \frac{m!}{k_0!k_1!...k_N!} a_0^{k_0}...a_N^{k_N}\mathbb{E}^{T_0}\left[\prod_{j=0}^{N} P(T_0,T_j)^{k_j}\Big|\mathcal{F}_t\right]$$

The algorithm for generating the following collection of sets

$$\{\{k_0,k_1,...,k_N\} : 0\le k_0,k_1,...,k_N\le N, k_0+k_1+...+k_N = M\}$$

is crucial in the implementation of our formulas.

Since the bond moments $\mathbb{E}^{T_0}\left[\prod_{k=1}^{m} P(T_0,T_{i_k})|\mathcal{F}_t\right]$ have closed-form expressions (see Theorem 2.5), we are able to obtain a closed-form formula for the swaptions. To sum up, we have the following theorems.

Theorem 2.6. *Suppose that the risk-neutral dynamics of the short rates follow the CIR2 model.*

$$r(t) = X_1(t) + X_2(t) + \delta_0.$$

where the Q-dynamics of $X(t) = (X_1(t), X_2(t))$ are given by the following SDEs:

$$dX_i(t) = K_i(\theta_i - X_i(t))dt + \sigma_i\sqrt{X_i(t)}dW_i(t)\ ,\ i = 1,2.$$

Let $c_n(t)$ be the swap cumulants which can be calculated by the swap moments $\{M_1(t), ..., M_n(t)\}$ and $C_n(t) = c_n(t)P(t, T_0)^n$ for $n \geq 1$. Let $q_0 = 1$, $q_1 = q_2 = 0$ and

$$q_n = \sum_{m=1}^{[\frac{n}{3}]} \sum_{\substack{k_1,...k_m \geq 3, \\ k_1+...+k_m=n,}} \frac{C_{k_1}...C_{k_m}}{m!k_1!...k_m!\sqrt{C_2}^n} \ , \ n \geq 3.$$

The risk-neutral price of the $T_0 \times (T_N - T_0)$-receiver swaption $SOV(t; T_0, T_n)$ is given by

$$SOV(t; T_0, T_n) = C_1 N\left(\frac{C_1}{\sqrt{C_2}}\right) + \sqrt{C_2}\phi\left(\frac{C_1}{\sqrt{C_2}}\right)\left[1 + \sum_{k=3}^{\infty}(-1)^k q_k H_{k-1}\left(\frac{C_1}{\sqrt{C_2}}\right)\right].$$

2.3 Numerical results

In this subsection, we provide numerical results for the method of pricing swaptions discussed in the last subsection. That is, we consider the following parameters in the CIR2 model:

Parameters	*Values*
S_0	1
δ_0	0.02
κ_1	0.2
κ_2	0.2
θ_1	0.03
θ_2	0.01
σ_1	0.04
σ_2	0.02
$X_1(0)$	0.04
$X_2(0)$	0.02

We consider $N = 5,00,000$ scenarios and 12 time steps per year for a Monte-Carlo simulation of the call option prices. We approximate the price of the call option using only first N terms in the Gram-Charlier expansions and denote them by $GC(N)$. The numerical results are summarized in Figures 1 - 12. From these figures, we see that the

GC3 , which is the Gram-Charlier expansions up to the third cumulants, is generally more accurate than GC6, which is the Gram-Charlier expansions up to the sixth cumulants. GC6 is slightly more accurate in the short tenor swaptions, but substantially less accurate for long tenor swaptions.

In order to confirm that the result given above is not merely an artifact of the selected configuration of parameter values, we repeat the testing procedure by using another set of parameter values:

$Parameters$	$Values$
S_0	1
δ_0	-0.02
κ_1	0.05
κ_2	0.5
θ_1	0.085
θ_2	0.01
σ_1	0.08
σ_2	0.05
$X_1(0)$	0.01
$X_2(0)$	0.01

The numerical results are summarized in Figures 13 - 24. From these figures, we again find that the GC3 is generally more accurate than GC6. Increasing the terms in the Gram-Charlier expansion does not appear to increase the accuracy of the approximation. This is due to the fact that the Gram-Charlier expansions is an orthogonal series. Also we note that it is difficult to obtain a precise estimate of the error under this approach.

2.4 Pricing Swaptions under the CIR2++ model

In this subsection, we discuss how the Gram-Charlier expansions can be used to calculate swaption prices under an extended version of the two-factor CIR (CIR2) model, known as the CIR2++ model. Recall that the CIR2 model is specified as

$$dX_1(t) = K_1(\theta_1 - X_1(t))dt + \sigma_1\sqrt{X_1(t)}dW_1(t);$$

$$dX_2(t) = K_2(\theta_1 - X_2(t))dt + \sigma_2\sqrt{X_2(t)}dW_2(t)$$

with the initial conditions given by

$$X(0) = (X_1(0), X_2(0)) \text{ are given,}$$

where W_1, W_2 are independent Q-Brownian motions.

In the CIR2++ model (See Brigo and Mercurio (2001) in [5]), the short rate, instead, is given by

$$r(t) = X_1(t) + X_2(t) + \psi(t).$$

where $\psi(t)$ is chosen, so as to fit the initial zero-coupon curve.

Let f_j be the the instantaneous forward rate given by the j^{th} SDE, $j = 1, 2$.
Let f_M be the market instantaneous forward rate. Then

$$\psi(t) = f_M(0,t) - f_1(0,t) - f_2(0,t).$$

Next we define the following

$$\Phi(u,v) = \frac{P^M(0,v)}{P^M(0,u)}\frac{P^{CIR}(0,u)}{P^{CIR}(0,v)}$$

where P_M is the market discount factor.

The price at time t of a zero-coupon bond maturing at time T and with a unit face value is given by

$$\bar{P}(t,T) = \Phi(t,T)P^{CIR}(t,T)$$

where

$$P^{CIR}(t,T) = e^{A(t,T)+B(t,T)\cdot X(t)}$$

$$\gamma_j = \sqrt{K_j^2 + 2\sigma_j^2}, j = 1,2$$

$$A(t,T) = -\sum_{j=1}^{2} K_j\theta_j\,[\frac{2}{\sigma_j^2}\ln\frac{(K_j+\gamma_j)(e^{\gamma_j(T-t)}-1)}{2\gamma_j} + \frac{2}{K_j-\gamma_j}(T-t)];$$

$$B_j(t,T) = \frac{-2(e^{\gamma_j(T-t)}-1)}{(K_j+\gamma_j)(e^{\gamma_j(T-t)}-1)+2\gamma_j}, j = 1,2$$

So, we can write

$$\bar{P}(t,T) = \frac{P^M(0,T)}{P^M(0,t)}\left[\frac{P^{CIR}(0,t)}{P^{CIR}(0,T)}P^{CIR}(t,T)\right].$$

It is clear that

$$\bar{P}(0,T) = \frac{P^M(0,T)}{P^M(0,t)}\left[\frac{P^{CIR}(0,t)}{P^{CIR}(0,T)}\,P^{CIR}(t,T)\right] = P^M(0,T).$$

Therefore, the discount factors derived from the model should match the initial term structure.

Consider a swaption with the expiry T_0 and the fixed rate K during a period $[T_0, T_N]$. The price of the underlying swap is given by

$$SV(t) = \sum_{i=0}^{N} a_i\bar{P}(t,T_i)$$

where

$$a_0 = -1\,;\; a_i = \frac{\bar{P}(t,T_0)-\bar{P}(t,T_N)}{\sum_{i=1}^{N}\bar{P}(t,T_N)}\;(i = 1,...,N-1) \text{ and } a_N = 1 + \frac{\bar{P}(t,T_0)-\bar{P}(t,T_N)}{\sum_{i=1}^{N}\bar{P}(t,T_N)}.$$

In particular, at time $t = 0$, the swap price is given by

$$SV(0) = \sum_{i=0}^{N} a_i\bar{P}(0,T_i) = \sum_{i=0}^{N} a_iP^M(0,T_i)$$

where

$$a_0 = -1\,;$$

$$a_i = \frac{P^M(0,T_0) - P^M(0,T_N)}{\sum_{i=1}^N P^M(0,T_N)} \ (i = 1, ..., N-1) \text{ and } a_N = 1 + \frac{P^M(0,T_0) - P^M(0,T_N)}{\sum_{i=1}^N P^M(0,T_N)}.$$

The m^{th} swap moment under the T_0-forward measure conditioned on $\mathcal{F}_t$ is given by

$$M_m^*(t) = \mathbb{E}^{T_0}[\{\sum_{i=0}^N a_i \bar{P}(t,T_i)\}^m | \mathcal{F}_t]$$

Note that

$$\left[\sum_{i=0}^N a_i \bar{P}(t,T_i)\right]^m = \sum_{0 \le i_1,...,i_m \le N} a_{i_1}...a_{i_m} \left[\prod_{k=1}^m \bar{P}(T_0, T_{i_k})\right].$$

So,

$$M_m^*(t) = \sum_{0 \le i_1,...,i_m \le N} a_{i_1}...a_{i_m} \mathbb{E}^{T_0} \left[\prod_{k=1}^m \bar{P}(T_0, T_{i_k}) \Big| \mathcal{F}_t\right]$$

We have to calculate the bond moment under the T_0-forward measure, which is defined as

$$\mu^{T_0}(t, T_0, \{T_{i_1}, T_{i_2}, ..., T_{i_m}\}) := \mathbb{E}^{T_0} \left[\prod_{k=1}^m \bar{P}(T_0, T_{i_k}) \Big| \mathcal{F}_t\right].$$

Observe that

$$\bar{P}(T_0, T_{i_k}) = \frac{P^M(0,T_{i_k})}{P^M(0,T_0)} \left[\frac{P^{CIR}(0,T_0)}{P^{CIR}(0,T_{i_k})} P^{CIR}(T_0, T_{i_k})\right].$$

We have

$$\begin{aligned} M_m^*(t) &= \sum_{0 \le i_1,...,i_m \le N} a_{i_1}...a_{i_m} \mathbb{E}^{T_0} \left[\prod_{k=1}^m \frac{P^M(0,T_{i_k})}{P^M(0,T_0)} \left[\frac{P^{CIR}(0,T_0)}{P^{CIR}(0,T_{i_k})} P^{CIR}(T_0, T_{i_k})\right] \Bigg| \mathcal{F}_t\right] \\ &= \left(\frac{P^{CIR}(0,T_0)}{P^M(0,T_0)}\right)^m \sum_{0 \le i_1,...,i_m \le N} a_{i_1}...a_{i_m} \prod_{k=1}^m \frac{P^M(0,T_{i_k})}{P^{CIR}(0,T_{i_k})} \mathbb{E}^{T_0} \left[\prod_{k=1}^m P^{CIR}(T_0, T_{i_k}) \Big| \mathcal{F}_t\right] \\ &= \left(\frac{P^{CIR}(0,T_0)}{P^M(0,T_0)}\right)^m \sum_{0 \le i_1,...,i_m \le N} a_{i_1}^*...a_{i_m}^* \mathbb{E}^{T_0} \left[\prod_{k=1}^m P^{CIR}(T_0, T_{i_k}) \Big| \mathcal{F}_t\right], \end{aligned}$$

where $a_{i_k}^* = a_{i_k} \frac{P^M(0,T_{i_k})}{P^{CIR}(0,T_{i_k})}$

Therefore, we have a closed-form formula for the swaption prices under the CIR2++ model.

Theorem 2.7. *Suppose that the risk-neutral dynamics of the short rates follow the CIR2++ model.*

$$r(t) = X_1(t) + X_2(t) + \psi(t).$$

where the Q-dynamics of $X(t) = (X_1(t), X_2(t))$ *are given by the following SDEs:*

$$dX_i(t) = K_i(\theta_i - X_i(t))dt + \sigma_i \sqrt{X_i(t)} dW_i(t) \ , \ i = 1, 2$$

and $\psi(t)$ *is chosen, so as to fit the initial zero-coupon curve.*

Let $c_n^*(t)$ *be the swap cumulants which can be calculated by the swap moments* $\{M_1^*(t), ..., M_n^*(t)\}$ *and* $C_n^*(t) = c_n^*(t)P(t,T_0)^n$ *for* $n \geq 1$*. Put* $q_0 = 1$, $q_1 = q_2 = 0$ *and*

$$q_n = \sum_{m=1}^{[\frac{n}{3}]} \sum_{\substack{k_1,...k_m \geq 3, \\ k_1+...+k_m = n,}} \frac{C_{k_1}^* ... C_{k_m}^*}{m! k_1! ... k_m! \sqrt{C_2^*}^n} \ , \ n \geq 3.$$

The risk neutral price of the $T_0 \times (T_N - T_0)$*-receiver swaption* $SOV(t; T_0, T_n)$ *is given by*

$$SOV(t; T_0, T_n) = C_1^* N\left(\frac{C_1^*}{\sqrt{C_2^*}}\right) + \sqrt{C_2^*}\phi\left(\frac{C_1^*}{\sqrt{C_2^*}}\right)\left[1 + \sum_{k=3}^{\infty} (-1)^k q_k H_{k-1}\left(\frac{C_1^*}{\sqrt{C_2^*}}\right)\right].$$

3 Applications of Gram-Charlier expansions to General Models

In Theorem A.2, we provide a procedure to calculate the survival function (hence, the cumulative distribution function) and the truncated first moment of a random variable when its cumulants (or moments) are known. Theoretically speaking, we are able to calculate the prices of the European-type derivatives of any diffusion process if we are able to calculate its moments. In this section, we discuss a general procedure with worked examples in details.

3.1 A Toy Example: Black-Scholes Model

In this subsection, we show how to use Gram-Charlier expansions to calculate the price of a European call option under the standard Black-Scholes (1973) model in [3]. The closed-form formula provides a useful benchmark for our approximation method.

Assume that the price process (Q-dynamics) of an asset follows a geometric Brownian motion in the Black-Scholes Model

$$\text{i.e. } dS_t = S_t(rdt + \sigma dW_t).$$

The solution of the SDE is given by

$$S_T = S_0 \exp\left((r - \frac{\sigma^2}{2})T + \sigma W_T)\right).$$

Now, S_t follows a log-normal distribution. We can obtain the moments, and hence cumulants, of S_t easily. In fact, we have

$$M_n(t) := E^Q[S_T^n|S_0] = S_0^n \exp\left((r - \frac{\sigma^2}{2})nT + \frac{n^2\sigma^2 T}{2}\right) \text{ for } n \geq 1.$$

Moreover, we can decompose the call option price as

$$E^Q[(S_T - K)^+] = E^Q[S_T I(S_T > K)] - K[Q(S_T > K))]$$

By using theorem A.2, we are ready to give a series expansion of the option price

Proposition 3.1. *Suppose that the risk-neutral dynamics of the stock price follow Heston's model.*

$$dS_t = S_t(rdt + \sigma dW_t)$$

with the initial conditions given by $S_0 = s_0$. *Suppose that we have a European call option with strike* K. *Let* c_n *be the* n^{th}*-cumulant of* S_t. *Let* $q_0 = 1$, $q_1 = q_2 = 0$ *and*

$$q_n = \sum_{m=1}^{[\frac{n}{3}]} \sum_{\substack{k_1,...k_m \geq 3, \\ k_1+...+k_m = n,}} \frac{c_{k_1}...c_{k_m}}{m!k_1!...k_m!\sqrt{c_2}^n} \text{ , } n \geq 3.$$

Then the price of the call option price is equal to the following infinite sum

$$\sqrt{c_2}\phi\left(\frac{c_1-K}{\sqrt{c_2}}\right)+c_1N\left(\frac{c_1-K}{\sqrt{c_2}}\right)$$
$$+\sum_{n=3}^{\infty}(-1)^{n-1}q_n\phi\left(\frac{c_1-K}{\sqrt{c_2}}\right)\left[KH_{n-1}\left(\frac{c_1-K}{\sqrt{c_2}}\right)-\sqrt{C_2}H_{n-2}\left(\frac{c_1-K}{\sqrt{c_2}}\right)\right]$$
$$-K\left[N\left(\frac{c_1-a}{\sqrt{c_2}}\right)+\sum_{k=3}^{\infty}(-1)^{k-1}q_kH_{k-1}\left(\frac{c_1-a}{\sqrt{c_2}}\right)\phi\left(\frac{c_1-a}{\sqrt{c_2}}\right)\right].$$

In the rest of this subsection, we present the result of a test on our Gram-Charlier approach. Here is a list of parameters used for the model:

Parameters	*Values*
S_0	100
K	$\{80, 80.1, ..., 119.9, 120.0\}$
σ	0.03
T	1 or 2
r	0.05

We use 7 terms in the Gram-Charlier expansion. The results are given in Figures 25 - 30. From these figures, we see that that the (relative) errors are generally very small. Specifically the errors are smaller for the out-of-money options. Also, it is more accurate if the time-to-expiry is longer.

3.2 Application to a Simplified Version of the Brennan and Schwarz Model

The Brennan and Schwarz (1982) model in (See [4]) is a two-factor model of interest rates. It is specified as

$$\begin{cases} dr_t = (a_1 + b_1(l_t - r_t))dt + \sigma_1 r_t dW_t^1 \\ dl_t = l_t(a_2 - b_2 r_t + c_2 l_t)dt + \sigma_2 l_t dW_t^2 \end{cases} \tag{21}$$

where a_i's and b_i's are constants.

To make the discussion more concrete, we assume that l_t is a constant process and rewrite the process of r_t as

$$dr_t = \kappa(\theta - r_t)dt + \sigma r_t dW_t \tag{22}$$

The first step of our approximation process is to calculate the moments of r_t. We first apply the Itô's lemma to the process (r_t^n):

$$\begin{aligned} dr_t^n &= nr_t^{n-1}dr_t + \frac{n(n-1)}{2}r_t^{n-2}(dr_t)^2 \\ &= nr_t^{n-1}[\kappa(\theta - r_t)dt + \sigma r_t dW_t] + \frac{n(n-1)}{2}r_t^{n-2}\sigma^2 r_t^2 dt \\ &= [n\kappa\theta r_t^{n-1} - n\kappa r_t^n]dt + \frac{n(n-1)}{2}r_t^n\sigma^2 dt + \sigma r_t dW_t. \end{aligned}$$

In the integral from, we have

$$r_t^n - r_0^n = n\kappa\theta\int_0^t r_s^{n-1}ds + [\frac{(n-1)\sigma^2}{2} - \kappa]\int_0^t nr_s^n ds + \sigma\int_0^t r_s dW_s.$$

Assume that the parameters in 22 behave well enough, so that r_t is square-integrable. The last term becomes a martingale. Let $F_n(t) = E[r_t^n]$. By Fubini's theorem, we have

$$F_n(t) = r_0^n + n\kappa\theta\int_0^t F_{n-1}(s)ds + [\frac{(n-1)\sigma^2}{2} - \kappa]\int_0^t nF_n(s)ds.$$

In other words, $F_n(t)$ can be solved recursively in the following system of ODEs:

$$F_n'(t) = n\kappa\theta F_{n-1}(t) + [\frac{(n-1)\sigma^2}{2} - \kappa]F_n(t) \; ; \; F_n(0) = r_0^n \text{ for } n \geq 1.$$

In the rest of this subsection, we present the result of a test on our Gram-Charlier approach. Here is a list of parameters we use for the model:

$Parameters$	$Values$
r_0	0.06
r	$\{0.001, 0.002, ...0.1\}$
T	5
κ	0.2
θ	0.05
σ	0.115

We approximate the distribution of r_t by using Theorem A.2. The moments of r_t are calculated by solving the system of ODEs discussed above with Mathematica. We use 7 terms in the Gram-Charlier expansion. The results are given in Figures 31 and 32. From these figures, we see that the approximation provides a fairly good fit to the model. The error in the middle region is relatively higher, and up to 0.02. This error is likely to be acceptable from the perspective of risk management.

4 Pricing Call Options under Heston's Model using Gram-Charlier expansions

4.1 Introduction to Heston's Model of Stochastic Volatility

We assume that the risk-neutral dynamics of the stock price follows Heston's (1993) stochastic volatility model in [10]. It is given by the following system of SDEs

$$\begin{cases} dS_t = S_t(rdt + \sqrt{V_t}dW_t^1) \\ dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^2 \end{cases} \tag{23}$$

with the initial conditions given by $S_0 = s_0$ and $V_0 = v_0 \geq 0$, where $\kappa, \theta, \sigma > 0$ and $dW_t^1 dW_t^2 = \rho dt$, $\rho \in [-1, 1]$.

Let $X_t = \ln S_t - rt$ be the logarithm of the discounted stock price. By Itô's lemma, we have

$$dX_t = -rdt + \frac{dS_t}{S_t} - \frac{1}{2S_t^2}(dS_t \cdot dS_t) = -\frac{1}{2}V_t dt + \sqrt{V_t}dW_t^1,$$

with the initial condition given by $X_o = x_o = \ln S_0$.

Thus, we can transform the system (28) into the following system

$$\begin{cases} dX_t = -\frac{1}{2}V_t dt + \sqrt{V_t} dW_t^1 \\ dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^2 \end{cases} \tag{24}$$

with the initial conditions given by $X_0 = x_0$ and $V_0 = v_0 \geq 0$.

We can show that the moment generating function of X_t (See [11]) is given by

$$M_t(u) = \mathbb{E}[e^{uX_t}] = e^{x_0 u} \left(\frac{e^{(\kappa - \sigma\rho t)/2}}{\cosh(P(u)t/2) + (\kappa - \sigma\rho u)\sinh(P(u)t/2)/P(u)} \right)^{2\kappa\theta/\sigma^2} \cdot \exp\left(-v_0 \frac{(u - u^2)\sinh(P(u)t/2)/P(u)}{\cosh(P(u)t/2) + (\kappa - \sigma\rho u)\sinh(P(u)t/2)/P(u)} \right) \tag{25}$$

where

$$P(u) = \sqrt{(\kappa - \rho c u)^2 + c^2(u - u^2)}.$$

Hence, the cumulants of X_t can be calculated by

$$c_n = \frac{d^n}{du^n}[\ln M_t(u)]\bigg|_{u=0} \quad \text{for } n = 1, 2, ...$$

In practice, the higher derivatives in the expression can be calculated reasonably fast by using any available symbolic calculation software.

4.2 Calculating Truncated Moment Generating Function using Gram-Charlier Expansions

Proposition 4.1. *Let Y be a random variable, such that it has a continuous density function $f(x)$ and finite cumulants $(c_k)_{k\in\mathbb{N}}$. Let $q_0 = 1$, $q_1 = q_2 = 0$ and*

$$q_n = \sum_{m=1}^{[\frac{n}{3}]} \sum_{\substack{k_1,...k_m \geq 3, \\ k_1+...+k_m = n,}} \frac{c_{k_1}...c_{k_m}}{m! k_1!...k_m! \sqrt{c_2}^n} \ , \ n \geq 3.$$

Suppose that e^{aY} is integrable where $a \in \mathbb{R}$. Then the following results are obtained

(a) *The truncated below moment generating function of Y is given by*

$$\mathbb{E}[e^{ax} I(Y \leq K)] = e^{aC_1} \sum_{n=0}^{\infty} q_n I_n \left(\frac{K - C_1}{\sqrt{C_2}}, a\sqrt{C_2} \right) \tag{26}$$

where $I_n = I_n(x, a)$ satisfies the following recurrence

$$I_0(x, a) = e^{\frac{b^2}{2}} N(x - a) \ ; \ I_n(x, b) = aI_{n-1}(x, a) - H_{n-1}(x)\phi(x)e^{ax}.$$

(b) *The truncated above moment generating function of Y is given by*

$$\mathbb{E}[e^{ax} I(Y \geq K)] = e^{aC_1} \sum_{n=0}^{\infty} q_n J_n \left(\frac{K - C_1}{\sqrt{C_2}}, a\sqrt{C_2} \right) \tag{27}$$

where $J_n = J_n(x, a)$ satisfies the following recurrence

$$J_0(x, a) = e^{\frac{b^2}{2}} N(a - x) \ ; \ J_n(x, a) = aJ_{n-1}(x, a) + H_{n-1}(x)\phi(x)e^{ax}.$$

Proof. We first prove part (a). Recall that

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right).$$

We have

$$\begin{aligned}
\mathbb{E}[e^{ax} I(Y \leq K)] &= \int_{-\infty}^{K} e^{ax} \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \\
&= \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} \int_{-\infty}^{K} e^{ax} H_n \left(\frac{x - c_1}{\sqrt{c_2}} \right) \phi \left(\frac{x - c_1}{\sqrt{c_2}} \right) dx \\
&= \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} \int_{-\infty}^{\frac{K - C_1}{\sqrt{C_2}}} H_n(y)\phi(y)e^{a\sqrt{C_2}y + aC_1} \sqrt{C_2} dy \\
&= \sum_{n=0}^{\infty} q_n e^{aC_1} \int_{-\infty}^{\frac{K - C_1}{\sqrt{C_2}}} H_n(y)\phi(y)e^{a\sqrt{C_2}y} dy
\end{aligned}$$

Let $I_n(x, a) = \int_{\infty}^{x} H_n(y)\phi(y)e^{ay}dy$. Write $I_n := I_n(x, a)$ for convenience. When $n = 0$, we have

$$\begin{aligned}
I_0 &= \int_{-\infty}^{x} H_0(y)\phi(y)e^{ay}dy \\
&= \int_{-\infty}^{x} \phi(y)e^{ay}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}e^{ay}dy \\
&= \frac{e^{\frac{a^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{(y-a)^2}{2}}dy \\
&= e^{\frac{a^2}{2}}N(x-a).
\end{aligned}$$

Note that

$$D[(D^{n-1}\phi(x))e^{ax}] = [D^n\phi(x)]e^{ax} + ae^{ax}[D^{n-1}\phi(x)].$$

We have

$$D[(-1)^{n-1}H_{n-1}(x)\phi(x)e^{ax}] = (-1)^n H_n(x)\phi(x)e^{ax} + (-1)^{n-1}aH_{n-1}(x)\phi(x)e^{ax}.$$

It follows that

$$H_{n-1}(x)\phi(x)e^{ax} = -\int_{-\infty}^{x} H_n(y)\phi(y)e^{ay}dy + a\int_{-\infty}^{x} H_{n-1}(y)\phi(y)e^{ay}dy.$$

Hence,

$$I_n = aI_{n-1} - H_{n-1}(x)\phi(x)e^{ax}.$$

Therefore, the proof of (a) is completed.

For the proof of part (b), we have

$$\begin{aligned}
\mathbb{E}[e^{ax}I(Y \geq K)] &= \int_K^{\infty} e^{ax} \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \\
&= \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} \int_K^{\infty} e^{ax} H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \\
&= \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} \int_{\frac{K-C_1}{\sqrt{C_2}}}^{\infty} H_n(y)\phi(y) e^{a\sqrt{C_2}y + aC_1} \sqrt{C_2} dy \\
&= \sum_{n=0}^{\infty} q_n e^{aC_1} \int_{\frac{K-C_1}{\sqrt{C_2}}}^{\infty} H_n(y)\phi(y) e^{a\sqrt{C_2}y} dy
\end{aligned}$$

Let $J_n(x,a) = \int_x^{\infty} H_n(y)\phi(y)e^{ay}dy$. Write $J_n := J_n(x,a)$ for convenience. When $n = 0$, we have

$$\begin{aligned}
J_0 &= \int_x^{\infty} H_0(y)\phi(y)e^{ay}dy \\
&= \int_x^{\infty} \phi(y)e^{ay}dy \\
&= \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{y^2}{2}} e^{ay} dy \\
&= \frac{e^{\frac{a^2}{2}}}{\sqrt{2\pi}} \int_x^{\infty} e^{-\frac{(y-a)^2}{2}} dy \\
&= e^{\frac{a^2}{2}} N(a-x).
\end{aligned}$$

Note that

$$D[(D^{n-1}\phi(x))e^{ax}] = [D^n\phi(x)]e^{ax} + ae^{ax}[D^{n-1}\phi(x)].$$

We have

$$D[(-1)^{n-1}H_{n-1}(x)\phi(x)e^{ax}] = (-1)^n H_n(x)\phi(x)e^{ax} + (-1)^{n-1} a H_{n-1}(x)\phi(x)e^{ax}.$$

It follows that

$$-H_{n-1}(x)\phi(x)e^{ax} = -\int_x^{\infty} H_n(y)\phi(y)e^{ay}dy + a\int_x^{\infty} H_{n-1}(y)\phi(y)e^{ay}dy.$$

Hence,

$$J_n = aJ_{n-1} + H_{n-1}(x)\phi(x)e^{ax}.$$

Therefore, the proof of (b) is completed. $\square$

4.3 Pricing Call Options under the Heston model

Let $X_t = \ln S_t - rt$ be the logarithm of the discounted stock price. We have $e^{X_t} = e^{-rt}S_t$. The price of the European call option with strike K is given by

$$C = \mathbb{E}[e^{-rt}(S_t - K)^+] = \mathbb{E}[(e^{X_t} - e^{-rt}K)^+].$$

Put $k = \ln K - rt$. We may rewrite the price as

$$C = \mathbb{E}[(e^{X_t} - e^k)^+].$$

Hence, the price of the call option can be calculated by the following formula

$$C = \mathbb{E}[e^{X_t} I(X_t > k)] - e^k \mathbb{E}[I(X_t > k)].$$

The first term on the right-hand side of the above expression is just a truncated Moment Generating Function (MGF) which can be calculated via equation (27) and the second term can be calculated by the formula given in Theorem A.2. Therefore, we are able to calculate the price of any European call option whenever the moments (or the cumulants) of the log-prices possess analytical formulas.

To sum up, we have the following formula:

Theorem 4.2. *Suppose that the risk-neutral dynamics of the stock price follow Heston's (1993) stochastic volatility model.*

$$\begin{cases} dS_t = S_t(rdt + \sqrt{V_t}dW_t^1) \\ dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^2 \end{cases} \tag{28}$$

with the initial conditions given by $S_0 = s_0$ and $V_0 = v_0 \geq 0$, where $\kappa, \theta, \sigma > 0$ and $dW_t^1 dW_t^2 = \rho dt$, $\rho \in [-1, 1]$. Suppose that we have a European call option with strike K.

Let $X_t = \ln S_t - rt$, $k = \ln K - rt$ *and* c_n *be the* n^{th}*-cumulant of* X_t.

Let $q_0 = 1$, $q_1 = q_2 = 0$ *and*

$$q_n = \sum_{m=1}^{[\frac{n}{3}]} \sum_{\substack{k_1,...k_m \geq 3, \\ k_1+...+k_m=n,}} \frac{c_{k_1}...c_{k_m}}{m!k_1!...k_m!\sqrt{c_2}^n} \ , n \geq 3.$$

The price of the call option price is equal to the following infinite sum

$$e^{c_1}\sum_{n=0}^{\infty} q_n J_n\left(\frac{k-c_1}{\sqrt{c_2}}, \sqrt{c_2}\right) - e^k\left[N\left(\frac{c_1-k}{\sqrt{c_2}}\right) + \sum_{n=3}^{\infty}(-1)^{n-1}q_n H_{n-1}\left(\frac{c_1-k}{\sqrt{c_2}}\right)\phi\left(\frac{c_1-k}{\sqrt{c_2}}\right)\right]$$

where $J_n = J_n(x,a)$ *satisfies the following recurrence:*

$$J_0(x,a) = e^{\frac{b^2}{2}}N(a-x) \ ; \ J_n(x,a) = aJ_{n-1}(x,a) + H_{n-1}(x)\phi(x)e^{ax}.$$

4.4 A Monte-Carlo Simulation Method for the Heston Model

In order to investigate the accuracy of our result, we calculate the options prices based on a Monte-Carlo method. Note that the second equation of the Heston system is a CIR-type mean-reverting process. Thus it is tempting to use an exact simulation method since the distribution of V_t is known as a non-central chi-square distribution. However, it turns out to be very cumbersome to include the correlation of the Brownian motions because the Cholesky decomposition is not applicable in this case.

Inspired by the result reported in Alfonsi (2005) in [1], we use the implicit scheme for $(\sqrt{V_t})$ and an exact simulation for (S_t). To make it clear, we first obtain the SDE for $(\sqrt{V_t})$ by Itô's lemma

$$d\sqrt{V_t} = \frac{\kappa\theta - \sigma^2/4}{2\sqrt{V_t}}dt - \frac{\kappa}{2}\sqrt{V_t}dt + \frac{\sigma}{2}dW_t^2.$$

Let the time grid be $\{t_0, ..., t_n\}$ where $t_0 = 0$, $t_n = T$ and $t_i = \frac{iT}{n}$ for $i = 1, ..n$. We obtain the following equation by impliciting the drift term:

$$\sqrt{V_{t_{i+1}}} - \sqrt{V_{t_i}} = \left(\frac{\kappa\theta - \sigma^2/4}{2\sqrt{V_{t_{i+1}}}} - \frac{\kappa}{2}\sqrt{V_{t_{i+1}}}\right)\frac{T}{n} + \frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}).$$

After this simplification, we obtain a quadratic equation in $\sqrt{V_{t_{i+1}}}$,

$$\left(1+\frac{\kappa T}{2n}\right)(\sqrt{V_{t_{i+1}}})^2 - \left[\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_i}\right]\sqrt{V_{t_{i+1}}} - \left(\frac{\kappa\theta - \sigma/4}{2}\right)\frac{T}{n} = 0,$$

which has only one positive root when $\sigma^2 < 4\kappa\theta$,

$$V_{t_{i+1}} = \left[\frac{\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}} + \sqrt{(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}})^2 + 4(1+\frac{\kappa T}{2n})(\frac{\kappa\theta - \sigma^2/4}{2})\frac{T}{n}}}{2(1+\frac{\kappa T}{2n})}\right]^2$$

Since $\frac{1}{(1+x)^2} \approx 1 - 2x$ when x is small, we have

$$V_{t_{i+1}} \approx \frac{1}{4}\left(1-\frac{kT}{n}\right)\left\{2\left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)^2 + 4\left(1+\frac{\kappa T}{2n}\right)\left(\frac{\kappa\theta - \sigma^2/4}{2}\right)\frac{T}{n} + \right.$$
$$\left. 2\left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)\sqrt{\left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)^2 + 4\left(1+\frac{\kappa T}{2n}\right)\left(\frac{\kappa\theta - \sigma^2/4}{2}\right)}\right\}$$

Moreover, note that for small $x, y > 0$, we have

$$x\sqrt{x^2+y} = x^2\sqrt{1+\frac{y}{x^2}} \approx x^2(1+\frac{y}{2x^2}) = x^2 + \frac{y}{2}.$$

It follows that

$$2x^2 + y + 2x\sqrt{x^2+y} \approx 4x^2 + 2y.$$

Thus, we can further approximate $V_{t_{i+1}}$ by

$$V_{t_{i+1}} \approx \frac{1}{4}\left(1-\frac{kT}{n}\right)\left\{4\left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)^2 + 8\left(1+\frac{\kappa T}{2n}\right)\left(\frac{\kappa\theta - \sigma^2/4}{2}\right)\frac{T}{n}\right\}.$$

Now we fix V_{t_i} and conserve the terms in $\frac{T}{n}$, $(W_{t_{i+1}} - W_{t_i})$ and $(W_{t_{i+1}} - W_{t_i})^2$ using a Taylor expansion

$$\begin{aligned}
V_{t_{i+1}} &\approx \left(1-\frac{kT}{n}\right)\left\{\left(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{V_{t_i}}\right)^2 + \left(1+\frac{\kappa T}{2n}\right)\left(\kappa\theta - \frac{\sigma^2}{4}\right)\frac{T}{n}\right\} \\
&\approx V_{t_i}\left(1-\frac{kT}{n}\right) + \sigma(W_{t_{i+1}} - W_{t_i})\sqrt{V_{t_i}} + \frac{\sigma^2}{4}(W_{t_{i+1}} - W_{t_i})^2 + \left(\kappa\theta - \frac{\sigma^2}{4}\right)\frac{T}{n} \\
&\approx V_{t_i}\left(1-\frac{kT}{2n}\right)^2 + \sigma(W_{t_{i+1}} - W_{t_i})\sqrt{V_{t_i}} + \left(\frac{\sigma(W_{t_{i+1}} - W_{t_i})}{2(1-\frac{kT}{2n})}\right)^2 + \left(\kappa\theta - \frac{\sigma^2}{4}\right)\frac{T}{n} \\
&= \left(\sqrt{V_{t_i}}\left(1-\frac{kT}{2n}\right) + \frac{\sigma(W_{t_{i+1}} - W_{t_i})}{2(1-\frac{kT}{2n})}\right)^2 + \left(\kappa\theta - \frac{\sigma^2}{4}\right)\frac{T}{n}
\end{aligned}$$

To sum up, we have the following algorithm for the Heston model

1. Set $S \leftarrow s0$, $V \leftarrow v0$.

2. Generate a pair of independent $Z_1, Z_2 \sim N(0,1)$.

3. Let $U_1 = \sqrt{1-\rho^2} Z_1 + \rho Z_2$ and $U_2 = Z_2$.

4. Generate

$$V \leftarrow \left[\sqrt{V} \left(1 - \frac{kT}{2n} \right) + \frac{\sigma(\sqrt{\frac{T}{n}} U_2)}{2(1 - \frac{kT}{2n})} \right]^2 + \left(\kappa\theta - \frac{\sigma^2}{4} \right) \frac{T}{n}.$$

5. Generate

$$S \leftarrow \exp\left(\left(r - \frac{\sigma^2}{2} \right) \frac{T}{n} + \sqrt{V} \sqrt{\frac{T}{n}} U_1 \right).$$

We assume the following parameters in the Heston model to demonstrate the mean behavior of the scenarios generated by the Alfonsi's scheme:

$Parameters$	$Values$
S_0	100
V_0	0.03
κ	$0.5, 1, 1.5$
θ	0.05
σ	0.30
ρ	-0.45
T	10
r	0.04

We use 250 time steps per year and generate 10,000 scenarios.The results are given in Figures 33 and 34.

5 Numerical results

Next we consider the following parameters in the Heston model:

Parameters	*Values*
S_0	100
V_0	0.03
K	$\{50, 51, ..., 149, 150\}$
κ	0.15
θ	0.05
σ	0.05
ρ	-0.55
T	1
r	0.04

We take $N = 1,000,000$ Scenarios and 250 time steps per year for the Monte-Carlo simulation of the call option prices. We approximate the price of the call option using only first N terms in the Gram-Charlier expansions and denote them by $GC(N)$. We also study $GC(ND)$ where $N = 3, 4, 5$. They are just $GC7$'s with $C_{N+1} = ... = C_7 = 0$ where $N = 3, 4, 5$.

Since the Fourier Transform (FT) approach is widely used in calculating the option price under the Heston model, we also incorporate the FT results in our graph for comparison purpose.

Selected Numerical results:

Value/ Strike	50	80	90	100	110	120	150
MC	51.9653	23.7206	15.5576	9.0765	4.6458	2.0766	0.0872
FT	51.9612	23.7138	15.5526	9.0761	4.6510	2.0835	0.0880
GC3	51.9603	23.7141	15.5635	9.0932	4.6667	2.0889	0.0694
GC4	51.9608	23.7216	15.5510	9.0620	4.6424	2.0883	0.0893
GC5	51.9608	23.7199	15.5469	9.0612	4.6469	2.0939	0.0871
GC3D	51.9604	23.7076	15.5648	9.1068	4.6755	2.0834	0.0674
GC4D	51.9609	23.7154	15.5523	9.0750	4.6508	2.0831	0.0874
GC5D	51.9609	23.7136	15.5482	9.0742	4.6553	2.0887	0.0852

We see from Figures 35 - 38 that the (relative) errors are generally very small for the out-of-money options. The GC4D and GC5D approach are generally better than other approximations. They occasionally outperform the FT approach.

We test the result with other parameters where the time-to-expiry is smaller than the previous one.

$Parameters$	$Values$
S_0	100
V_0	0.03
K	$\{50, 51, ..., 149, 150\}$
κ	0.15
θ	0.05
σ	0.05
ρ	-0.55
T	4
r	0.04

Selected Numerical results:

Value/ Strike	50	80	90	100	110	120	150
MC	57.5787	34.5536	28.0928	22.4256	17.5883	13.5663	5.6970
FT	57.5982	34.5743	28.1104	22.4405	17.6010	13.5764	5.7030
GC3	57.5589	34.5966	28.1521	22.4947	17.6604	13.6328	5.6958
GC4	57.6126	34.5736	28.0705	22.3704	17.5222	13.5110	5.7302
GC5	57.6134	34.5410	28.0416	22.3582	17.5335	13.5458	5.7918
GC3D	57.5517	34.5983	28.1902	22.5629	17.7389	13.6991	5.6607
GC4D	57.6065	34.5750	28.1026	22.4278	17.5882	13.5668	5.7007
GC5D	57.6073	34.5424	28.0737	22.4155	17.5995	13.6015	5.7623

We see from Figures 39 - 42 that the (relative) errors are generally very small for the out-of-money options. The GC4D approach is generally better than other approximations.

It outperforms the FT approach when the option is in the at-time money region. Increasing the number of terms in the approximation formula does not appear to increase the accuracy systematically since the Gram-Charlier expansions are known to be orthogonal series. Emprical results show that $GC4D$ outperforms other methods in general.

6 Conclusion

This paper discussed several important applications of Gram-Charlier expansions in pricing swaptions and European call options in finance. It is important to stress that the Gram-Charlier expansions can be used in any affine-term structure model. Our work on the extension of this method from the CIR2 model to the CIR2++ model can be generalized to any affine-term structure++ model (i.e. models with the fitting of the initial term structure). Empirical results show that GC3 (the Gram-Charlier expansions up to the third cumulants) gives the most efficient and accurate approximation for the swaption prices.

We discussed a procedure on how to apply the Gram-Cahrlier approach to general models in section 3. The models are reasonably simple. For example, the drift and diffusion terms are polynomials. Moments can be found by solving a system of ODEs, which is derived by Fubini's Theorem and martingale properties of It$\hat{o}$ integrals as shown in subsection 3.2. This allows us to calculate prices of any European-type derivatives.

For the Heston model of stochastic volatlity, the European call option price has traditionally been obtained by a Fourier Transform derived in Carr and Madan (1999) in [7]. This method is proven to be both accurate and efficient. For a given set of parameters, we are able to use a fast Fourier transform to calculate the option prices of different strikes. While the logarithm of the strike prices is assumed to be equally spaced, the strike prices themselves cannot be equally spaced. However, in our method, cumulants are fixed whenever the parameters are given for the model. Thus, the option prices with different strikes can be calculated in a parallel manner since the Gram-Charlier expansions can be easily implemented in this case. For example, we are able to calculate 10,000 option prices with arbitrary strikes within 0.003 seconds. Therefore, our approach is more efficient if the parameters are already calibrated or given in advanced.

In principle, we can extend our approach to any stochastic volatility models with a reasonable degree of complexity. That is, if the moment generating functions (or moments themselves) can be derived for a given model, then our approach can be readily applied to it.

However there are a few important limitations to our approach. First the main underlying assumption in the Gram-Charlier expansions is that the cumulants of the random variable are finite. This assumption is stronger than expected in stochastic modeling. For example, pure jump processes such as the variance gamma and so on, do not process this property in general. In these cases, our approach is not expected to outperform the FT approach. Second, the error in the approximations is hard to be estimated rigorously in our approach since the Gram-Charlier expansions are known to be orthogonal series. In fact there seems to be no guarantee that adding finitely more terms to the expansion will imporve the accuracy of the approximation. In sum, further rigorous testing procedures for the accuracy and efficiency of the Gram-Charlier expansions are warranted for pricing purposes.

References

[1] A. Alfonsi, *On the Discretization Schemes for the CIR (and Bessel Squared) Processes*, Monte Carlo Methods and Applications , Volume 11 (4), 2005

[2] T. Björk, *Arbitrage Theory in Continuous Time*, Oxford Finance, 2009

[3] F. Black, M. Scholes, *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy 81 (3): 637654, 1973.

[4] M. Brennan , E. Schwartz, *An Equilibrium model of Bond Pricing and a Test of Market Efficiency*, Journal of Financial and Quantitative Analysis, 17, 3, 301-29, 1982

[5] D. Brigo, F. Mercurio, *On Deterministic Shift Extensions of Short Rate Models, 2001, working paper*

[6] D. Brigo, F. Mercurio, *Interest Rate Models Theory and Practice with Smile, Inflation and Credit*, (2nd ed. 2006 ed.), Springer Verlag.

[7] P. Carr, D. B. Madan, *Option Valuation using the Fast Fourier Transform*, J. Comput. Finance, 2(4):6173, 1999.

[8] P. Collin-Dufresne, R. Goldstein, *Pricing Swaptions Within an Affine Framework, The Journal of Derivatives, Vol. 10, No. 1, pp. 9-26, 2002*

[9] J.C. Cox, J.E. Ingersoll and S.A. Ross, *A Theory of the Term Structure of Interest Rates*, Econometrica 53, 385-407, 1985.

[10] Heston, S. *A Closed-form Solution for Options with Stochastic volatility with Applications to Bonds and Currency Options*, Review of Financial Studies 6, 327343, 1993.

[11] S. del Bano Rollin, A. Ferreiro-Castilla, and F.Utzet, *A New Look at the Heston Characteristic Function, working paper, 2009*

[12] K. Tanaka, T. Yamada, T. Watanabe, *Applications of GramCharlier Expansion and Bond Moments for Pricing of Interest Rates and Credit Risk*, Quantitative Finance, Volume 10, Issue 6, 2010

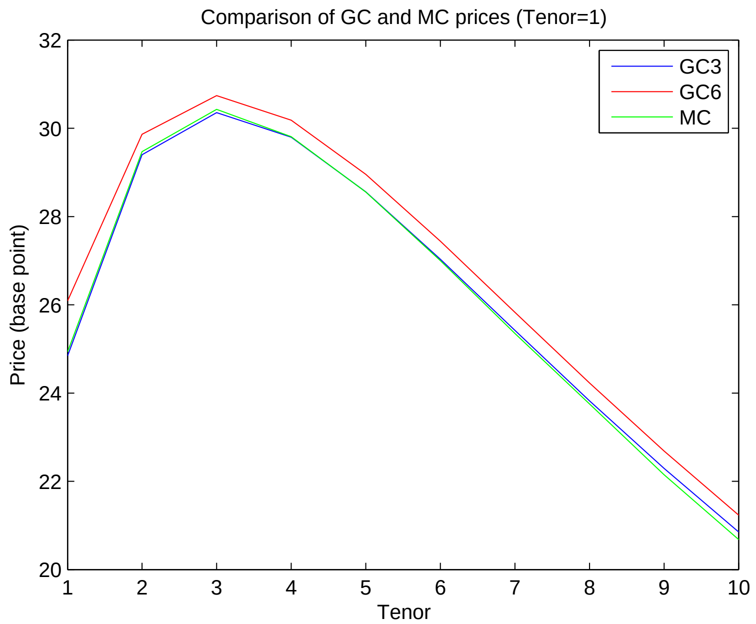

Figure 1: **C**omparison of swaption prices (Tenor = 1)

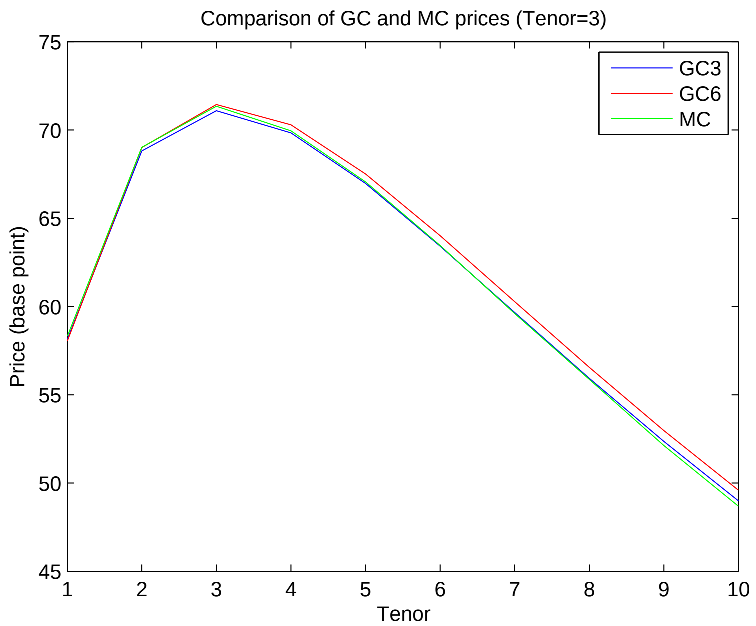

Figure 2: **C**omparison of swaption prices (Tenor = 3)

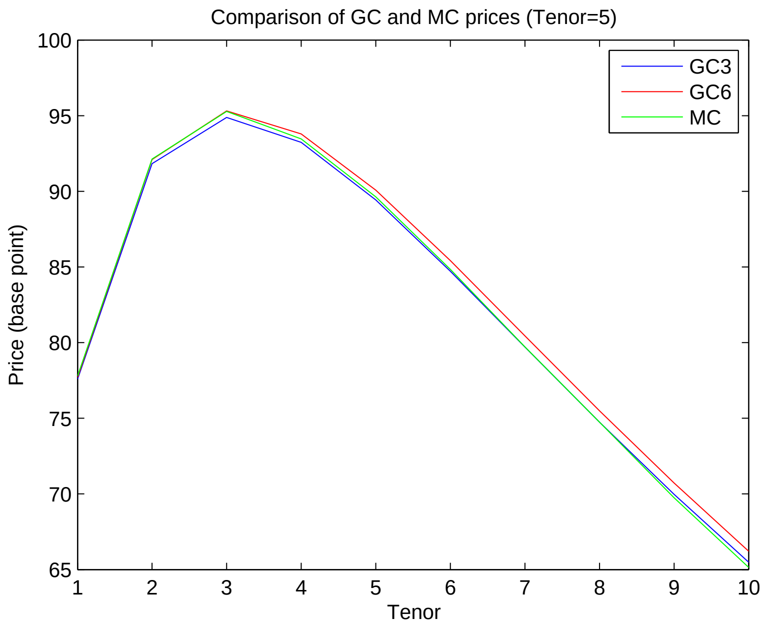

Figure 3: Comparison of swaption prices (Tenor = 5)

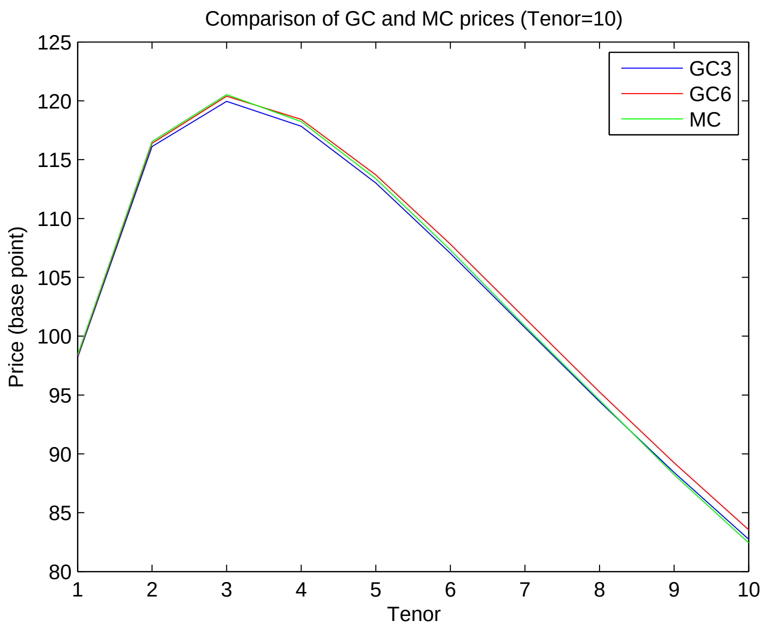

Figure 4: Comparison of swaption prices (Tenor = 10)

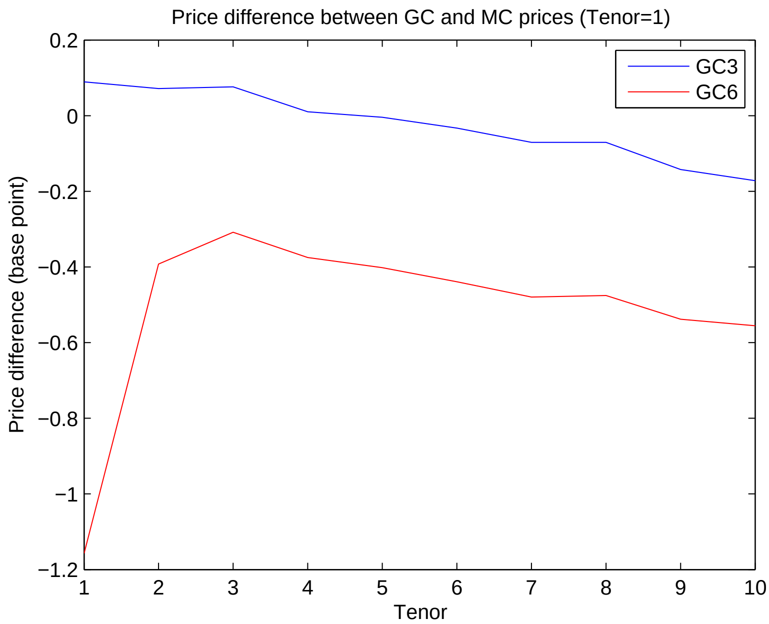

Figure 5: Pricing Errors of swaption prices (Tenor = 1)

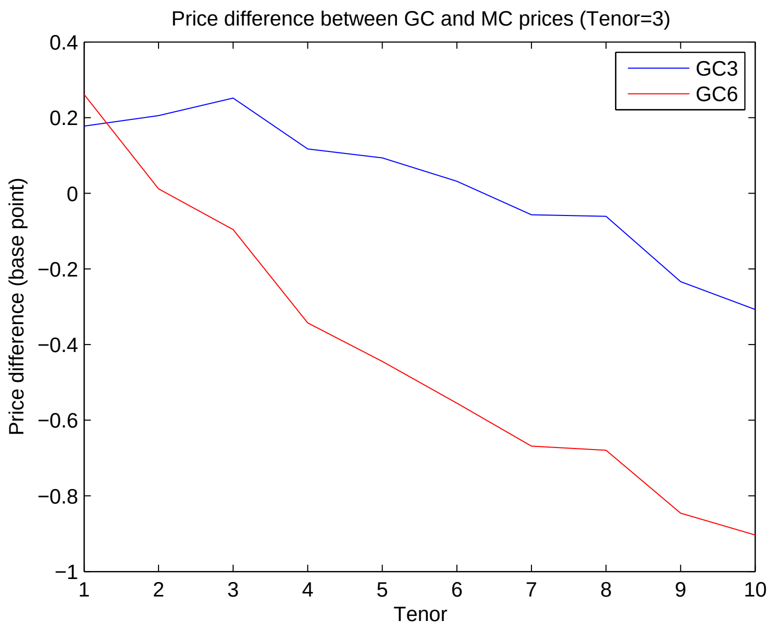

Figure 6: Pricing Errors of swaption prices (Tenor = 3)

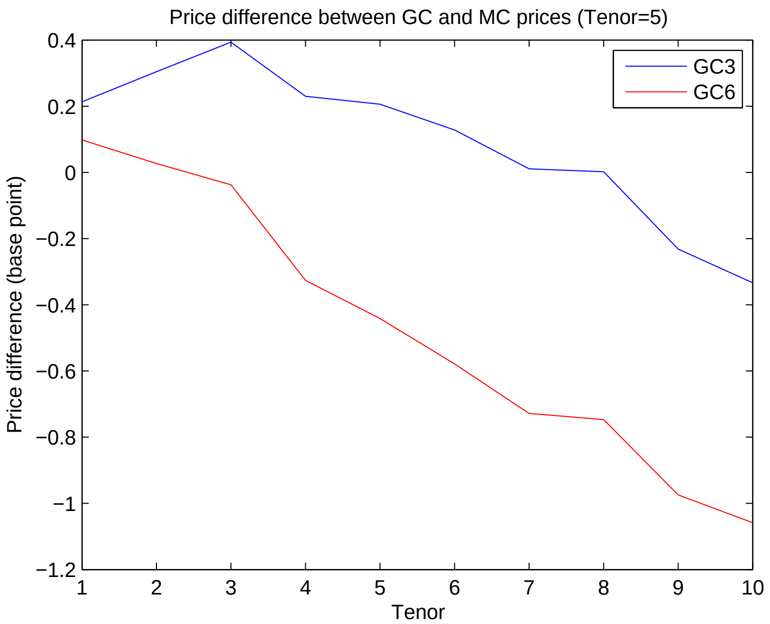

Figure 7: **P**ricing Errors of swaption prices (Tenor = 5)

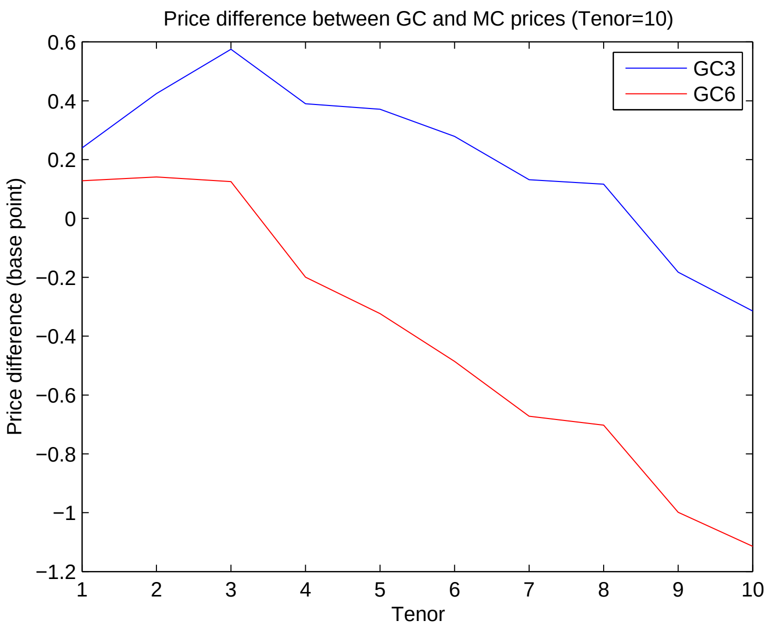

Figure 8: **P**ricing Errors of swaption prices (Tenor = 10)

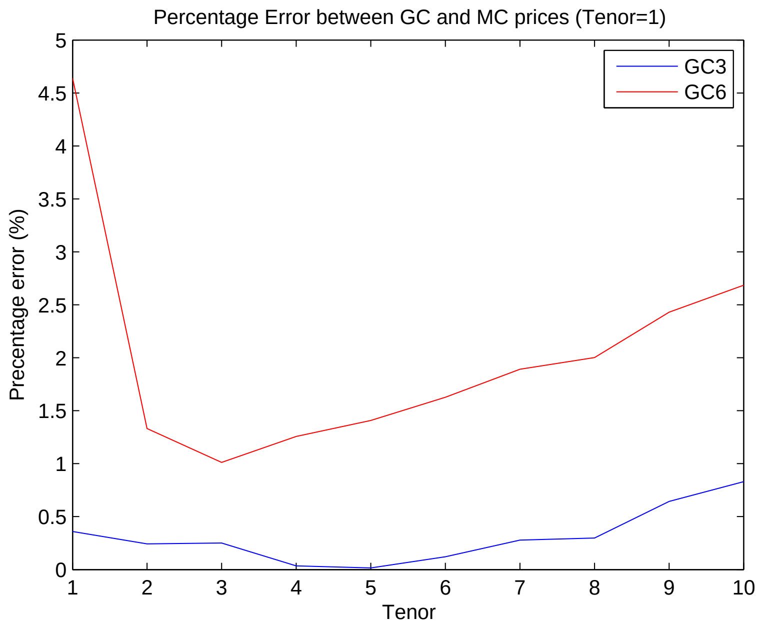

Figure 9: Percentage Errors of swaption prices (Tenor = 1)

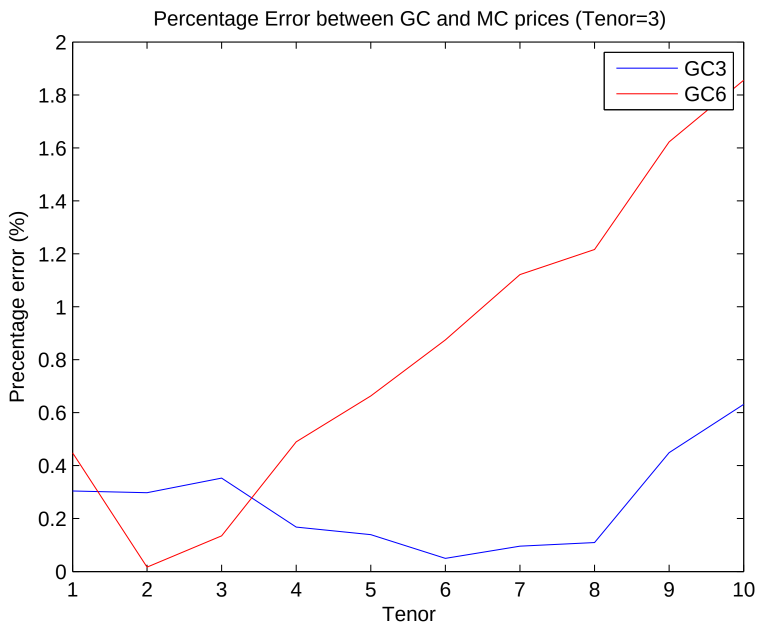

Figure 10: Percentage Errors of swaption prices (Tenor = 3)

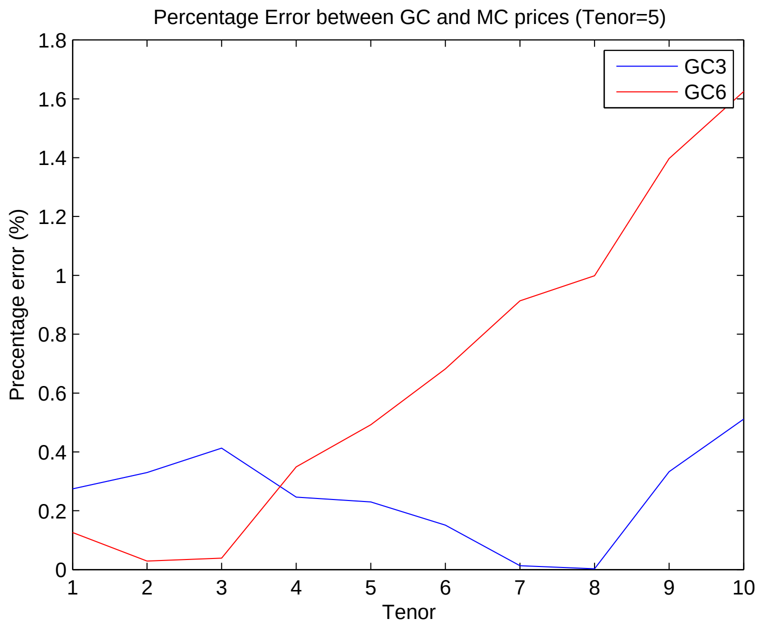

Figure 11: **P**ercentage Errors of swaption prices (Tenor = 5)

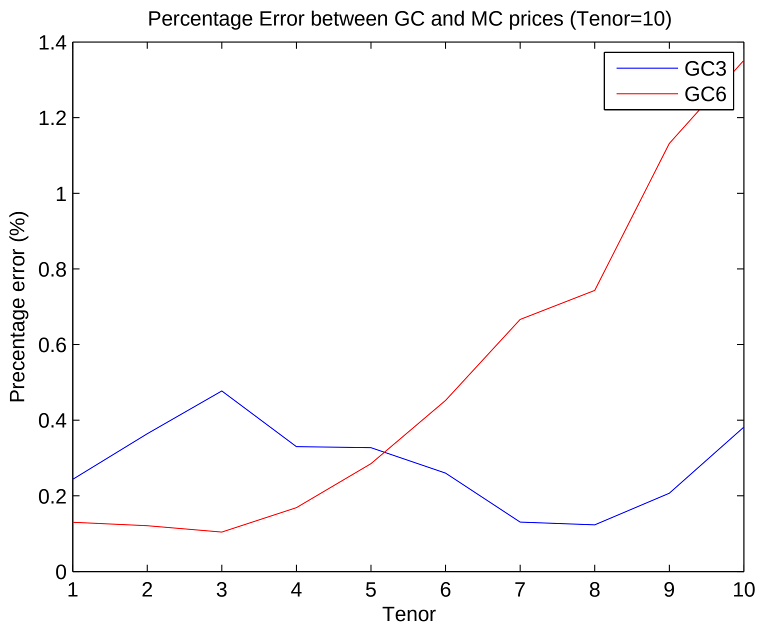

Figure 12: **P**ercentage Errors of swaption prices (Tenor = 10)

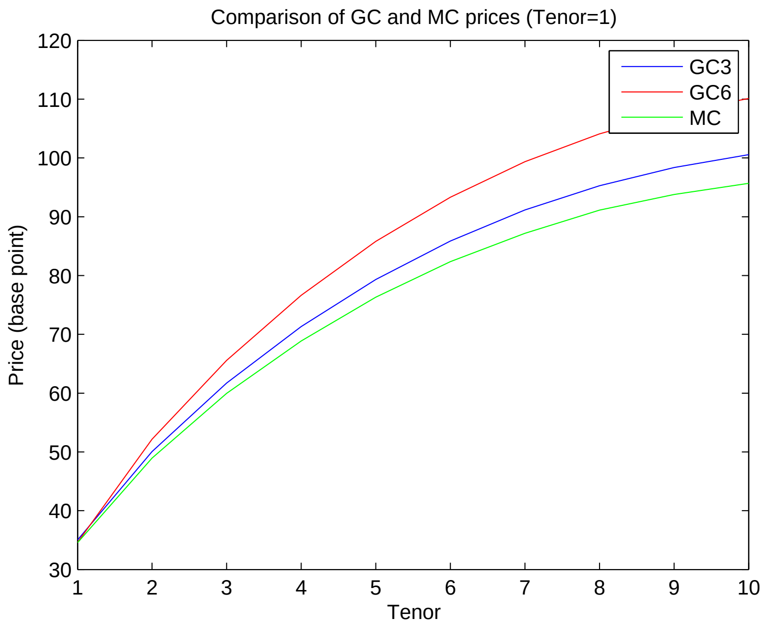

Figure 13: **C**omparison of swaption prices (Tenor = 1)

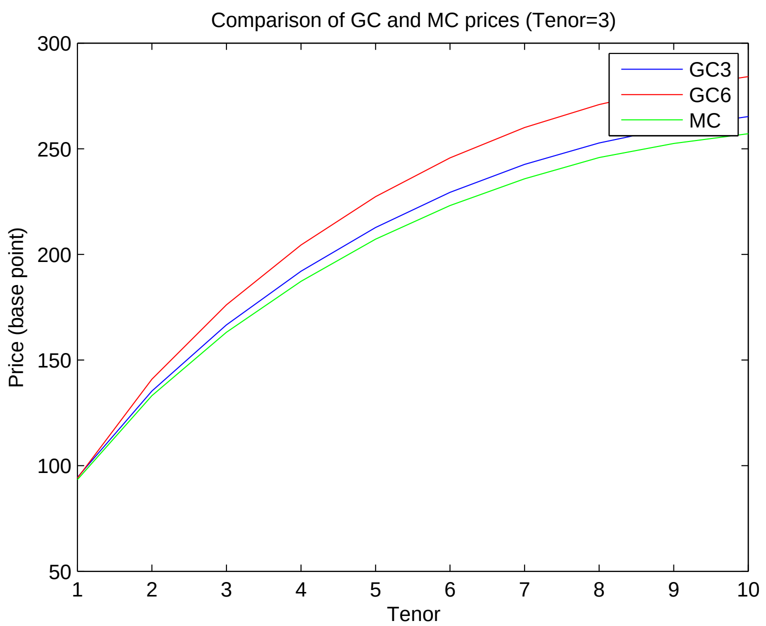

Figure 14: **C**omparison of swaption prices (Tenor = 3)

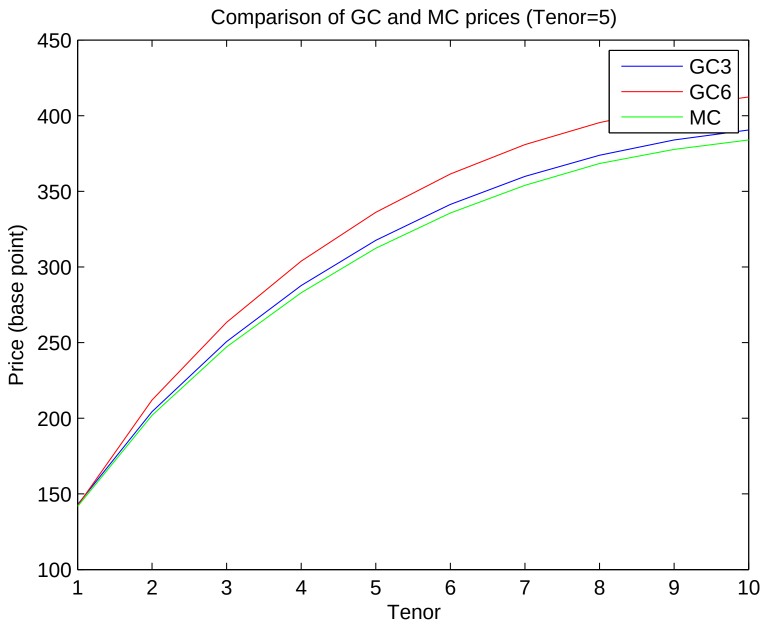

Figure 15: **C**omparison of swaption prices (Tenor = 5)

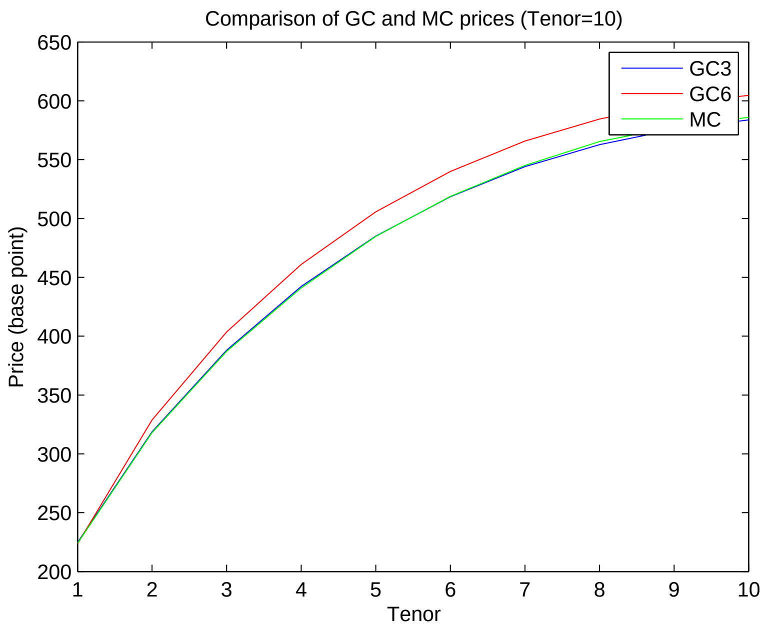

Figure 16: **C**omparison of swaption prices (Tenor = 10)

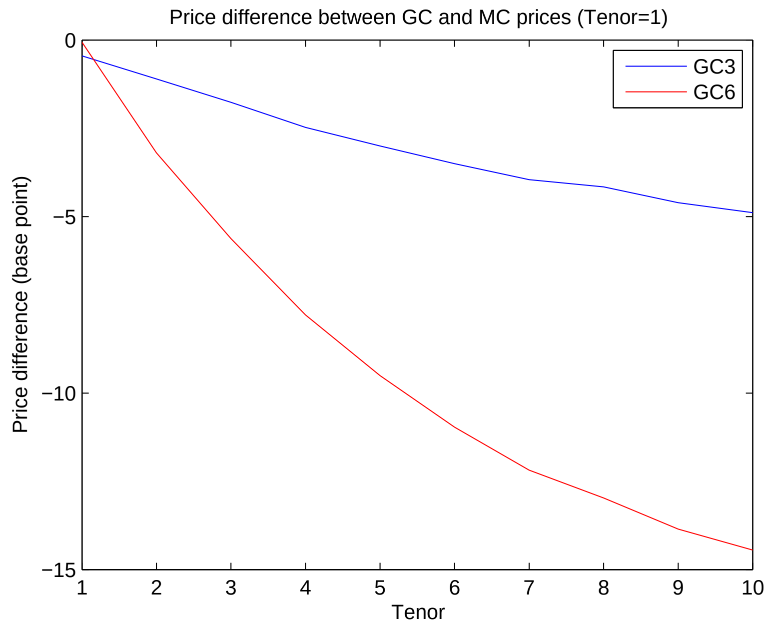

Figure 17: Pricing Errors of swaption prices (Tenor = 1)

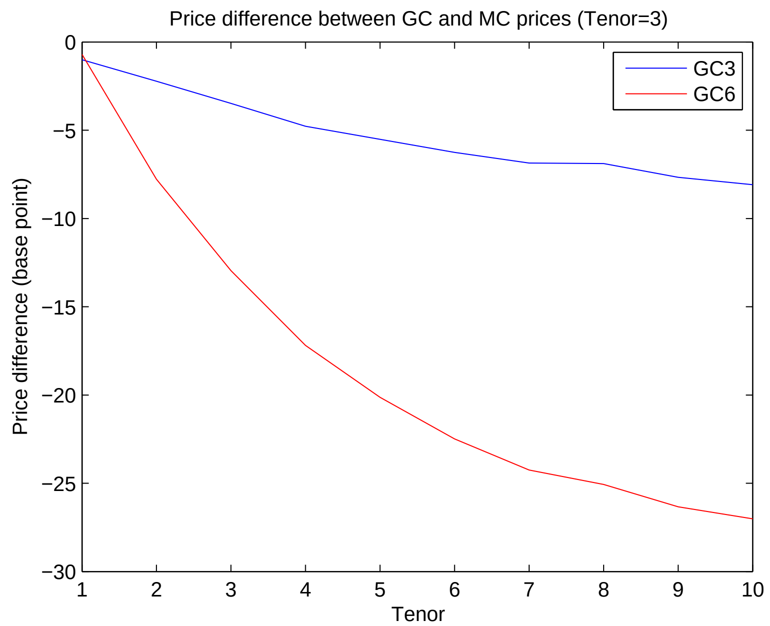

Figure 18: Pricing Errors of swaption prices (Tenor = 3)

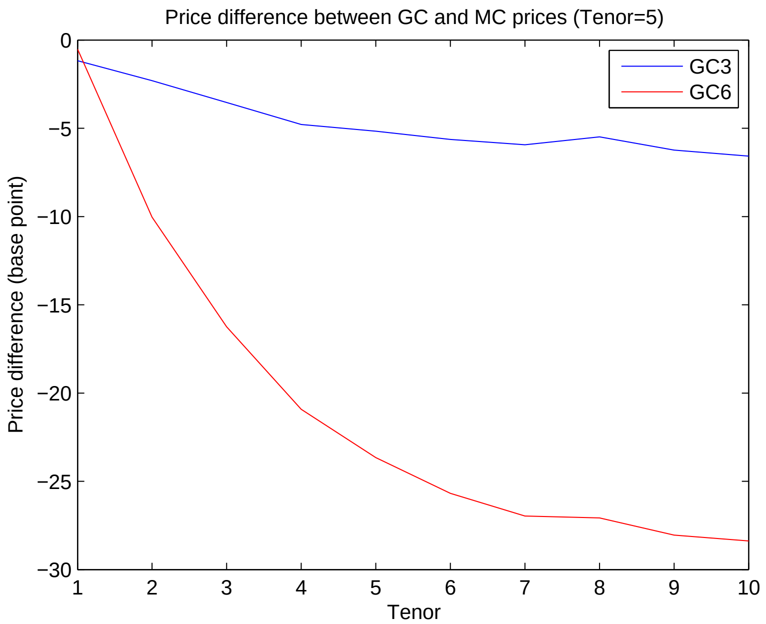

Figure 19: Pricing Errors of swaption prices (Tenor = 5)

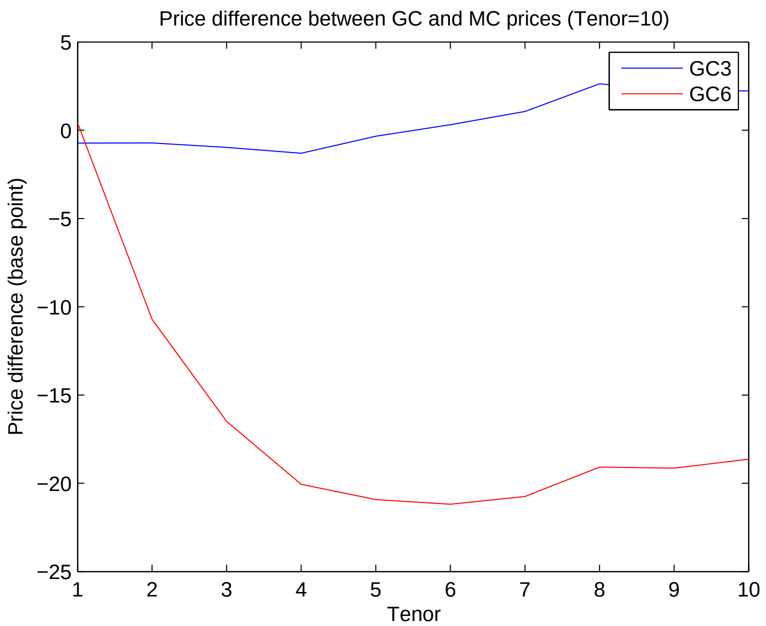

Figure 20: Pricing Errors of swaption prices (Tenor = 10)

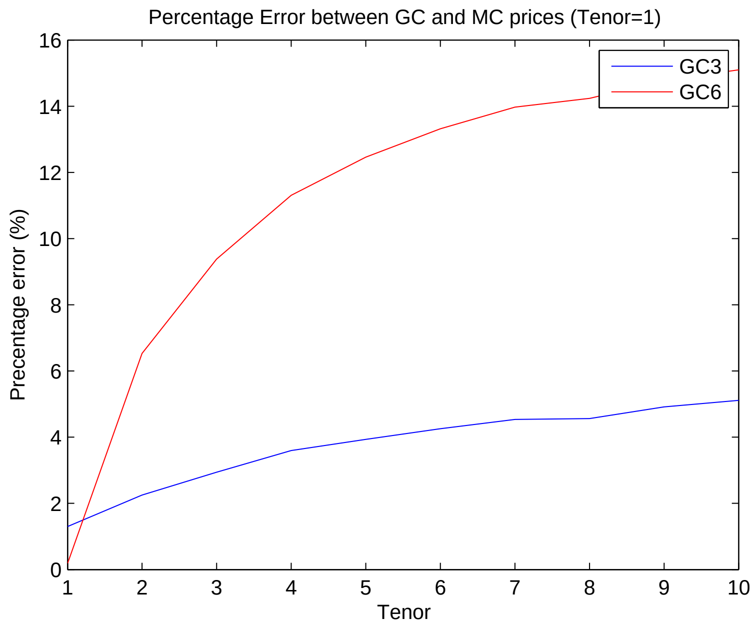

Figure 21: Percentage Errors of swaption prices (Tenor = 1)

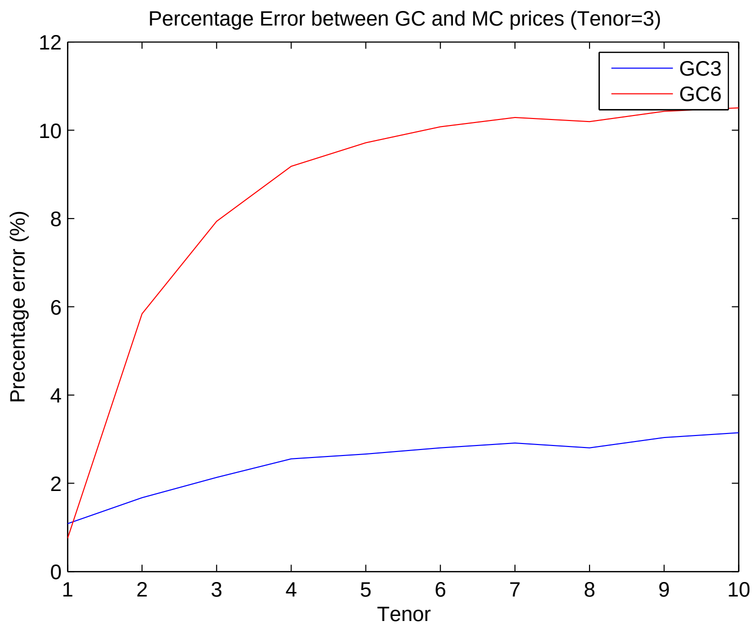

Figure 22: Percentage Errors of swaption prices (Tenor = 3)

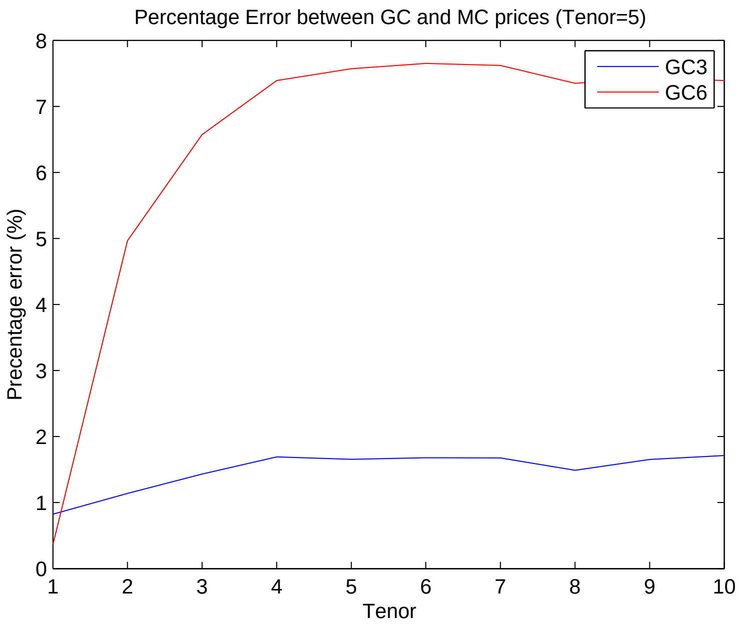

Figure 23: Percentage Errors of swaption prices (Tenor = 5)

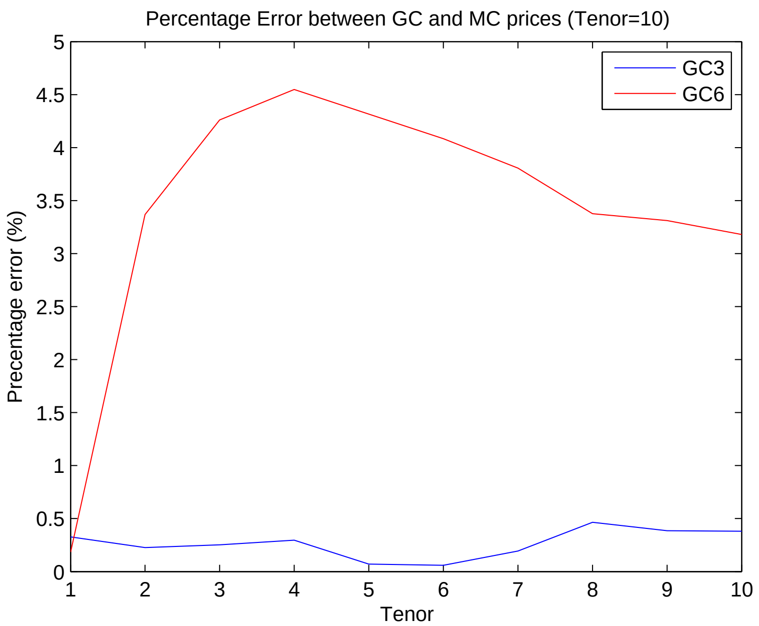

Figure 24: Percentage Errors of swaption prices (Tenor = 10)

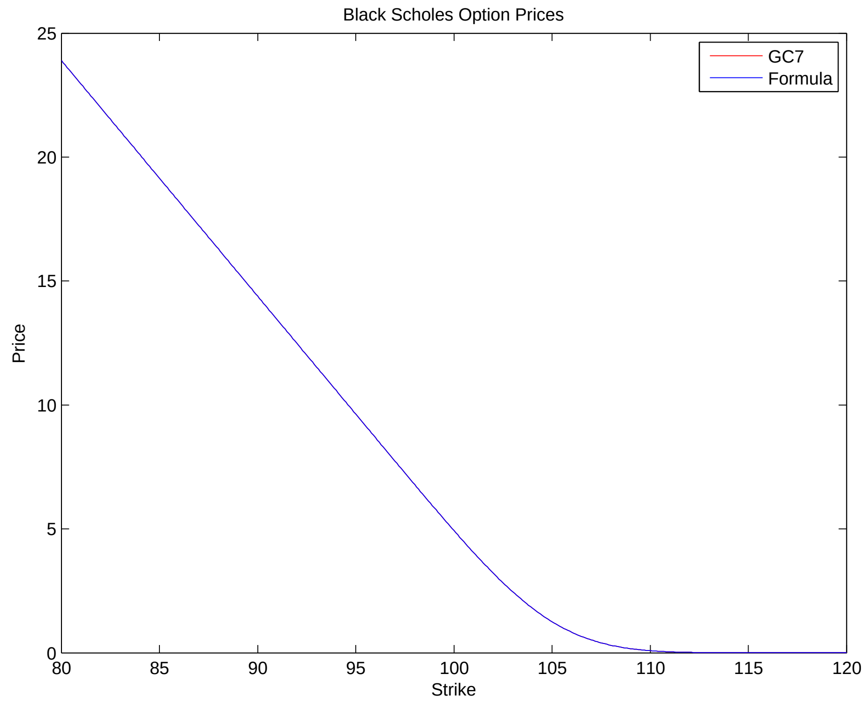

Figure 25: Black-Scholes Call Option Prices (T = 1)

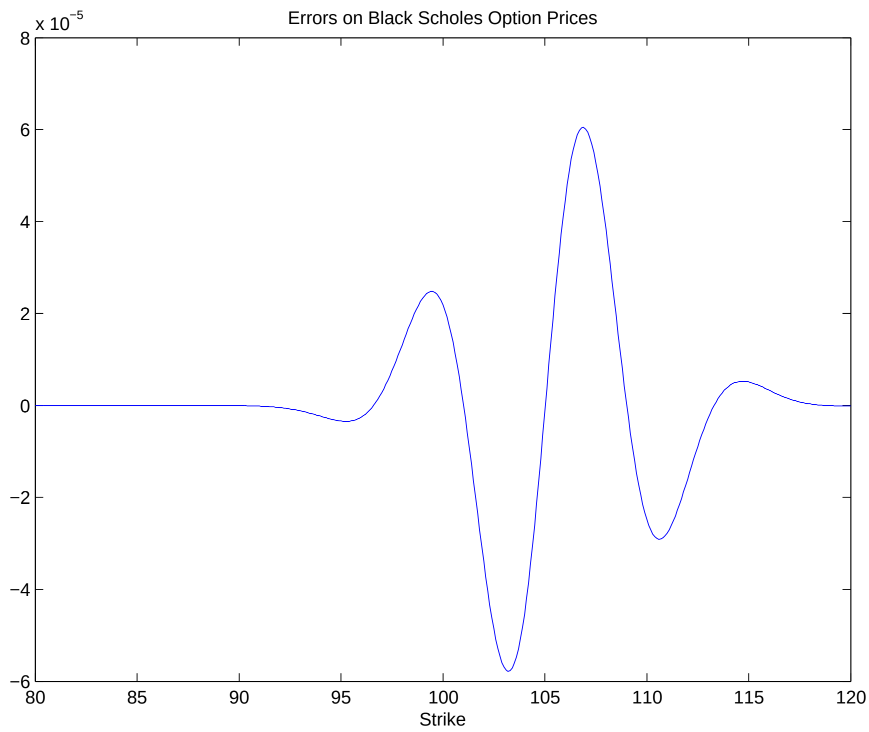

Figure 26: Black-Scholes Call Option Prices (T = 1)

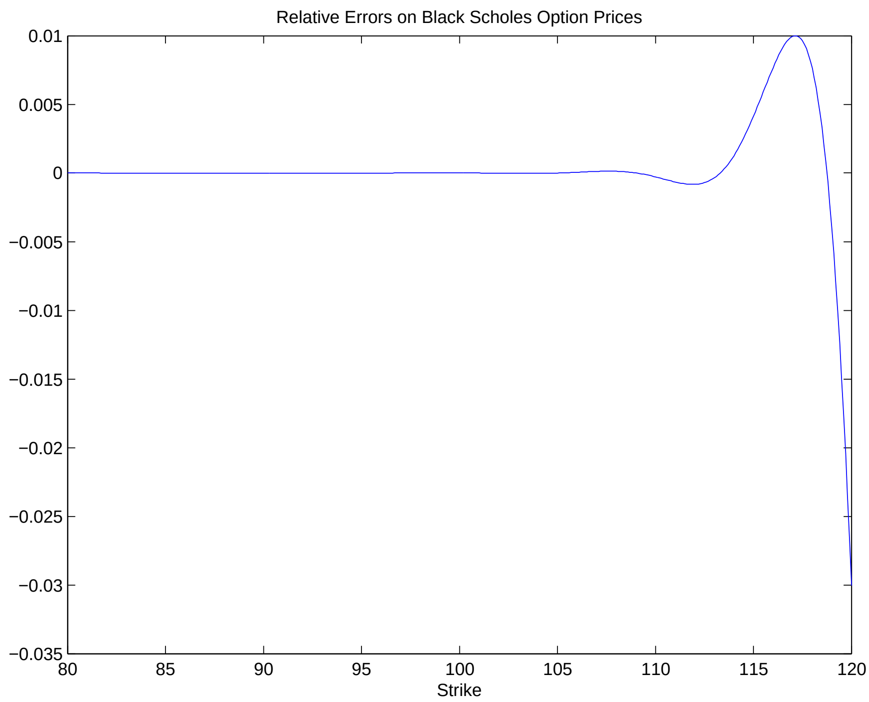

Figure 27: Black-Scholes Call Option Prices (T = 1)

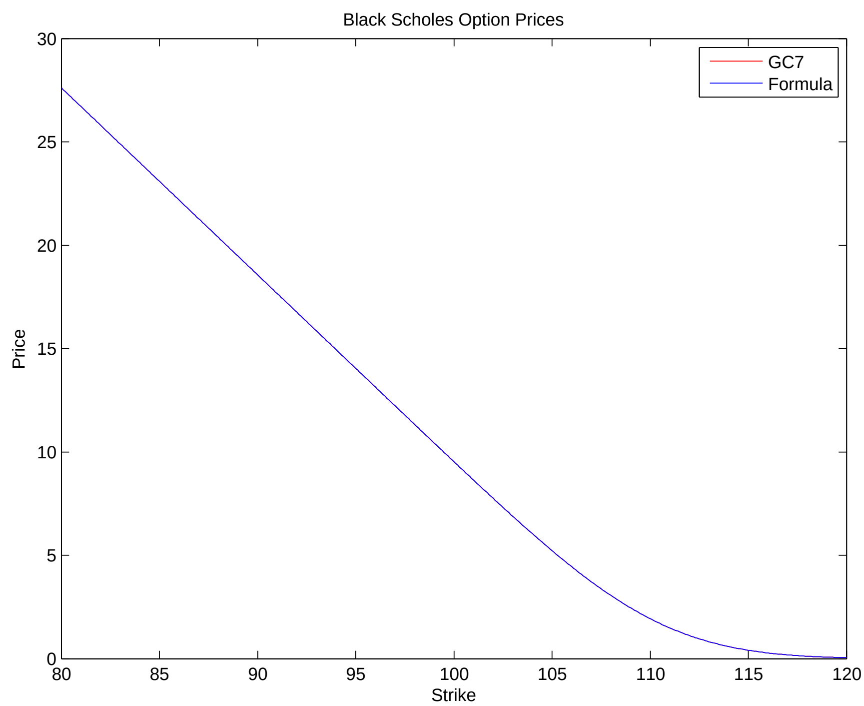

Figure 28: Black-Scholes Call Option Prices (T = 2)

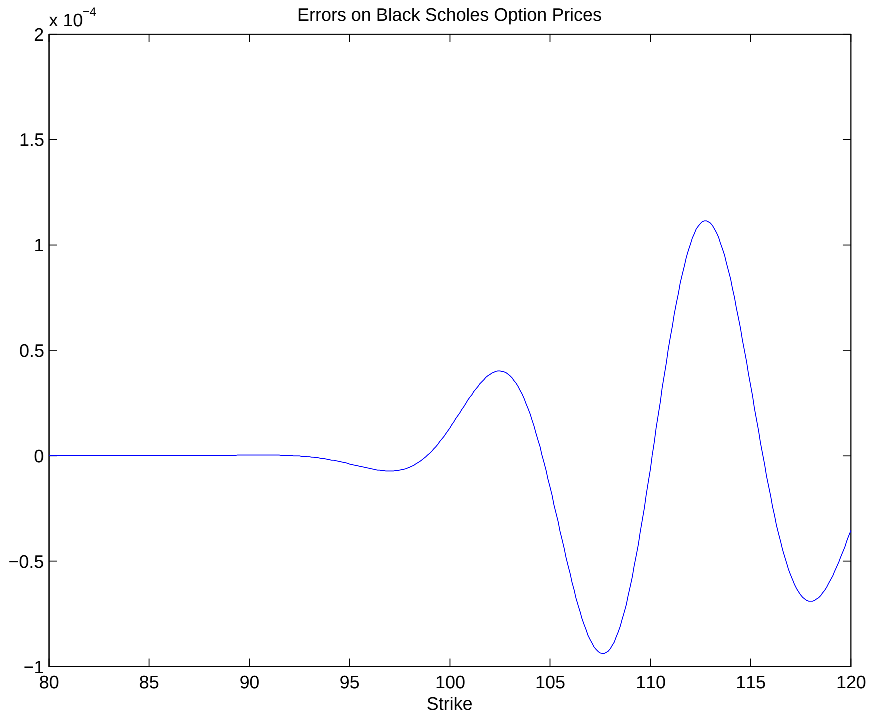

Figure 29: Black-Scholes Call Option Prices (T = 2)

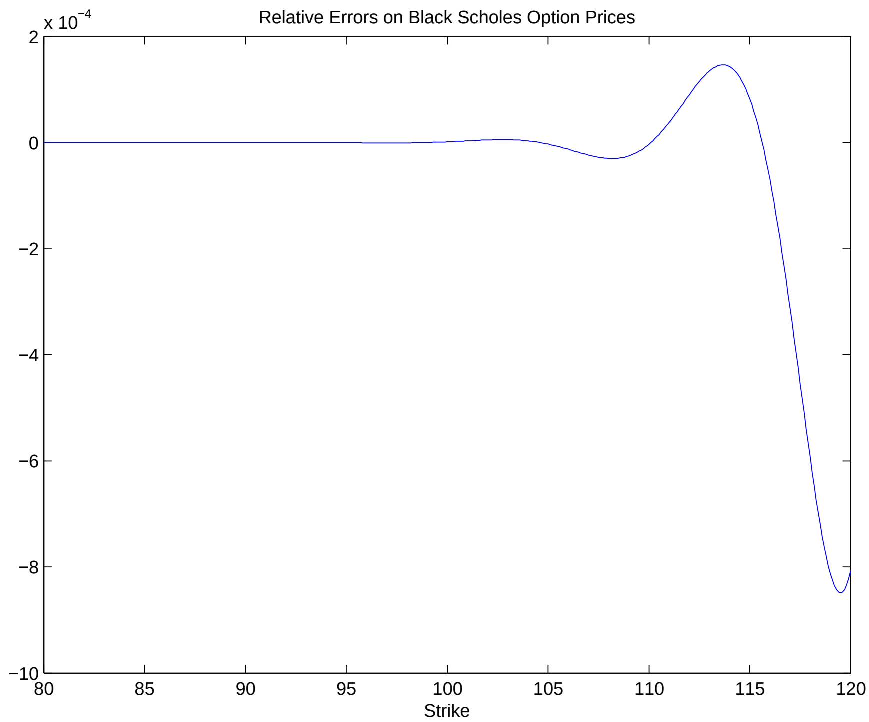

Figure 30: Black-Scholes Call Option Prices (T = 2)

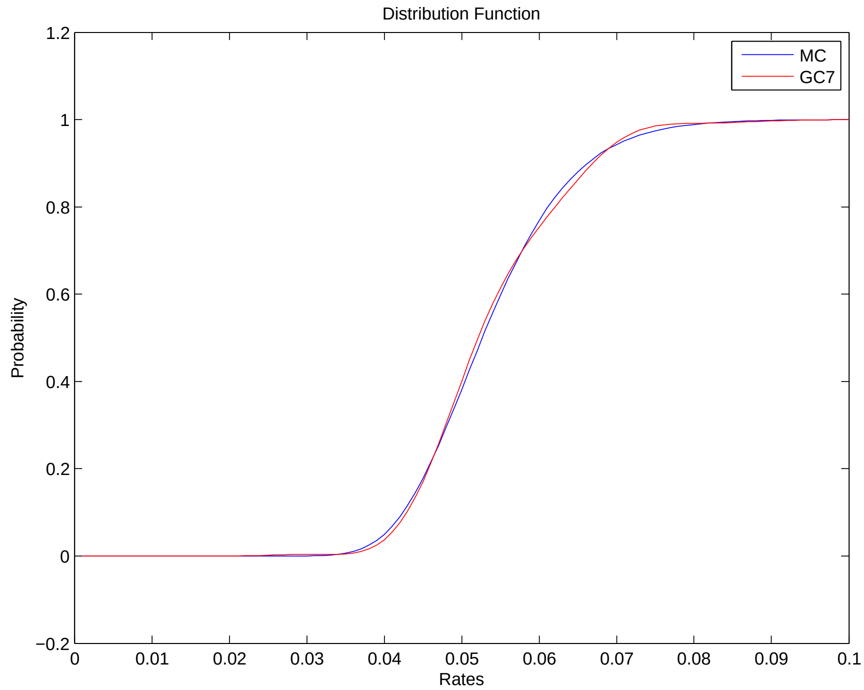

Figure 31: Comparison of Distribution Functions

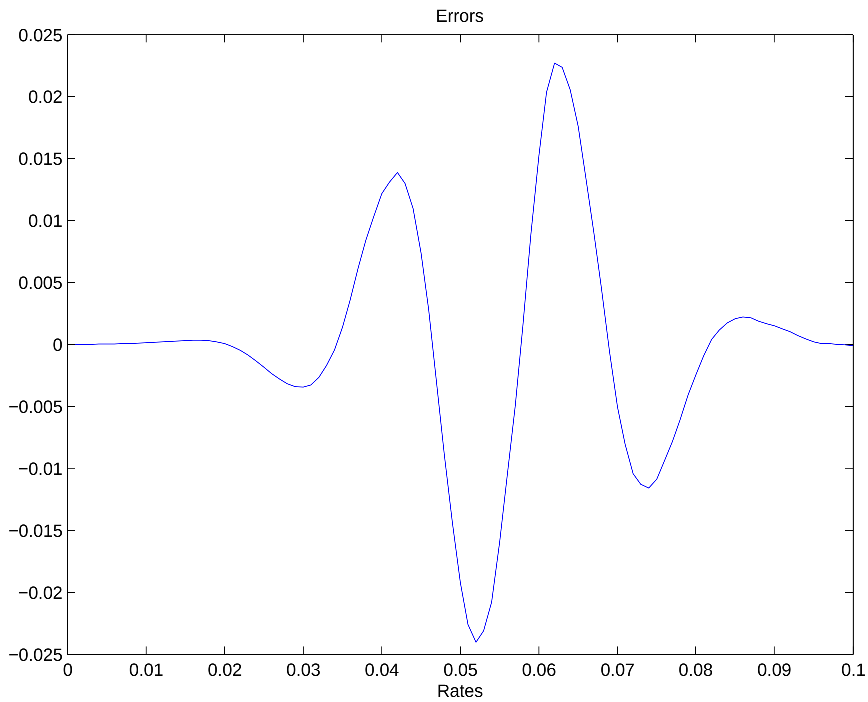

Figure 32: Error of Distribution Function

Figure 33: Mean reversion levels of Heston's Model using Alfonsi scheme

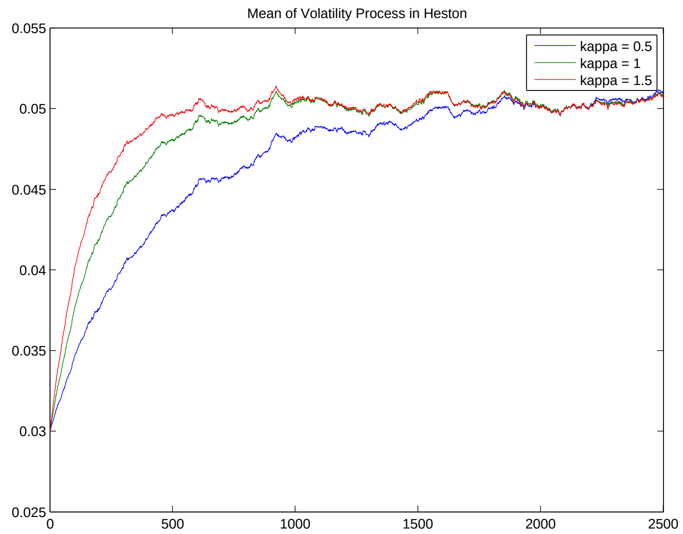

Figure 34: Mean reversion speeds of Heston's Model using Alfonsi scheme

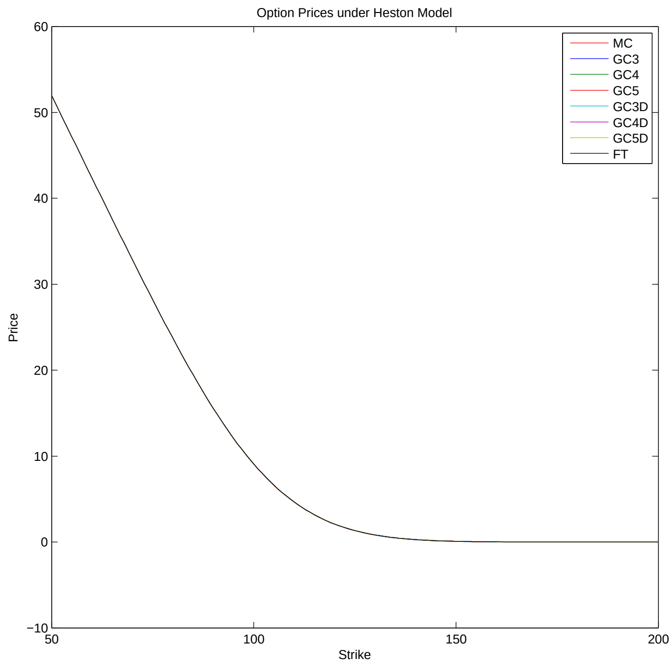

Figure 35: Call Option Prices under Heston's Model (T = 1)

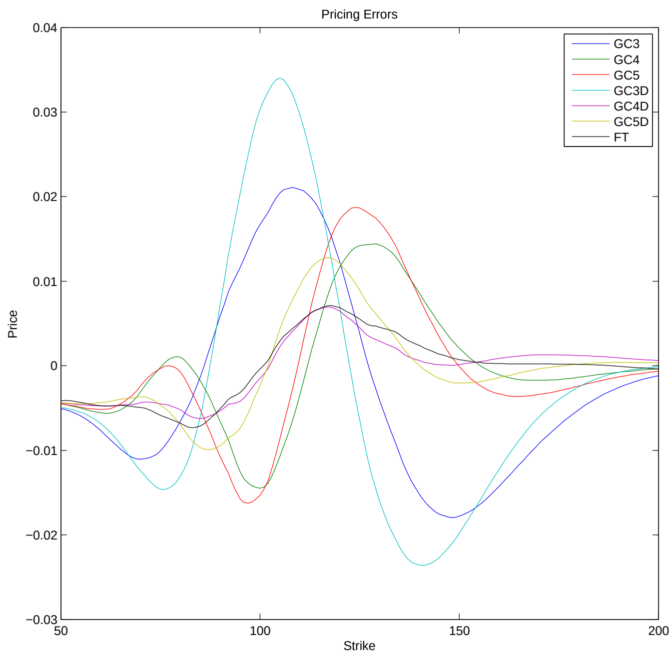

Figure 36: Pricing Errors of Call Options under Heston's Model (T = 1)

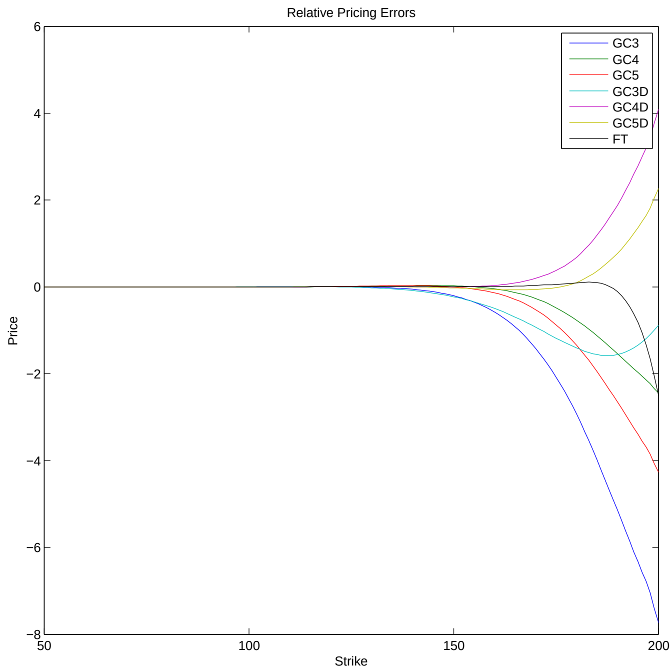

Figure 37: Relative Errors of Call Option Prices under Heston's Model (T = 1)

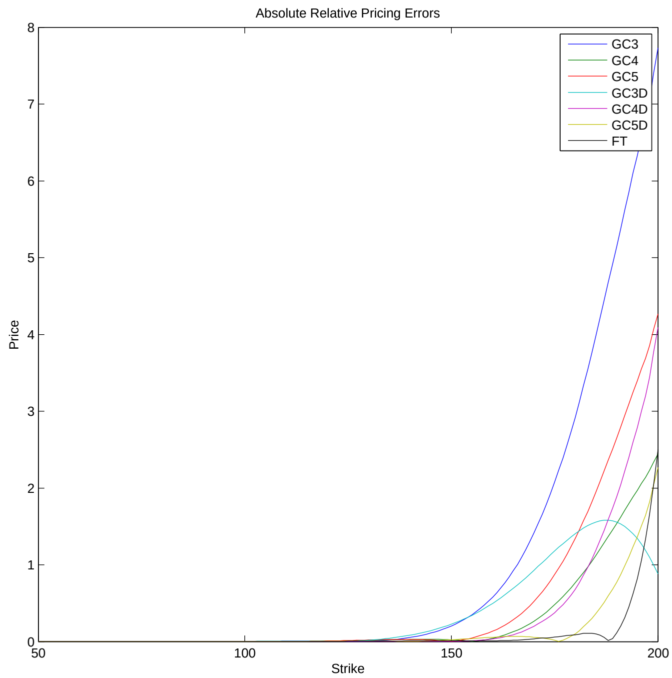

Figure 38: Absolute Relative Errors of Call Option Prices under Heston's Model (T = 1)

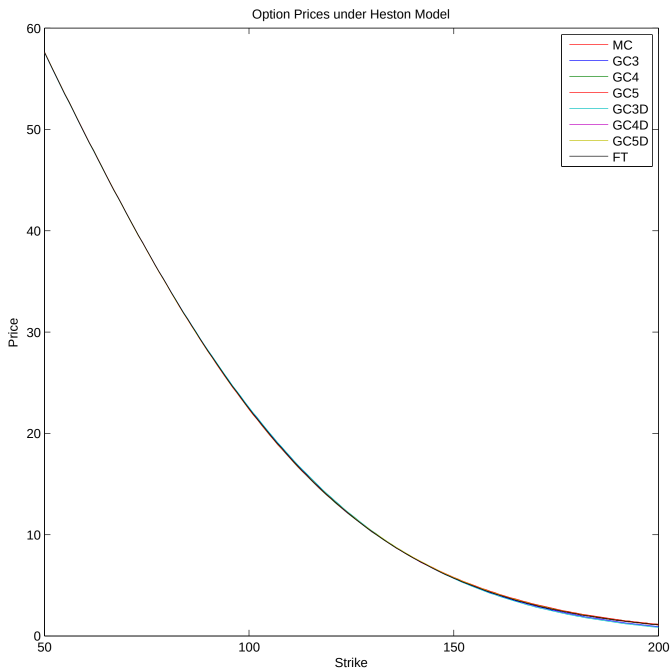

Figure 39: Call Option Prices under Heston's Model (T = 4)

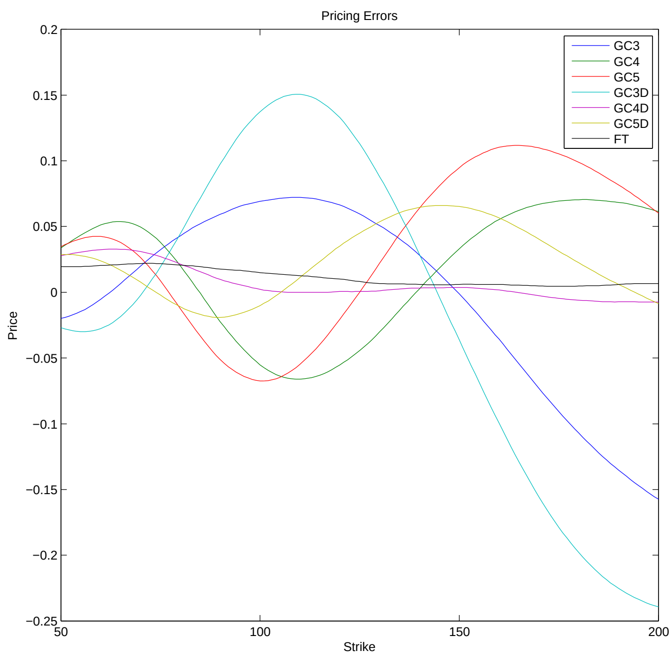

Figure 40: Pricing Errors of Call Options under Heston's Model (T = 4)

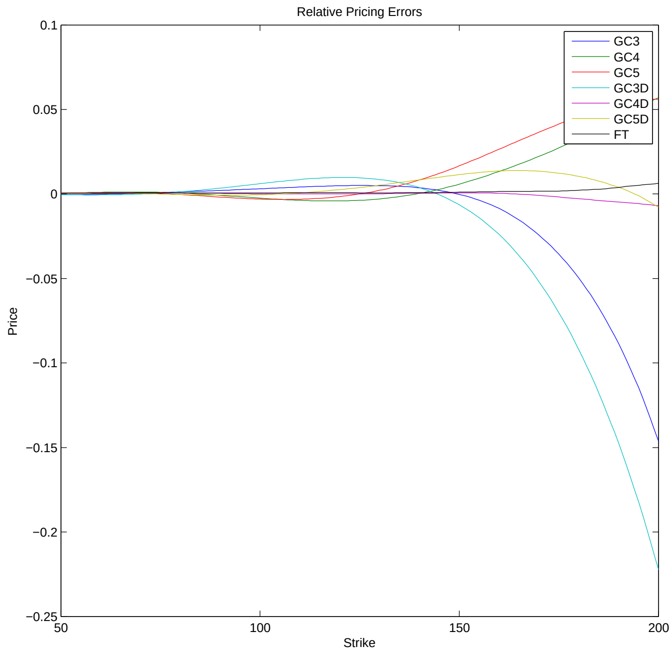

Figure 41: Relative Errors of Call Option Prices under Heston's Model (T = 4)

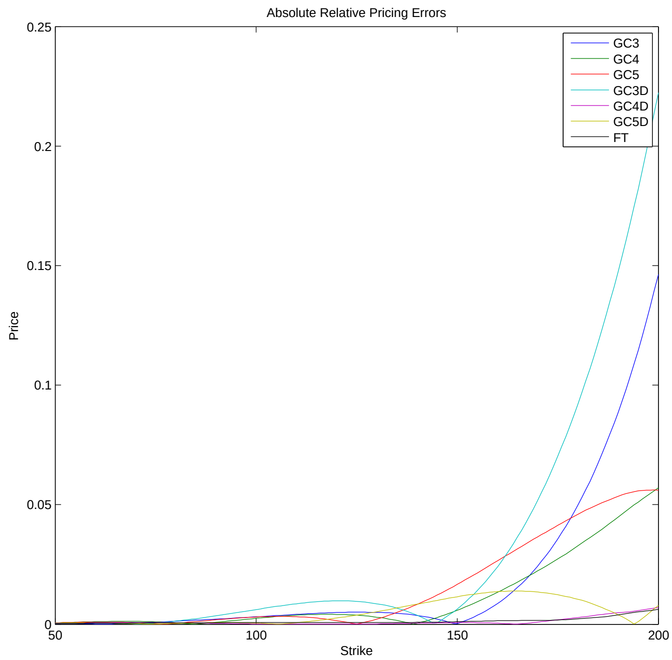

Figure 42: Absolute Relative Errors of Call Option Prices under Heston's Model (T = 4)

APPENDIX

A Preliminaries

In Part A of the Appendix, we collect preliminary results useful in understanding both the concepts and the derivations related to the Gram-Charlier expansions used throughout the paper.

A.1 Hermite polynomials

Let $\phi(x)$ be the density function of the standard normal distribution $N(0,1)$. Throughout this section, Hermite polynomials are defined as

$$H_n(x) = (-1)^n \phi(x)^{-1} D^n \phi(x) \text{ with } H_0(x) \equiv 1$$

where

$$n \in \mathbb{N} \text{ and } \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

The proof of the following lemma is elementary, but may not be immediately obvious.

Lemma A.1. *We have the following formula*

$$\int_x^{\infty} \phi(y) H_n(y)\, dy = x\phi(x) H_{n-1}(x) + \phi(x) H_{n-2}(x).$$

Proof. Note that $D^n\phi(x) = (-1)^n H_n(x)\phi(x)$. By using integration by parts, we have

$$D((D^{n-1}\phi(x))x) = [D^n(\phi(x))]x + D^{n-1}\phi(x) = (-1)^n x H_n(x)\phi(x) + D^{n-1}\phi(x).$$

Therefore,

$$-(D^{n-1}\phi(x))x = \int_x^{\infty} (-1)^n y H_n(y)\phi(y) dy - D^{n-2}\phi(x).$$

Hence,

$$\int_x^{\infty} (-1)^n y H_n(y)\phi(y) dy = -(D^{n-1}\phi(x))x + D^{n-2}\phi(x)$$

$$\text{i.e. } \int_x^{\infty} (-1)^n y H_n(y)\phi(y) dy = (-1)\cdot(-1)^{n-1} x\phi(x) H_{n-1}(x) + (-1)^{n-2}\phi(x) H_{n-1}(x).$$

□

We now give the Gram-Charlier expansion of a probability density function and show how to use it to calculate the cumulative distribution function and the truncated expectation. The primary reference for this discussion can be found in [8].

Proposition A.2. *Let Y be a random variable, such that it has a continuous density function $f(x)$ and finite cumulants $(c_k)_{k\in\mathbb{N}}$. Then the following hold:*

(a) *f is given by the following expansion*

$$f(x) = \sum_{n=0}^{\infty} \frac{q_n}{\sqrt{c_2}} H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right)\phi\left(\frac{x-c_1}{\sqrt{c_2}}\right)$$

where $q_0 = 1$, $q_1 = q_2 = 0$,

$$q_n = \sum_{m=1}^{[\frac{n}{3}]} \sum_{\substack{k_1,\ldots k_m \geq 3,\\ k_1+\ldots+k_m=n,}} \frac{c_{k_1}\ldots c_{k_m}}{m!k_1!\ldots k_m!\sqrt{c_2}^n}, \; n \geq 3.$$

(b) *for any $a \in \mathbb{R}$,*

$$\mathbb{E}[I(Y>a)] = N\left(\frac{c_1-a}{\sqrt{c_2}}\right) + \sum_{k=3}^{\infty}(-1)^{k-1} q_k H_{k-1}\left(\frac{c_1-a}{\sqrt{c_2}}\right)\phi\left(\frac{c_1-a}{\sqrt{c_2}}\right)$$

(c) *for any $a \in \mathbb{R}$,*

$$\mathbb{E}[Y\,I(Y>a)] = \sqrt{c_2}\,\phi\left(\frac{c_1-a}{\sqrt{c_2}}\right) + c_1 N\left(\frac{c_1-a}{\sqrt{c_2}}\right)$$

$$+\sum_{n=3}^{\infty}(-1)^{n-1} q_n \phi\left(\frac{c_1-a}{\sqrt{c_2}}\right)\left[aH_{n-1}\left(\frac{c_1-a}{\sqrt{c_2}}\right) - \sqrt{C_2}H_{n-2}\left(\frac{c_1-a}{\sqrt{c_2}}\right)\right].$$

Proof. (a)

$$\begin{aligned}
&G_Y(t) := \mathbb{E}(e^{itY})\\
&= \int_{-\infty}^{\infty} e^{itx} f(x)dx\\
&= \int_{-\infty}^{\infty} e^{it(c_1+\sqrt{c_2}x)} f(c_1+\sqrt{c_2}x)d(c_1+\sqrt{c_2}x)\\
&= e^{itc_1}\int_{-\infty}^{\infty} e^{i\sqrt{c_2}x}\sqrt{c_2} f(c_1+\sqrt{c_2}x)dx.
\end{aligned}$$

Since

$$m(t) = e^{\ln m(t)} = e^{\sum_{k=0}^{\infty}\left[\frac{d^k}{dt^k}(\ln m(t))\right]_{t=0}\frac{t^k}{k!}} = e^{\sum_{k=1}^{\infty}\frac{c_k t^k}{k!}}$$

we have

$$\begin{aligned} & G_Y(t) \\ = & e^{\sum_{k=1}^{\infty}\frac{c_k (it)^k}{k!}} \\ = & e^{ic_1 t} e^{-\frac{c_2 t^2}{2} + \sum_{k=3}^{\infty}\frac{c_k (it)^k}{k!}} \\ = & e^{ic_1 t} e^{-\frac{c_2 t^2}{2} + \sum_{k=3}^{\infty}\frac{c_k (-1)^k}{k!\sqrt{c_2}^k}(-i\sqrt{c_2}t)^k} \\ = & e^{ic_1 t}\int_{-\infty}^{\infty} e^{i\sqrt{c_2}tx}\left[e^{\sum_{k=3}^{\infty}\frac{c_k(-1)^k}{k!\sqrt{c_2}^k}D^k}\right](\phi(x))\,dx. \end{aligned}$$

Then

$$\begin{aligned} & \left[e^{\sum_{k=3}^{\infty}\frac{c_k(-1)^k}{k!\sqrt{c_2}^k}D^k}\right](\phi(x)) \\ = & \left\{1 + \sum_{m=1}^{\infty}\frac{1}{m!}\left[\sum_{k=3}^{\infty}\frac{c_k(-1)^k}{k!\sqrt{c_2}^k}D^k\right]^m\right\}(\phi(x)) \\ = & \left\{1 + \sum_{m=1}^{\infty}\frac{1}{m!}\left[\sum_{k_1,\ldots k_m \geq 3}\frac{c_{k_1}\ldots c_{k_m}(-1)^{k_1+\ldots+k_m}}{k_1!\ldots k_m!\sqrt{c_2}^{k_1+\ldots+k_m}}D^{k_1+\ldots+k_m}\right]\right\}(\phi(x)) \\ = & \left\{1 + \left[\sum_{m=1}^{\infty}\sum_{k_1,\ldots k_m \geq 3}\frac{c_{k_1}\ldots c_{k_m}(-1)^{k_1+\ldots+k_m}}{m!k_1!\ldots k_m!\sqrt{c_2}^{k_1+\ldots+k_m}}D^{k_1+\ldots+k_m}\right]\right\}(\phi(x)) \\ = & \left[1 + \sum_{n=3}^{\infty}\sum_{m=1}^{[\frac{n}{3}]}\sum_{\substack{k_1,\ldots k_m \geq 3,\\ k_1+\ldots+k_m = n,}}\frac{c_{k_1}\ldots c_{k_m}}{m!k_1!\ldots k_m!\sqrt{c_2}^n}H_n(x)\right]\phi(x) \end{aligned}$$

Therefore,

$$
\begin{aligned}
& G_Y(t) \\
= {} & e^{ic_1t}\int_{-\infty}^{\infty} e^{i\sqrt{c_2}tx}\phi(x)dx \\
& + e^{ic_1t}\int_{-\infty}^{\infty} e^{i\sqrt{c_2}tx}\left[\sum_{n=3}^{\infty}\sum_{m=1}^{[\frac{n}{3}]}\sum_{\substack{k_1,\dots k_m\geq 3,\\ k_1+\dots+k_m=n,}} \frac{c_{k_1}\dots c_{k_m}}{m!k_1!\dots k_m!\sqrt{c_2}^n}H_n(x)\phi(x)\right]dx
\end{aligned}
$$

The rest follows from the inverse Fourier transform and is straightforward.

(b).

$$
\begin{aligned}
& \mathbb{E}[I(Y\leq a)] \\
= {} & \int_{-\infty}^{a}\sum_{n=0}^{\infty}\frac{q_n}{\sqrt{c_2}}H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right)\phi\left(\frac{x-c_1}{\sqrt{c_2}}\right)dx \\
= {} & \sum_{n=0}^{\infty}\int_{-\infty}^{a}\frac{q_n}{\sqrt{c_2}}H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right)\phi\left(\frac{x-c_1}{\sqrt{c_2}}\right)dx \\
= {} & \sum_{n=0}^{\infty}\int_{-\infty}^{\frac{a-c_1}{\sqrt{c_2}}} q_nH_n(y)\phi(y)\,dy \\
= {} & \int_{-\infty}^{\frac{a-c_1}{\sqrt{c_2}}}\phi(y)\,dy + \sum_{n=3}^{\infty}q_n\int_{-\infty}^{\frac{a-c_1}{\sqrt{c_2}}}H_n(y)\phi(y)\,dy \\
= {} & N\left(\frac{a-c_1}{\sqrt{c_2}}\right) + \sum_{n=3}^{\infty}q_n\int_{-\infty}^{\frac{a-c_1}{\sqrt{c_2}}}(-1)^nD^n\phi(y)\,dy \\
= {} & N\left(\frac{a-c_1}{\sqrt{c_2}}\right) + \sum_{n=3}^{\infty}q_n(-1)^nD^{n-1}\phi\left(\frac{a-c_1}{\sqrt{c_2}}\right) \\
= {} & N\left(\frac{a-c_1}{\sqrt{c_2}}\right) + \sum_{n=3}^{\infty}q_n(-1)^n\cdot(-1)^{n-1}H_{n-1}\left(\frac{a-c_1}{\sqrt{c_2}}\right)\phi\left(\frac{a-c_1}{\sqrt{c_2}}\right) \\
= {} & N\left(\frac{a-c_1}{\sqrt{c_2}}\right) - \sum_{n=3}^{\infty}q_nH_{n-1}\left(\frac{a-c_1}{\sqrt{c_2}}\right)\phi\left(\frac{a-c_1}{\sqrt{c_2}}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}[I(Y > a)] \\
&= 1 - \left[N\left(\frac{a - c_1}{\sqrt{c_2}}\right) - \sum_{n=3}^{\infty} q_n H_{n-1}\left(\frac{a - c_1}{\sqrt{c_2}}\right) \phi\left(\frac{a - c_1}{\sqrt{c_2}}\right) \right] \\
&= N\left(\frac{c_1 - a}{\sqrt{c_2}}\right) + \sum_{n=3}^{\infty} q_n H_{n-1}\left(\frac{a - c_1}{\sqrt{c_2}}\right) \phi\left(\frac{a - c_1}{\sqrt{c_2}}\right) \\
&= N\left(\frac{c_1 - a}{\sqrt{c_2}}\right) + \sum_{n=3}^{\infty} (-1)^{n-1} q_n H_{n-1}\left(\frac{c_1 - a}{\sqrt{c_2}}\right) \phi\left(\frac{c_1 - a}{\sqrt{c_2}}\right).
\end{aligned}
$$

(c).

$$
\begin{aligned}
&\mathbb{E}[YI(Y \leq a)] \\
&= \int_a^\infty \sum_{n=0}^\infty \frac{q_n}{\sqrt{c_2}}\, x\, H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \\
&= \sum_{n=0}^\infty \int_a^\infty \frac{q_n}{\sqrt{c_2}}\, x\, H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \\
&= \sum_{n=0}^\infty q_n \Bigg[\int_a^\infty \left(\frac{x-c_1}{\sqrt{c_2}}\right) H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \\
&\qquad + \frac{c_1}{\sqrt{c_2}} \int_a^\infty H_n\left(\frac{x-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{x-c_1}{\sqrt{c_2}}\right) dx \Bigg] \\
&= \sum_{n=0}^\infty q_n \sqrt{c_2} \left[\int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty y H_n(y)\phi(y)\, dy + \frac{c_1}{\sqrt{c_2}} \int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty H_n(y)\phi(y)\, dy \right] \\
&= \sqrt{c_2} \left[\int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty y\phi(y)\, dy + \frac{c_1}{\sqrt{c_2}} \int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty \phi(y)\, dy \right] \\
&\qquad + \sum_{n=3}^\infty q_n \sqrt{c_2} \left[\int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty y H_n(y)\phi(y)\, dy + \frac{c_1}{\sqrt{c_2}} \int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty H_n(y)\phi(y)\, dy \right] \\
&= \sqrt{c_2} \int_{\frac{a-c_1}{\sqrt{c_2}}}^\infty y\phi(y)\, dy + c_1 N\left(\frac{c_1-a}{\sqrt{c_2}}\right) \\
&\qquad + \sum_{n=3}^\infty q_n \sqrt{c_2} \left[\frac{a-c_1}{\sqrt{c_2}} H_{n-1}\left(\frac{a-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{a-c_1}{\sqrt{c_2}}\right) + H_{n-2}\left(\frac{a-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{a-c_1}{\sqrt{c_2}}\right) \right] \\
&\qquad + \sum_{n=3}^\infty q_n \sqrt{c_2} \left[\frac{c_1}{\sqrt{c_2}} H_{n-1}\left(\frac{a-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{a-c_1}{\sqrt{c_2}}\right) \right] \\
&= \sqrt{c_2}\, \phi\left(\frac{a-c_1}{\sqrt{c_2}}\right) + c_1 N\left(\frac{c_1-a}{\sqrt{c_2}}\right) \\
&\qquad + \sum_{n=3}^\infty q_n \sqrt{c_2} \left[\frac{a}{\sqrt{c_2}} H_{n-1}\left(\frac{a-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{a-c_1}{\sqrt{c_2}}\right) + H_{n-2}\left(\frac{a-c_1}{\sqrt{c_2}}\right) \phi\left(\frac{a-c_1}{\sqrt{c_2}}\right) \right]
\end{aligned}
$$

The second last equality follows from Lemma A.1 □

Remark. In principle, we are able to develop a general formula for $\mathbb{E}[Y^n I(Y > a))]$ for any natural number n.

A.2 An important system of Riccati equations

Consider the following Ricatti equation which will be useful in the sequel.

$$\frac{dy}{dx} = 1 + ky - \frac{\sigma^2 y}{2}, y(T) = y_0.$$

Consider an auxiliary equation

$$\lambda^2 - \frac{2k}{\sigma^2}\lambda - \frac{2}{\sigma^2} = 0$$

The roots of this quadratic equation is given by

$$\lambda_+ = \frac{k+\gamma}{\sigma^2}, \lambda_- = \frac{k-\gamma}{\sigma^2} \text{ where } \gamma = \sqrt{k^2 + 2\sigma^2}.$$

$$\begin{aligned}
\frac{dy}{dx} = 1 + ky - \frac{\sigma^2 y}{2} \Rightarrow \int \frac{dy}{1 + ky - \frac{\sigma^2 y^2}{2}} &= \int dt \\
\Rightarrow \int \frac{dy}{y^2 - \frac{2k}{\sigma^2} y - \frac{2}{\sigma^2}} &= -\frac{\sigma^2 t}{2} + C \\
\Rightarrow \frac{1}{\lambda_+ - \lambda_-} \int \frac{1}{y - \lambda_+} - \frac{1}{y - \lambda_-} dy &= -\frac{\sigma^2 t}{2} + C \\
\Rightarrow \frac{\sigma^2}{2\gamma} \ln \frac{y - \lambda_+}{y - \lambda_-} &= -\frac{\sigma^2 t}{2} + C \\
\Rightarrow \frac{y - \lambda_+}{y - \lambda_-} &= De^{-\gamma t}.
\end{aligned}$$

By using the terminal condition, we have

$$D = e^{\gamma T} \frac{y_0 - \lambda_+}{y_0 - \lambda_-}.$$

Therefore,

$$\frac{y - \lambda_+}{y - \lambda_-} = \frac{y_0 - \lambda_+}{y_0 - \lambda_-} e^{\gamma(T-t)}.$$

Let $y_0^* = \frac{y_0 - \lambda_+}{y_0 - \lambda_-}$. Then the solution of the differential equation is given by

$$y = \lambda_+ + (\lambda_+ - \lambda_-) \frac{y_0^* e^{\gamma(T-t)}}{1 - y_0^* e^{\gamma(T-t)}}.$$

or

$$y = \frac{1}{\sigma^2} \left[K + \gamma + \frac{2\gamma y_0^* e^{\gamma(T-t)}}{1 - y_0^* e^{\gamma(T-t)}} \right].$$

Next, suppose that we are given the following system of Riccati equations

$$\frac{dx}{dt} = \delta_0 - K_1\theta_1 y - K_2\theta_2 z, \tag{29}$$

$$\frac{dy}{dt} = 1 + K_1 y - \frac{1}{2}\sigma_1^2 y^2, \tag{30}$$

$$\frac{dz}{dt} = 1 + K_2 z - \frac{1}{2}\sigma_2^2 z^2, \tag{31}$$

$$x(T) = x_0, y(T) = y_0, z(T) = z_0. \tag{32}$$

By the above, we have

$$y = \frac{1}{\sigma_1^2}\left[K_1 + \gamma_1 + \frac{2\gamma_1 y_0^* e^{\gamma_1(T-t)}}{1 - y_0^* e^{\gamma_1(T-t)}}\right].$$

$$z = \frac{1}{\sigma_2^2}\left[K_2 + \gamma_2 + \frac{2\gamma_2 z_0^* e^{\gamma_2(T-t)}}{1 - z_0^* e^{\gamma_2(T-t)}}\right].$$

where $\gamma_j = \sqrt{K^2 + \sigma^2}$, $j = 1, 2$.

Now,

$$\begin{aligned} x = \delta_0 t - \frac{K_1\theta_1}{\sigma_1^2}[(K_1 + \gamma_1)t + 2\ln|1 - y_0^* e^{\gamma_1(T-t)}|] \\ - \frac{K_2\theta_2}{\sigma_2^2}[(K_2 + \gamma_2)t + 2\ln|1 - z_0^* e^{\gamma_2(T-t)}|] + C \end{aligned}$$

Therefore,

$$\begin{aligned} C = x_0(T) - \delta_0 T + \frac{K_1\theta_1}{\sigma_1^2}[(K_1 + \gamma_1)T + 2\ln|1 - y_0^*|] \\ + \frac{K_2\theta_2}{\sigma_2^2}[(K_2 + \gamma_2)T + 2\ln|1 - z_0^*|] \end{aligned}$$

Hence, we have

$$\begin{aligned} x = x_0 - \delta_0(T - t) - \frac{K_1\theta_1}{\sigma_1^2}\left[(K_1 + \gamma_1)(T - t) + 2\ln\left|\frac{1 - y_0^* e^{\gamma_1(T-t)}}{1 - y_0^*}\right|\right] \\ - \frac{K_2\theta_2}{\sigma_2^2}\left[(K_2 + \gamma_2)(T - t) + 2\ln\left|\frac{1 - z_0^* e^{\gamma_1(T-t)}}{1 - z_0^*}\right|\right]. \end{aligned}$$

B Swaptions

B.1 Change of Numeraire - Forward Measures

The primary reference for the discussion in this subsection can be found in [2, Chapter 10, 26].

Assumptions:

- The market model consists of asset prices $S_0, ..., S_n$, where S_0 is assumed to be strictly positive.
- Under the real-world measure, the S-dynamics are of the following form

  $$dS_i = S_i(t)\alpha_i(t)dt + S_i(t)\sigma_i(t)d\bar{W}(t)$$

  where α_i, σ_i are adapted processes and $\bar{W}$ is a standard Brownian motion.

Lemma B.1. *Let β be a strictly positive Itô's process and let $Z = \frac{S}{\beta}$. Then h is S-self-financing if and only if h is Z-self-financing,*

$$\textit{i.e. } dV^S(t,h) = h(t) \cdot dS(t) \textit{ if and only if } dV^Z(t,h) = h(t) \cdot dS(t)$$

where $V^S(t,h) = h(t) \cdot S(t)$ and $V^Z(t,h) = h(t) \cdot Z(t)$.

Proof.

$$\begin{aligned}
dV^Z(t,h) &= d\left[\frac{V^S(t,h)}{\beta(t)}\right] \\
&= \frac{dV^S(t,h)}{\beta(t)} + V^S(t,h)d\left(\frac{1}{\beta(t)}\right) + dV^S(t,h) \cdot d\left(\frac{1}{\beta(t)}\right) \\
&= \frac{h(t)dS(t)}{\beta(t)} + h(t)S(t)d\left(\frac{1}{\beta(t)}\right) + h(t)dS(t,h) \cdot d\left(\frac{1}{\beta(t)}\right) \\
&= h(t)\left[\frac{dS(t)}{\beta(t)} + S(t)d\left(\frac{1}{\beta(t)}\right) + dS(t,h) \cdot d\left(\frac{1}{\beta(t)}\right)\right] \\
&= h(t)d\left[\frac{S(t,h)}{\beta(t)}\right] \\
&= h(t)dZ(t).
\end{aligned}$$

□

As a result, the model is S-arbitrage-free if and only if it is Z-arbitrage-free.

Now, let us recall the Fundamental Theorems of Asset Pricing

Theorem B.2. *Under the assumption, the following hold:*

(a) *The market model is free of arbitrage if and only if there exists a probability measure $Q^0 \sim P$ such that*

$$\frac{S_0(t)}{S_0(t)}, \frac{S_1(t)}{S_0(t)}, \ldots, \frac{S_0(t)}{S_n(t)}$$

are Q_0-martingales.

(b) *If the market is arbitrage-free, then any sufficiently integrable T-claim must be priced according to the formula*

$$\Pi(t; X) = S_0(t)\, \mathbb{E}^{Q_0}\left[\frac{X}{S_0(t)} \middle| \mathcal{F}_t \right]$$

where $\mathbb{E}^{Q_0}$ denotes expectation under Q^0.

Let S_0, S_1 be strictly positive assets in an arbitrage-free market. Then there exist probability measures Q^0, Q^1, such that for any choice of sufficiently integrable T-claim,

$$\Pi(0; X) = S_0(0)\, \mathbb{E}^{Q_0}\left[\frac{X}{S_0(T)} \right] = S_1(0)\, \mathbb{E}^{Q_1}\left[\frac{X}{S_1(T)} \right].$$

Denote by $L_0^1(T)$ the Radon-Nikodym derivative

$$L_0^1(T) = \frac{dQ^1}{dQ^0} \text{ on } \mathcal{F}_T.$$

Then, we have

$$\begin{aligned} \Pi(0; X) &= S_1(0)\, \mathbb{E}^{Q_0}\left[\frac{X}{S_1(T)} \cdot L_0^1(T) \right] \\ \Rightarrow S_0(0)\, \mathbb{E}^{Q_0}\left[\frac{X}{S_0(T)} \right] &= S_1(0)\, \mathbb{E}^{Q_0}\left[\frac{X}{S_1(T)} \cdot L_0^1(T) \right] \\ \Rightarrow \frac{S_0(0)}{S_0(T)} &= \frac{S_1(0)}{S_1(T)} \cdot L_0^1(T) \\ \Rightarrow L_0^1(T) &= \frac{S_0(0)}{S_1(0)} \frac{S_1(T)}{S_0(T)}. \end{aligned}$$

Proposition B.3. *Assume that Q^0 is a martingale measure for the numeraire S_0 (on $\mathcal{F}_T$) and assume that S_1 is a positive asset price process, such that $\frac{S_1}{S_0}$ is a Q^0-martingale. Define Q^1 on $\mathcal{F}_t$ by the likelihood process*

$$L_0^1(t) = \frac{S_0(0)}{S_1(0)}\frac{S_1(t)}{S_0(t)}, \ 0 \leq t \leq T.$$

Then Q^1 is a martingale measure for S_1.

Proof. If Π is an arbitrage-free price process, then $\frac{\Pi}{S_0}$ is also an arbitrage-free price process. Hence,

$$\begin{aligned}
\mathbb{E}^{Q_1}\left[\frac{\Pi(t)}{S_1(t)}\middle|\mathcal{F}_s\right] &= \frac{\mathbb{E}^{Q_0}\left[\frac{\Pi(t)}{S_1(t)}\cdot L_0^1(t)\middle|\mathcal{F}_s\right]}{L_0^1(s)} \\
&= \frac{\mathbb{E}^{Q_0}\left[\frac{\Pi(t)}{S_1(t)}\cdot \frac{S_0(0)}{S_1(0)}\frac{S_1(t)}{S_0(t)}\middle|\mathcal{F}_s\right]}{L_0^1(s)} \\
&= \frac{\frac{S_0(0)}{S_1(0)}\cdot \mathbb{E}^{Q_0}\left[\frac{\Pi(t)}{S_0(t)}\middle|\mathcal{F}_s\right]}{L_0^1(s)} \\
&= \frac{\frac{S_0(0)}{S_1(0)}\cdot \frac{\Pi(s)}{S_0(s)}}{L_0^1(s)} = \frac{\Pi(s)}{S_1(s)}
\end{aligned}$$

□

We are now ready to define the notion of forward measures.

Definition B.4. Let $P(0,t)$ be the price process of a zero coupon bond maturing at time t. The *risk-neutral measure* Q is defined as the martingale measure for the numeraire process $P(0,t)$.

Proposition B.5. *For any T-claim X, we have*

$$\Pi(t;X) = \mathbb{E}^{Q}[P(t,T)X|\mathcal{F}_t]$$

where $\mathbb{E}^T$ denotes expectation under Q^T.

Proof. By the First Fundamental Theorem and the definition of Q^T, we have

$$\frac{\Pi(t;X)}{P(0,t)} = \mathbb{E}^Q \left[\frac{X}{P(0,T)} \middle| \mathcal{F}_t \right].$$

□

Definition B.6. Let $P(t,T)$ be the price process of a zero coupon bond maturing at time T. Suppose that we are given a bond market model with a fixed martingale measure Q. For a fixed T, the *T-forward measure* Q^T is defined as the martingale measure for the numeraire process $P(t,T)$.

Proposition B.7. *For any T-claim X, we have*

$$\Pi(t;X) = P(t,T)\,\mathbb{E}^T[X|\mathcal{F}_t]$$

where $\mathbb{E}^T$ denotes expectation under Q^T.

Proof. By the First Fundamental Theorem of Asset Pricing and the definition of Q^T, we have

$$\Pi(t;X) = P(t,T)\,\mathbb{E}^T \left[\frac{X}{P(T,T)} \middle| \mathcal{F}_t \right].$$

□

Lemma B.8. *Let Q be the risk-neutral measure. The short rate, r, is deterministic if and only if $Q = Q^T$.*

Proof. If $Q = Q^T$, it is easy to see that $B(t) = e^{\int_0^t r(s)ds}$ is deterministic. Conversely, if r is deterministic, then

$$P(t,T) = \mathbb{E}^Q \left[e^{-\int_t^T r(s)ds} \right] = e^{-\int_t^T r(s)ds} = \frac{B(t)}{B(T)}.$$

Hence, $\frac{dQ^t}{dQ} \equiv 1$. □

The following result tells us that the forward measure is the measure that makes the present forward rate an unbiased estimator of the future short rate.

Lemma B.9. *Assume that, for any $T > 0$, $\frac{r(T)}{B(T)}$ is integrable. Then for all fixed T, $f(t,T)$ is a Q^T-martingale,*

$$i.e. \;\; f(t,T) = \mathbb{E}^T[r(T)|\mathcal{F}_t].$$

Proof. Let $X = r(T)$. Note that

$$\Pi(t,X) = \mathbb{E}^Q\left[r(T)e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right] = P(t,T)\,\mathbb{E}^T\left[r(T)|\mathcal{F}_t\right].$$

It follows that

$$\begin{aligned}
\mathbb{E}^T\left[r(T)|\mathcal{F}_t\right] &= \frac{1}{p(t,T)}\mathbb{E}^Q\left[r(T)e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right] \\
&= -\frac{1}{P(t,T)}\mathbb{E}^Q\left[\frac{\partial}{\partial T}e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right] \\
&= -\frac{1}{P(t,T)}\frac{\partial}{\partial T}\mathbb{E}^Q\left[e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right] \\
&= -\frac{1}{P(t,T)}\frac{\partial}{\partial T}P(t,T) \\
&= -\frac{\partial}{\partial T}\ln P(t,T) \\
&= f(t,T).
\end{aligned}$$

□

B.2 Interest rate swaps and swaptions

The interest rate swap is one of the simplest interest rate derivatives. This is basically a scheme where we exchange a payment stream at a fixed rate of interest, known as swap rates, for a payment stream at a floating rate (LIBOR rate $L(T_{i-1},T_i)$). Typically, an interest rate swap is a forward swap settled in arrears, which will be defined below.

Let N be the nominal principal and R be the swap rate. By assumption, we have a number of equally spaced dates $T_0, T_1, ..., T_n$ and payment occurs at $T_1, ..., T_n$.

Let $\delta = T_i - T_{i-1}$, $i = 1, 2, ..., n$.

At time T_i, the swap receiver (or, fixed rate receiver) will receive

$$N\delta R \text{ (on the fixed rate leg)}$$

and will pay

$$N\delta L(T_{i-1}, T_i) \text{ (on the floating rate leg).}$$

Hence, the net cash flow is

$$N\delta[R - L(T_{i-1}, T_i)].$$

Therefore, at time T, the no-arbitrage price of the $T_0 \times (T_N - T_0)$ receiver's swap is the present value of the cash flow, which is given by

$$\begin{aligned} SV(t; T_0, T_n) &= N\sum_{i=1}^{n}[\delta R - \delta L(T_{i-1}, T_i)] \times P(t, T_i) \\ &= N[\delta R \sum_{i=1}^{n} P(t, T_i) - \sum_{i=1}^{n} \delta \times \frac{P(t, T_{i-1)} - P(t, T_i)}{\delta P(t, T_i)} \times P(t, T_i)] \\ &= N[\delta R \sum_{i=1}^{n} P(t, T_i) - \sum_{i=1}^{n}[P(t, T_{i-1)} - P(t, T_i)] \\ &= N[-P(t, T_0) + \delta R \sum_{i=1}^{n} P(t, T_i) + P(t, T_n)]. \end{aligned}$$

We write $SV(T_0, T_n) = SV(T_0; T_0, T_n)$.

By definition, the swap rate R is chosen, so that the value of the swap equals zero at the time when the contract is made.

Proposition B.10. *The forward (par) swap rate* $R(t; T_0, T_n)$ *is given by*

$$R(t; T_0, T_n) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^{n} P(t, T_i)}.$$

Definition B.11. A $T_0 \times (T_N - T_0)$ *receiver swaption* with swaption strike K is a contract which at the expiry date T_0, gives the holder the right but not the obligation to enter into a swap with the fixed swap rates K and payment dates $T_1, ..., T_N$. We will call T_0 the *swaption expiry* and $T_N - T_0$ the *tenor* of the swaption.

At time T_0, the payoff of the swaption is given by $\mathbb{I}_{SV(T_0,T_n)>0}SV(T_0,T_n)$. As a result, the no-arbitrage price of the swaption $SOV(t)$ at time t is given by

$$SOV(t;T_0,T_n) = P(t,T_0)\,\mathbb{E}^{T_0}[\mathbb{I}_{SV(T_0,T_n)>0}SV(T_0,T_n)|\mathcal{F}_t]$$

where the expectation is taken under the T_0-forward measure.

For more details, the reader is referred to [2, Chapter 27].

B.3 Implied Black's Volatilities for Swaptions

The discussion in this subsection is based in part of the material found in [2, Chapter 27].

Definition B.12. let $S(t;T_0,T_n)$ be the following process

$$S(t;T_0,T_n) = \sum_{i=1}^{n} \delta P(t,T_i).$$

It is referred to as the accrual factor.

The forward swap rate can be expressed by

$$R(t;T_0,T_n) = \frac{P(t,T_0) - P(t,T_n)}{S(t;T_0,T_n)}.$$

Suppose that the swap rate is K and the nominal principal N is 1. The price of the $T_0 \times (T_N - T_0)$ receiver's swap can be expressed by

$$SV(t;T_0,T_n) = S(t;T_0,T_n)[K - R(t;T_0,T_n)].$$

Therefore, the swaption can be regarded as a put option on $R(t;T_0,T_n)$ with a strike price K when expressed in the numeraire $S(t;T_0,T_n)$. The market convention is to compute swaption prices by using the Black-76 formula and to quote prices in terms of the implied Black volatilities.

Definition B.13. The Black-76 formula for a $T_0 \times (T_N - T_0)$ *receiver swaption* with a swaption strike K defined as

$$SOV(t;T_0,T_n) = S(t;T_0,T_n)[KN(-d_2) - R(t;T_0,T_n)N(-d_1)],$$

where

$$d_1 = \frac{1}{\sigma(T_0, T_n)\sqrt{T_0 - t}} \left[\ln(\frac{R(t, T_0, T_n)}{K}) + \frac{1}{2}\sigma(T_0, T_n)^2 (T_0 - t) \right],$$
$$d_2 = d_1 - \sigma(T_0, T_n)\sqrt{T_0 - t}.$$

The constant $\sigma(T_0, T_n)$ is known as the Black's volatility. Given a market price for the swaption, the Black volatility implied by the Black formula is referred to as the implied Black volatility.