THE RIMINI CENTRE FOR ECONOMIC ANALYSIS

WP 48_07

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Dynamic Advertising with Spillovers: Cartel vs Competitive Fringe[1]

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[1]We thank George Leitmann and Georges Zaccour for helpful comments and discussion. The usual disclaimer applies.

Abstract

A differential oligopoly game with advertising is investigated, where different dynamics occur between two groups of agents, the former playing a competitive Nash game and the latter cooperating as a cartel. Sufficient conditions for stability and a qualitative analysis of the profit ratio and social welfare at equilibrium are provided. A threshold value for the size of the competitive fringe is pointed out by a suitable numerical simulation.

JEL Classification: C73, D43, D92, L13, M37

Keywords: Advertising, Differential games, Oligopoly, Collusion

1 Introduction

A relatively large amount of literature stemming from the pioneering contributions of Vidale and Wolfe (1957) and Nerlove and Arrow (1962) has investigated the strategic aspects of advertising campaigns with the tools of optimal control and differential game theory.[1]

Here we want to extend the analysis of this issue to consider a plausible scenario that, to the best of our knowledge, has been overlooked thus far; namely, joint advertising investments by a subset of the overall population of firms operating in a given industry. From a practical point of view, this might reflect a realistic situation where several firms split the cost of buying a page on magazines or newspapers, or a spot on TV channels, to advertise their products jointly. This implies that the firms involved in such a project save upon advertising costs, which is privately convenient; additionally, it may have the socially desirable effect of diminishing the wasteful duplication of efforts typically associated with competition. Incidentally, it is worth noting that this is a perspective which has not been given the attention it would deserve by the current legislation, where there is an abundant debate on the viability of comparative advertising but none (or very little) on the effects of joint advertising.

Building upon a model introduced by Cellini and Lambertini (2003) and further investigated by Cellini, Lambertini and Leitmann (2005) and Cellini, Lambertini and Mantovani (2007), we propose a dynamic game describing a Cournot oligopoly where firms belong to either a competitive fringe or a cartel. Each firm in the competitive fringe invests individually and non-cooperatively in advertising, while the cartel invests in a single advertising campaign whose costs are symmetrically distributes over cartel members.

[1]For exhaustive overviews, see Erickson (1991), Feichtinger, Hartl and Sethi (1994), Jørgensen and Zaccour (2004), *inter alia*.

Our results can be summarised as follows. The game is a linear state one, and therefore produces a strongly time consistent (i.e., subgame perfect) equilibrium under open-loop decision rules.[2] Then, having identified the steady state equilibrium and the related stability conditions, we assess the relative performance of a firm belonging to the fringe against that of a cartel member, identifying a threshold level of the relative density of firms in the two groups that establishes the optimal cartel size from a private (profit) standpoint. As to the social welfare performance, our analysis indicates that a benevolent regulator (unaffected by distributional concerns) is indifferent between two polar situations where the cartel is either very large or very small. This is the intuitive consequence of trading off the benefits of cost saving upon advertising investments against the inefficiency induced by lower reservation prices and sales. Finally, we propose an illustrative example based upon an appropriate numerical simulation.

The remainder of the paper is structured as follows. The model is outlined in section 2. Sections 3 and 4 are devoted to the analysis of the profit and welfare performances, respectively. The numerical simulation is carried out in section 5. Section 6 contains some concluding remarks.

2 The setup of the model

Our aim is to compare the optimal strategies of N single-good firms in an oligopoly market with Cournot competition, product differentiation and advertising investments to increase consumers' reservation prices. The game takes place in continuous time, with $t \in [0, \infty)$. The instantaneous utility

[2] For early differential games of advertising showing this property, see Leitmann and Schmitendorf (1978) and Feichtinger (1983). For an overview, see Dockner *et al.* (2000, ch. 7).

function of the representative consumer is borrowed from Spence (1976):

$$U(t) = \sum_{i=1}^{N} A_i(t) x_i(t) - \frac{1}{2}\left[B \sum_{i=1}^{N} x_i^2(t) + D \sum_{i} \sum_{j \neq i} x_i(t) x_j(t)\right] \qquad (1)$$

where $A_i(t) > 0$ is the reservation price for good i, whose purchased quantity is $x_i(t)$, and the constants B and D are positive parameters such that $0 < D \leq B$. In particular, D describes the degree of substitutability between any pair of different products: if $D = B$, products are homogeneous, while if $D = 0$ products are being sold on completely independent markets. Hence, parameter D can be considered as an inverse measure of the degree of product differentiation between any two varieties. The precise implications of this feature of the utility function (1) will be discussed below.

The maximisation of (1) s.t. the budget constraint $Y(t) \geq \sum_{i=1}^{N} p_i(t) x_i(t)$, where $Y(t)$ is the representative consumer's nominal income and $p_i(t)$ is the market price of variety i, yields the market demand for each variety. Each firm outside the cartel faces a demand function of the kind:

$$p_i(t) = A_i(t) - Bx_i(t) - D \sum_{j \neq i} x_j(t), \quad i = 1, \ldots, l, \qquad (2)$$

whereas the cooperating firms face the following:

$$p_m(t) = A_C(t) - Bx_m(t) - D \sum_{h \neq m} x_h(t), \quad m = l+1, \ldots, N. \qquad (3)$$

Accordingly, instantaneous consumer surplus is

$$CS(t) = U(t) - \sum_{i=1}^{N} p_i(t) x_i(t). \qquad (4)$$

We now turn to the description of the other side of the market, consisting in a population of N profit-seeking firms. While quantity-setting is noncooperative, the advertising investment behaviour is mixed: l firms constitute a competitive fringe behaving noncooperatively, whereas the remaining $N - l$ agents form an advertising cartel maximising the joint profits of the members in this phase.

The reservation prices evolve over time according to the following differential equations:

$$\dot{A}_i(t) = k_i(t) + \gamma \sum_{j=1,\dots,l,\ j \neq i} k_j(t) - \delta A_i(t), \quad i = 1, \dots, l; \tag{5}$$

$$\dot{A}_C(t) = k_C(t) - \delta A_C(t). \tag{6}$$

The first dynamics, eq. (5), refers to any firm i belonging to the competitive fringe choosing advertising efforts noncooperatively, while the second one, eq. (6), concerns the set of the $N - l$ agents in the cartel. The control variables are the quantities $x_i(t)$ and the advertising efforts $k_i(t)$ and $k_C(t)$; the state variables are the reservation prices (or market sizes): $A_i(t)$ for the i-th firm and $A_C(t)$ for all cartel's members. It is worth noting that (5) and (6) are indeed decoupled across the two groups of players, and this may appear as a special case of a more general setting where advertising efforts spill over across groups. However, we have chosen to take this route for two reasons. The first is that, clearly, the present structure simplifies calculations; the second is that it can be interpreted economically as of a situation where, say, TV ads of either group do not mention the existsnce of the rival group's products.[3]

The objective functions are discounted by the factor $e^{-\rho t}$, where $\rho \geq 0$ is assumed to be constant and common to all the N players. Further parameters appear in the equations of motion (5): in particular, the spillover constant $\gamma \in [-1, 1]$ captures the external effects of a firm's advertising of a firm on

[3]In fact, one might think that cross-groups advertising spillovers should be strictly negative, as either group would suggest consumers not to buy the rivals' varieties. Yet this can be ruled out *a priori* in view of the current legislation prohibiting it as an illegal anticompetitive behaviour.

the market size of all the remaining ones within the competitive fringe. γ may take negative values when an effect of comparative advertising occurs. Such a spillover effect clearly does not appear in the cartel's dynamics (6), as the cartel members operate a common advertising campaign. Additionally, we assume away cross effects between the competitive fringe and the cartel. Finally, $\delta \geq 0$ denotes the depreciation affecting every market size as time passes.

The instantaneous profit functions of firms, belonging respectively to the fringe and the cartel, look as follows:

$$\pi_i(t) = \begin{cases} (p_i(t) - c)x_i(t) - \alpha k_i^2(t), & i = 1, \ldots, l, \\ \\ (p_i(t) - c)x_i(t) - \dfrac{\alpha k_C^2(t)}{N - l}, & i = l + 1, \ldots, N, \end{cases}$$

where $cx_i(t)$ represents the linear production cost of the i-th agent, $c > 0$, and $\alpha > 0$ is a constant scaling the marginal cost of the advertising activity. Note that in the cooperative case, α is divided by the number of participants in the cartel, since the advertising effort is equally shared by all the players in the coalition. In order to simplify next calculations, from now on we will fix $B = 1$.

Firms control output levels and advertising efforts so as to maximise their discounted profit flows. Hence, the infinite horizon problem can be stated as follows:

$$\max_{x_i(t), k_i(t)} J_i \equiv \int_0^{\infty} \pi_i(t) e^{-\rho t} dt, \tag{7}$$

subject to the set of state equations composed by (5) and (6), and the set of initial conditions $A_i(0) = A_{i0}$ and $A_C(0) = A_{C0}$.

For future reference, we may now define the instantaneous social welfare function, as follows:

$$SW\left(t\right) = CS\left(t\right) + \sum_{i=1}^{N} \pi_i(t). \tag{8}$$

That is, social welfare is given by the sum of consumer surplus and industry profits.

From a technical standpoint, we can look upon this setup as a combination of two different optimal control problems, which we will confront by applying Pontryagin's maximum principle, under open-loop decision rules. We begin from the l non-cooperating firms. The current value Hamiltonian, the first order conditions (FOCs) and the adjoint equation for this first part of the problem are, respectively:

$$H_i^{NC}(\cdot,t) = e^{-\rho t}\left[(p_i(t)-c)x_i(t) - \alpha k_i^2(t) + \lambda_i(t)\left(k_i(t) + \gamma\sum_{j\neq i} k_j(t) - \delta A_i(t)\right)\right]; \tag{9}$$

$$\frac{\partial H_i^{NC}(\cdot,t)}{\partial x_i} = 0 \implies -c + A_i(t) - 2x_i(t) - D\sum_{j\neq i} x_j(t) = 0; \tag{10}$$

$$\frac{\partial H_i^{NC}(\cdot,t)}{\partial k_i} = 0 \implies -2\alpha k_i(t) + \lambda_i(t) = 0; \tag{11}$$

$$\begin{aligned} -\frac{\partial H_i^{NC}(\cdot,t)}{\partial A_i} &= \dot{\lambda}_i(t) - \rho\lambda_i(t) \implies \\ \dot{\lambda}_i(t) &= (\rho+\delta)\lambda_i(t) - x_i(t). \end{aligned} \tag{12}$$

From (11) we obtain $k_i(t) = \lambda_i(t)/(2\alpha)$, which can be differentiated w.r.t. time to yield the control dynamics $\dot{k}_i(t) = \dot{\lambda}_i(t)/(2\alpha)$. Then, using (12) and $\lambda_i(t) = 2\alpha k_i(t)$, the latter equation can be rewritten as

$$\dot{k}_i(t) = (\rho+\delta)k_i(t) - \frac{x_i(t)}{2\alpha},\ i = 1,\ldots,l. \tag{13}$$

Now consider the $N-l$ firms forming the cartel, whose specific maximization problem features the following current value Hamiltonian, FOCs and adjoint equation:

$$H_i^{C}(\cdot,t) = e^{-\rho t}\left[(p_i(t)-c)x_i(t) - \frac{\alpha k_C^2(t)}{N-l} + \mu_i(t)(k_C(t) - \delta A_C(t))\right]; \tag{14}$$

$$\frac{\partial H_i^C(\cdot,t)}{\partial x_i} = 0 \implies -c + A_C(t) - 2x_i(t) - D\sum_{j\neq i} x_j(t) = 0; \tag{15}$$

$$\frac{\partial H_i^C(\cdot,t)}{\partial k_C} = 0 \implies -\frac{2\alpha k_C(t)}{N-l} + \mu_i(t) = 0; \tag{16}$$

$$\begin{aligned} -\frac{\partial H_i^C(\cdot,t)}{\partial A_C} &= \dot{\mu}_i(t) - \rho\mu_i(t) \implies \\ \dot{\mu}_i(t) &= (\rho+\delta)\mu_i(t) - x_i(t), \end{aligned} \tag{17}$$

whereby, after some simple manipulations along the same line followed for the competitive fringe, we obtain the state dynamics

$$\dot{k}_C(t) = (\rho+\delta)k_C(t) - \frac{(N-l)x_i(t)}{2\alpha},\ i = l+1,\ldots,N-l. \tag{18}$$

Before proceeding any further with the solution of the game, we state a preliminary result that justifies our choice of pursuing the open-loop solution:

Lemma 1. *The game is a linear state one. Therefore, its open-loop solution is strongly time consistent.*

Proof. Straightforward, from the observation that Hamiltonian functions (9) and (14) are linear in the state variables. For further details, see Cellini and Lambertini (2003) and Cellini, Lambertini and Leitmann (2005). □

Now, relying upon the *a priori* symmetry across firms in the two groups, we may relabel the state and control variables as follows:

$$x_i := x_{NC},\ \ k_i := k_{NC},\ \ A_i := A_{NC},\ \ i = 1,\ldots,l, x_i := x_C,\ \ i = l+1,\ldots,N. \tag{19}$$

Hence, the conditions for the l non-cooperating firms are:

$$\begin{cases} -c + A_{NC} - 2x_{NC} - D[(l-1)x_{NC} + (N-l)x_C] = 0 \\ \\ \dot{k}_{NC} - (\delta+\rho)k_{NC} + \dfrac{x_{NC}}{2\alpha} = 0 \\ \\ \dot{A}_{NC} = k_{NC} + \gamma[(l-1)k_{NC}] - \delta A_{NC} \end{cases} , \tag{20}$$

whereas the corresponding set for the $N-l$ cartelized firms turns out to be:

$$\begin{cases} -c + A_C - 2x_C - D[lx_{NC} + (N-l-1)x_C] = 0 \\ \\ \dot{k}_C - (\delta+\rho)k_C + \dfrac{(N-l)x_C}{2\alpha} = 0 \\ \\ \dot{A}_C = k_C - \delta A_C \end{cases} . \tag{21}$$

After some calculations, which we omit for brevity, we obtain the following expressions for the quantity control variables:

$$x_{NC} = \frac{-(2 + D(N-l-1))A_{NC} + D(N-l)A_C + c(2-D)}{(D-2)(DN-D+2)}, \tag{22}$$

$$x_C = \frac{DlA_{NC} - [2 + D(l-1)]A_C + c(2-D)}{(D-2)(DN-D+2)}, \tag{23}$$

Hence, the related dynamical system is:

$$\begin{pmatrix} \dot{k}_{NC} \\ \\ \dot{A}_{NC} \\ \\ \dot{k}_C \\ \\ \dot{A}_C \end{pmatrix} = J \cdot \begin{pmatrix} k_{NC} \\ \\ A_{NC} \\ \\ k_C \\ \\ A_C \end{pmatrix} + \begin{pmatrix} \dfrac{c}{F(D,N,\alpha)} \\ \\ 0 \\ \\ \dfrac{(N-l)c}{F(D,N,\alpha)} \\ \\ 0 \end{pmatrix} , \tag{24}$$

the Jacobian matrix being:

$$J = \begin{pmatrix} \delta+\rho & \dfrac{2+D(N-l-1)}{(D-2)F(D,N,\alpha)} & 0 & -\dfrac{D(N-l)}{(D-2)F(D,N,\alpha)} \\ \\ \gamma(l-1)+1 & -\delta & 0 & 0 \\ \\ 0 & -\dfrac{D(N-l)l}{(D-2)F(D,N,\alpha)} & \delta+\rho & \dfrac{(N-l)(2+D(l-1))}{(D-2)F(D,N,\alpha)} \\ \\ 0 & 0 & 1 & -\delta \end{pmatrix},$$

where $F(D,N,\alpha) = 2(DN - D + 2)\alpha$.

Obviously, the only possible meaningful equilibrium points for (24) must have all positive coordinates. On the other hand, when a dynamic system involves such a large number of parameters, it is difficult to find suitable conditions to this aim. For instance, the ratio between advertising effort and reservation price for the non-cooperating firms at the equilibrium is $[\gamma(l-1)+1]/\delta$, so that we should consider the limitation $\gamma > -1/(l-1)$. However, sufficient conditions may be established as in the following Proposition, where we set $\theta := \delta(\delta+\rho)$:

Proposition 2. *If* $\det(J) \neq 0$ *and the following conditions hold:*

$$DN - D + 2 > 0,$$

$$min\left\{\frac{F(D,N,\alpha)}{N}, \frac{\gamma}{2D\alpha(\gamma+1)}\right\} > \theta > max\left\{\frac{1+\gamma(l-1)}{2(2-D)\alpha}, \frac{N-l}{2(2-D)\alpha}, \theta_0\right\},$$

where

$$\theta_0 = \frac{(\gamma-1)(2-D) + 2\gamma N - DN(1-N\gamma)}{[DN(1+\gamma) - 2(\gamma-1)]F(D,N,\alpha)},$$

then system (24) admits only one steady state $(k_{NC}^{SS}, A_{NC}^{SS}, k_C^{SS}, A_C^{SS})$ *with positive coordinates.*

Proof. If $\det(J) \neq 0$, (24) admits only one equilibrium point $(k_{NC}^{SS}, A_{NC}^{SS}, k_C^{SS}, A_C^{SS})$:

$$k_{NC}^{SS} = \frac{c\delta H(l,N,D,\alpha,\theta)}{G(l,N,D,\alpha,\gamma,\theta)},$$

$$A_{NC}^{SS} = \frac{c(\gamma(l-1)+1)H(l,N,D,\alpha,\theta)}{G(l,N,D,\alpha,\gamma,\theta)},$$

$$k_C^{SS} = \frac{c\delta(N-l)K(l,N,D,\alpha,\gamma,\theta)}{G(l,N,D,\alpha,\gamma,\theta)},$$

$$A_C^{SS} = \frac{c(N-l)K(l,N,D,\alpha,\gamma,\theta)}{G(l,N,D,\alpha,\gamma,\theta)},$$

where

$$\begin{aligned}
H(l,N,D,\alpha,\theta) &= (D-2)N - DN^2 + (DN-D+2)l + (2-D)\theta F(D,N,\alpha),\\
K(l,N,D,\alpha,\gamma,\theta) &= 2(\gamma - 1 - \gamma l + \theta F(D,N,\alpha))+\\
&\quad -(N-1+\gamma-\gamma l + (l-1)\gamma N + \theta F(D,N,\alpha))D,\\
G(l,N,D,\alpha,\gamma,\theta) &= (N-\theta F(D,N,\alpha))[2(\gamma-1+\theta F(D,N,\alpha))+\\
&\quad -(N-1+\gamma-\gamma N+\theta F(D,N,\alpha))D] + (-D\theta F(D,N,\alpha)\\
&\quad + (2+D(N-1-\theta F(D,N,\alpha))\gamma)l^2+\\
&\quad + [(\gamma-1+(1-\gamma N+\theta(1+\gamma)F(D,N,\alpha))N+\\
&\quad -2(\theta F(D,N,\alpha)-1+(N+1-\theta F(D,N,\alpha))\gamma)]l.
\end{aligned}$$

By imposing the positivity of each numerator and denominator of the coordinates, we achieve some inequalities on the parameters, yielding the sufficient conditions of the hypothesis. □

Concerning the stability properties of the dynamic system, we are in a position to prove:

Proposition 3. *If*

$$\theta^2 + \left(\frac{(N-l)[2+D(l-1)] + [\gamma(l-1)+1][2+D(N-l-1)]}{(D-2)F(D,N,\alpha)}\right)\theta +$$

$$-\frac{(N-l)[\gamma(l-1)+1]}{(D-2)F(D,N,\alpha)} < 0,$$

the steady state of system (24) is a saddle point.

Proof. The Jacobian matrix of (24) is of the following kind:

$$J = \left(\begin{array}{ccc|cc} & & & 0 & \phi \\ & J_1 & & & \\ & & & 0 & 0 \\ \hline 0 & & l\phi & & \\ & & & & J_2 \\ 0 & & 0 & & \end{array} \right),$$

where $\phi := -D(N-l)/\left[(D-2)F(D,N,\alpha)\right]$. By expressing its characteristic polynomial $p_J(\lambda)$ through Laplace's expansion, we have:

$$p_J(\lambda) = \lambda^4 - \mathrm{tr}(J)\lambda^3 + [\mathrm{tr}(J_1)\mathrm{tr}(J_2) + \det(J_1) + \det(J_2)]\lambda^2 +$$

$$-[\mathrm{tr}(J_1)\det(J_2) + \mathrm{tr}(J_2)\det(J_1)]\lambda + \det(J_1)\det(J_2) - C,$$

where $C := D^2[\gamma(l-1)+1](N-l)^2 l/\left[(D-2)^2 F^2(D,N,\alpha)\right]$. A sufficient condition to have a negative eigenvalue and consequently a saddle point for (24) is obviously:

$$\det(J_1)\det(J_2) - C < 0 \iff \ldots \iff$$

$$\iff \theta^2 + \left(\frac{(N-l)[2+D(l-1)] + [\gamma(l-1)+1][2+D(N-l-1)]}{(D-2)F(D,N,\alpha)}\right)\theta +$$

$$-\frac{(N-l)[\gamma(l-1)+1]}{(D-2)F(D,N,\alpha)} < 0.$$

This proves the claim. □

3 Comparison between profit levels

In this section, we investigate the relationship between profits π_C^{SS} and π_{NC}^{SS}, that is, the instantaneous profits accruing at the steady state equilibrium to the fringe and a cartel firm, respectively. By substituting the values found in previous section, we have:

$$\pi_{NC}^{SS} = \frac{\alpha c^2 \delta^2 (N - l - 2\alpha(2-D)\theta)^2 (4\alpha(\delta+\rho)^2 - 1)}{P(l, N, D, \alpha, \gamma, \theta)}; \tag{25}$$

$$\pi_C^{SS} = \frac{\alpha c^2 \delta^2 (2\alpha(2-D)\theta + \gamma - 1 - \gamma l)^2 (l - N + 4\alpha(\delta+\rho)^2)}{P(l, N, D, \alpha, \gamma, \theta)}, \tag{26}$$

where

$$P(l, N, D, \alpha, \gamma, \theta) = ((1+\gamma(l-1))(l-N) + 2\alpha(2((l-1)(\gamma-1)+N) + D(\gamma-1-(\gamma+1)l^2 +$$

$$+(l + (l-1)\gamma)N)\theta - 4\alpha^2(2-D)(2-D+DN)\theta^2). \tag{27}$$

As an intermediate step, we prove:

Lemma 4. *If* $\alpha > \dfrac{N-l}{4(\delta+\rho)^2}$, *the profit ratio at equilibrium is nonnegative.*

Proof. The ratio

$$\frac{\pi_{NC}^{SS}}{\pi_C^{SS}} = \frac{(N - l - 2\alpha(2-D)\theta)^2 (4\alpha(\delta+\rho)^2 - 1)}{(2\alpha(2-D)\theta + \gamma - 1 - \gamma l)^2 (l - N + 4\alpha(\delta+\rho)^2)}$$

is nonnegative if the expressions $4\alpha(\delta+\rho)^2 - 1$ and $l - N + 4\alpha(\delta+\rho)^2$ have the same sign. The condition $N - l > 1$ implies $l - N + 4\alpha(\delta+\rho)^2 < -1 + 4\alpha(\delta+\rho)^2$, so if $\alpha > (N-l) / [4(\delta+\rho)^2]$, both expressions are positive. □

The next step consists in assessing whether the profit ratio is above or below one. Consider the profit ratio at equilibrium as a function of the size of the competitive fringe l:

$$pr(l) := \frac{\pi_{NC}^{SS}(l)}{\pi_C^{SS}(l)} \tag{28}$$

defined on the parametric interval where it cannot be negative and it verifies the assumption $l \in (N - 4\alpha(\delta+\rho)^2,\ N-1)$. Basically, if $pr(l) > 1$ a firm belonging to the competitive fringe outperforms a firm belonging to the cartel, whereas if $pr(l) < 1$ the opposite applies.

First of all, $pr(l)$ admits the vertical asymptote $l = \widetilde{l} = \frac{1}{\gamma}(\gamma - 1 + 2\alpha(2 - D)\theta)$ and vanishes at $l = \overline{l} = N - 2\alpha(2-D)\theta$. On this basis, we may state:

Proposition 5. *Take* $\alpha > \frac{1}{2(2-D)\theta}$*; then:*

a) if $\overline{l} > \widetilde{l}$*, there exists a value* $l^* \in (\widetilde{l},\ \overline{l})$ *such that* $pr(l^*) = 1$*, and* $pr(l) > 1$ *for* $\widetilde{l} < l < l^*$ *and* $pr(l) < 1$ *for* $l^* < l < \overline{l}$*;*

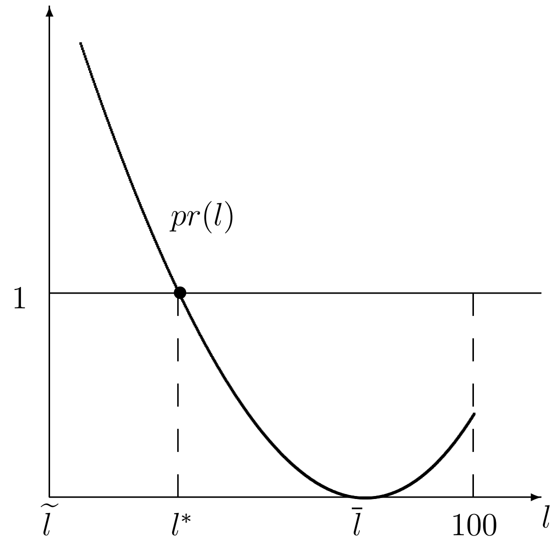
b) if $\overline{l} < \widetilde{l}$*, there exist two values* l^* *and* l^{**}*,* $l^* < \overline{l} < l^{**}$*, such that* $pr(l^*) = pr(l^{**}) = 1$ *and two related neighbourhoods in which* $pr(l)$ *takes values both larger and smaller than 1.*

Proof. The hypothesis on α implies that the zero $\overline{l}$ of $pr(l)$ belongs to the interval where the function is defined. Two different possible situations may take place, depending on the position of $\overline{l}$ with respect to the asymptote $l = \widetilde{l}$. In case (a), since $pr(l) \geq 0$ in the domain being considered, and since

$$\lim_{l \longrightarrow \widetilde{l}+} pr(l) = +\infty,$$

$\overline{l}$ is a minimum for $pr(l)$, which is necessarily monotonically decreasing over the interval $(\widetilde{l},\ \overline{l})$. Therefore in that interval there exists an l^* such that $pr(l^*) = 1$.

Figure 1. Case a): $pr(l)$ intersecting the level 1 in $l = l^*$



As far as case (b) is concerned, since

$$\lim_{l \longrightarrow N-4\alpha(\delta+\rho)^{2+}} pr(l) = \lim_{l \longrightarrow \widetilde{l}^-} pr(l) = +\infty,$$

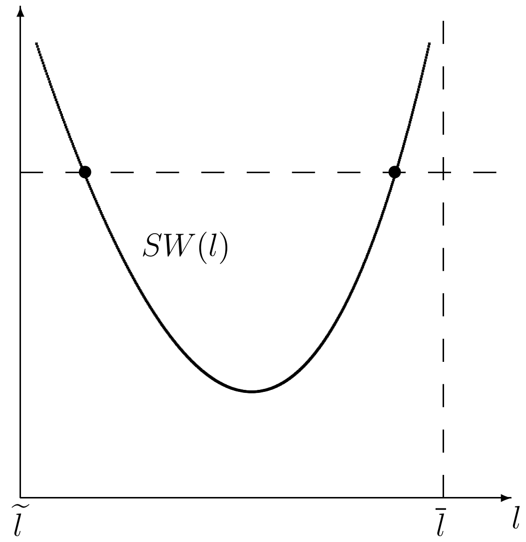
then $\bar{l}$ is a minimum included in the interval bounded by the two asymptotes. Although two further values may exist such that $pr'(l) = 0$, the curve intersects the level $pr(l) = 1$ in at least two points corresponding to l^* and l^{**}, so two subintervals may be delimited where the relative weight of profits changes. □

4 Welfare assessment

Here we address the social welfare implications of the advertising cartel. Plugging the equilibrium expressions of advertising efforts, quantities and reservation prices into (8), one obtains the relevant expression describing the social welfare level in steady state.

Our task is to evaluate the behaviour of steady state social welfare for $l \in \left(\widetilde{l}, \bar{l}\right)$, taking as given $\{\alpha, \delta, \gamma, \rho, c, D, N\}$. This is done in figure 2, showing that social welfare is convex in l over the relevant range.

Figure 2. Steady state social welfare



The qualitative properties of the social welfare function illustrated in figure 2 imply that there are two distinct values of l achieving any given level of social welfare, *all else equal*. The relevant consequence of this result is that an antitrust agency aiming at the maximisation of social welfare w.r.t. the size of the advertising cartel would be indifferent between a very large cartel or a very small one. This of course holds under the assumption that the antitrust authority does not worry about the distribution of surplus between firms and consumers, but it is only concerned about the size of the total pie of that surplus. This may be fully justified provided consumers are, at the same time, shareholders of the firms operating in this industry.[4]

[4]The current antitrust legislation, both in the EU and in the US, is ambiguous in this respect. See, for instance, Motta (2004).

The shape of social welfare w.r.t. l can be explained by looking at total industry output and advertising efforts at the steady state equilibrium. All else equal, both $X^{SS} = lx_{NC}^{SS} + (N-l)\, x_C^{SS}$ and $K^{SS} = lk_{NC}^{SS} + k_C^{SS}$ are convex over the range $\left(\widetilde{l}, \bar{l}\right)$, taking a minimum inside such a range. The same applies to A_{NC}^{SS} and A_C^{SS}. Hence, we may outline the following: higher levels of k_{NC}^{SS} and k_C^{SS} entail higher levels of A_{NC}^{SS} and A_C^{SS}; in turn, higher levels of A_{NC}^{SS} and A_C^{SS} entail levels of x_{NC}^{SS} and x_C^{SS}. In summary, the regulator must trade off any costs saving for a reduction of reservation prices and output levels (or conversely). Therefore, a situation where the cartel is very large (respectively, very small) is preferable to any intermediate situation as far as social welfare is concerned. The only appreciable difference between having a large cartel or a large competitive fringe lies in the relative concentration of the advertising investment and output across groups of firms in the industry. However, this is clearly immaterial if the regulator has no such concerns and only looks at the aggregate welfare performance.

5 Numerical simulations

In this section, the results of some numerical simulations carried out with Mathematic@ 5.0 are shown in order to better illustrate a few essential features of the model. Several parameters are preliminarily fixed: the population of players consists of $n = 100$ firms, and the other parameters are $c = 1/30$, $\delta = 1/7$, $\rho = 1/8$, $\alpha = 12$, and $\gamma = 9/10$. Additionally, $1 < l < 99$ and $0 < D < 1$.

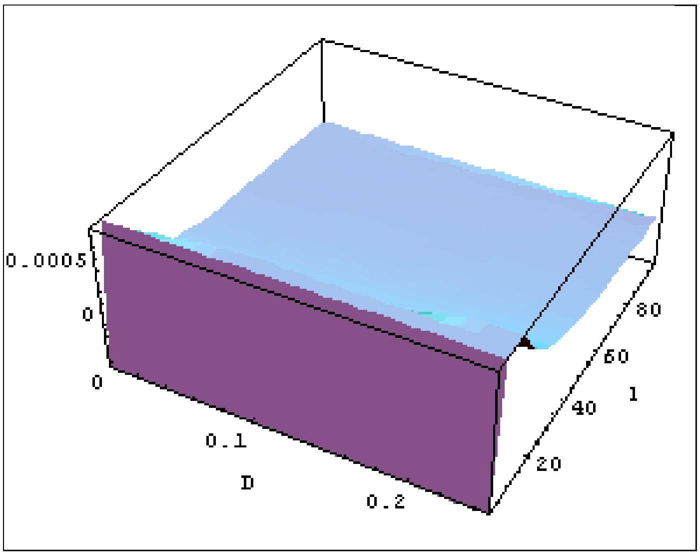


Figure 3. k_{NC}^{ss} as a function of l and D. It keeps positive for $(l, D) \in [1, 99] \times [0, 0.25]$.

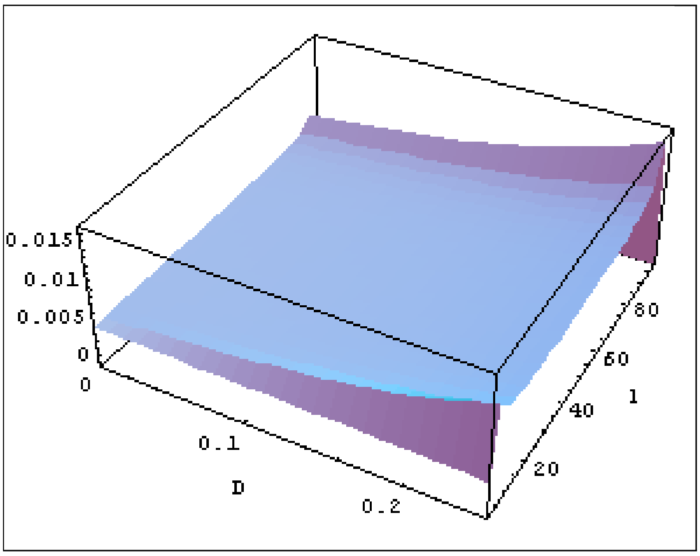


Figure 4. k_{C}^{ss} as a function of l and D. It keeps positive for $(l, D) \in [1, 99] \times [0, 0.25]$.

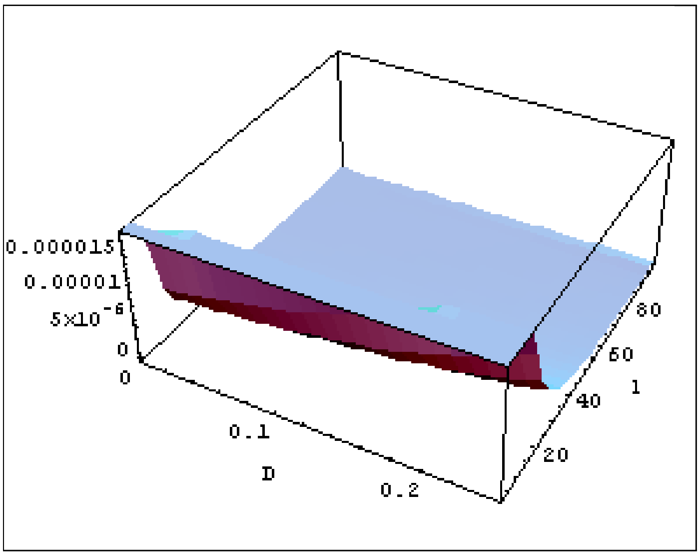


Figure 5. π_{NC}^{SS} as a function of l and D. Note that it is everywhere decreasing in l.

If we fix $D = 0.2$, we can also plot $pr(l)$, which takes nonnegative values for large l as in Figure 7. By focusing on the intersection between the level $pr(l) = 1$ and the graph in Figure 7, we can determine the value $l^* \sim 96.5$, corresponding to Proposition 4(a). If we specifically fix $l = 97$, we obtain the following positive controls and states at equilibrium:

$$k_{NC}^{SS}(97) = 0.000145, \quad k_C^{SS}(97) = 0.027682, \tag{29}$$

$$x_{NC}^{SS}(97) = 0.000931, \quad x_C^{SS}(97) = 0.059319, \tag{30}$$

$$A_{NC}^{SS}(97) = 0.088679, \quad A_C^{SS}(97) = 0.193778. \tag{31}$$

The same social welfare level associated with $l^* \sim 96.5$ is also attainable in correspondence of $\widehat{l} \sim 4.2$, with the social welfare level taking a minimum at $l \sim 56.7$.

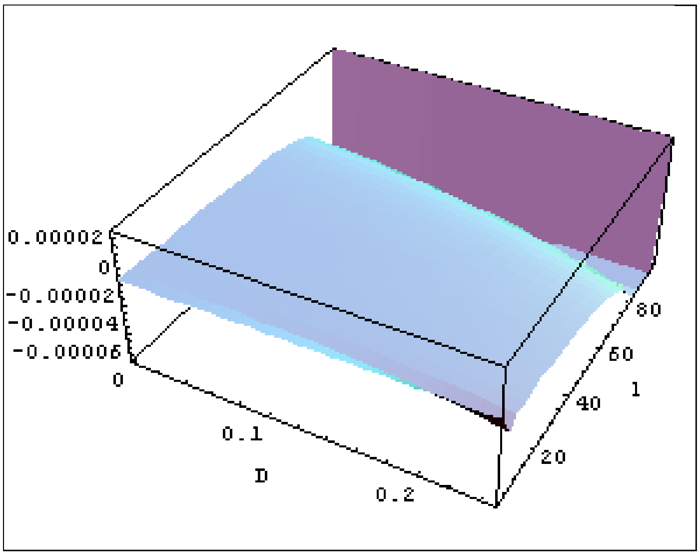


Figure 6. π_C^{SS} as a function of l and D. Note that it is positive only for values of l close to the boundary of the considered domain.

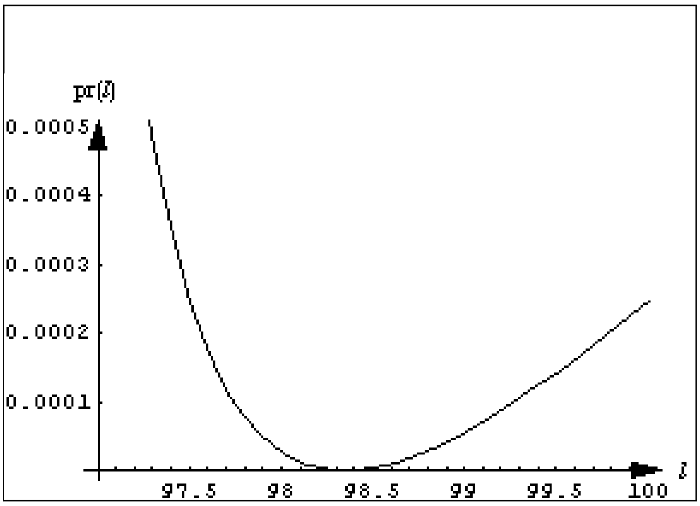


Figure 7. The profit ratio $pr\,(l) = \pi_{NC}^{SS}(l)/\pi_C^{SS}(l)$.

6 Concluding remarks

Adopting a differential game approach, we have shown that, in designing the optimal amount of investment in advertising, an industry may find it efficient to activate a cartel aimed at splitting the cost of advertising among its members. We have investigated the optimal size of such a cartel, as well as its implications on social welfare. In particular, the latter appears to be non-monotone w.r.t. the size of the cartel (or, conversely, the competitive fringe), whereby a regulator should shape competition policy so as to avoid intermediate industry structures and aim instead at attaining rather *extreme* structures where the cartel is either very large or very small.

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