THE RIMINI CENTRE FOR ECONOMIC ANALYSIS

WP 38_07

GABRIELE FIORENTINI
Universituy of Firenze, Italy
and
The Rimini Centre for Economic Analysis, Rimini, Italy

ENRIQUE SENTANA
CEMFI, Madrid, Spain

"ON THE EFFICIENCY AND CONSISTENCY OF LIKELIHOOD ESTIMATION IN MULTIVARIATE CONDITIONALLY HETEROSKEDASTIC DYNAMIC REGRESSION MODELS"

The Rimini Centre for Economic Analysis
Legal address: Via Angherà, 22 – Head office: Via Patara, 3 - 47900 Rimini (RN) – Italy
www.rcfea.org - secretary@rcfea.org

On the efficiency and consistency of likelihood estimation in multivariate conditionally heteroskedastic dynamic regression models*

Gabriele Fiorentini
Università di Firenze, Viale Morgagni 59, I-50134 Firenze, Italy
The Rimini Centre for Economic Analysis
<fiorentini@ds.unifi.it>

Enrique Sentana
CEMFI, Casado del Alisal 5, E-28014 Madrid, Spain
<sentana@cemfi.es>

February 2003
Revised: July 2007

Abstract

We rank the efficiency of several likelihood-based parametric and semiparametric estimators of conditional mean and variance parameters in multivariate dynamic models with *i.i.d.* spherical innovations, and show that Gaussian pseudo maximum likelihood estimators are inefficient except under normality. We also provide conditions for partial adaptivity of semiparametric procedures, and relate them to the consistency of distributionally misspecified maximum likelihood estimators. We propose Hausman tests that compare Gaussian pseudo maximum likelihood estimators with more efficient but less robust competitors. We also study the efficiency of sequential estimators of the shape parameters. Finally, we provide finite sample results through Monte Carlo simulations.

Keywords: Adaptivity, ARCH, Elliptical Distributions, Financial Returns, Hausman tests, Semiparametric Estimators, Sequential Estimators.

JEL: C13, C14, C12, C51, C52

*We would like to thank Dante Amengual, Manuel Arellano, Nour Meddahi, Javier Mencía, Olivier Scaillet, Paolo Zaffaroni, participants at the European Meeting of the Econometric Society (Stockholm, 2003), the Symposium on Economic Analysis (Seville, 2003), the CAF Conference on Multivariate Modelling in Finance and Risk Management (Sandbjerg, 2006), the Second Italian Congress on Econometrics and Empirical Economics (Rimini, 2007), as well as audiences at AUEB, Bocconi, Cass Business School, CEMFI, CREST, EUI, Florence, NYU, RCEA, Roma La Sapienza and Queen Mary for useful comments and suggestions. Of course, the usual caveat applies.

1 Introduction

Many empirical studies with financial time series data indicate that the distribution of asset returns is usually rather leptokurtic, even after controlling for volatility clustering effects. Nevertheless, the Gaussian pseudo-maximum likelihood (PML) estimators advocated by Bollerslev and Wooldridge (1992) remain consistent for the conditional mean and variance parameters in those circumstances, so long as those moments are correctly specified.

However, a non-normal distribution may be indispensable when one is interested in features of the distribution of asset returns beyond its conditional mean and variance. For instance, empirical researchers and financial market practitioners are often interested in the so-called Value at Risk of an asset, which is the positive threshold value V such that the probability of the asset suffering a reduction in wealth larger than V equals some pre-specified level $\varkappa < 1/2$. In addition, they are sometimes interested in the probability of the joint occurrence of several extreme events, which is regularly underestimated by the multivariate normal distribution, especially in larger dimensions. This naturally leads one to specify a parametric leptokurtic distribution for the standardised innovations, such as the multivariate student t analysed in Fiorentini, Sentana and Calzolari (2003) (FSC), and to estimate the conditional mean and variance parameters jointly with the parameters characterising the shape of the assumed distribution by maximum likelihood (ML). However, while ML will often yield more efficient estimators of the conditional mean and variance parameters than Gaussian PML if the assumed conditional distribution is correct, it may end up sacrificing consistency when it is not, as shown by Newey and Steigerwald (1997).

If one were mostly interested in the first two conditional moments, the semiparametric (SP) estimators of Engle and Gonzalez-Rivera (1991) and Gonzalez-Rivera and Drost (1999) would offer an attractive solution because they are sometimes both consistent and partially efficient, as proved by Linton (1993), Drost and Klaassen (1997), Drost, Klaassen and Werker (1997), or Sun and Stengos (2006). However, they suffer from the curse of dimensionality, which severely limits their use in multivariate models. To avoid this problem, Hodgson and Vorkink (2003) and Hafner and Rombouts (2007) have recently discussed elliptically symmetric semiparametric (SSP) estimators, which retain univariate rates for their nonparametric part regardless of the cross-sectional dimension of the data, but which are unfortunately less robust.

One of the main objectives of our paper is to study in detail the trade-offs between efficiency and consistency of the conditional mean and variance parameters that arise in this context. While many of the aforementioned papers provide detailed analyses of one of these issues, especially in univariate models, or in models with no mean, to our knowledge we are the first to simultaneously analyse all the hard choices than an empirical researcher faces in practice. Furthermore, we do so in a multivariate framework with non-zero means, in which some of the earlier results seem misleadingly simple. Moreover, we explicitly look at the efficiency ranking of the feasible ML procedure that jointly estimates the shape parameters, as well as the infeasible ML, SSP, SP and PML estimators considered in the existing literature. We also provide conditions for partial adaptivity of the SSP and SP procedures, which we relate to the conditions for the consistency of the corresponding parametric ML estimators when the conditional distribution is misspecified. Finally, we propose simple Hausman tests that compare the feasible ML and SSP estimators to the Gaussian PML ones to assess the validity of the distributional assumptions.

But given that practitioners often want to go beyond the first two conditional moments, one cannot simply treat the shape parameters as nuisance parameters. For that reason, we also consider sequential estimators of the shape parameters, which can be easily obtained from the standardised innovations evaluated at the Gaussian PML estimators, and assess their asymptotic efficiency relative to their feasible ML counterpart. In particular, we consider a sequential ML estimator, as well as sequential method of moments (MM) estimators based on higher order moment parameters such as the coefficient of multivariate excess kurtosis.

The rest of the paper is organised as follows. In section 2, we present closed-form expressions for the score vector, Hessian and conditional information matrices of a log-likelihood function based on a spherically symmetric assumption for the innovations, and derive the efficiency bounds of the Gaussian PML estimator and both SP estimators, as well as the sequential estimators of the shape parameters. Then, in section 3 we compare the efficiency of the different estimators of the conditional mean and variance parameters, discuss two specific models of practical interest, and obtain some general results on partial adaptivity. In section 4, we compare the relative efficiency of the different estimators of the shape parameters, while in section 5 we first study the consistency of the conditional mean parameters when the conditional distribution is misspecified, and then introduce the Hausman tests. A Monte Carlo evaluation of the different parameter estimators and testing procedures can be found in section 6. Finally, we present our conclusions in section 7. Proofs and auxiliary results are gathered in appendices.

2 Theoretical background

2.1 The model

In a multivariate dynamic regression model with time-varying variances and covariances, the vector of N dependent variables, $\mathbf{y}_t$, is typically assumed to be generated as:

$$\begin{aligned}
\mathbf{y}_t &= \boldsymbol{\mu}_t(\boldsymbol{\theta}_0) + \boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta}_0)\varepsilon_t^*, \\
\boldsymbol{\mu}_t(\boldsymbol{\theta}) &= \boldsymbol{\mu}(\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}), \\
\boldsymbol{\Sigma}_t(\boldsymbol{\theta}) &= \boldsymbol{\Sigma}(\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}),
\end{aligned}$$

where $\boldsymbol{\mu}()$ and $vech[\boldsymbol{\Sigma}()]$ are $N \times 1$ and $N(N+1)/2 \times 1$ vector functions known up to the $p \times 1$ vector of true parameter values $\boldsymbol{\theta}_0$, $\mathbf{z}_t$ are k contemporaneous conditioning variables, I_{t-1} denotes the information set available at $t-1$, which contains past values of $\mathbf{y}_t$ and $\mathbf{z}_t$, $\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})$ is some particular "square root" matrix such that $\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{1/2\prime}(\boldsymbol{\theta}) = \boldsymbol{\Sigma}_t(\boldsymbol{\theta})$, and $\boldsymbol{\varepsilon}_t^*$ is a martingale difference sequence satisfying $E(\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0) = \mathbf{0}$ and $V(\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0) = \mathbf{I}_N$. Hence,

$$\left.\begin{array}{l} E(\mathbf{y}_t|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0) = \boldsymbol{\mu}_t(\boldsymbol{\theta}_0) \\ V(\mathbf{y}_t|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0) = \boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0) \end{array}\right\}. \tag{1}$$

To complete the model, we need to specify the conditional distribution of $\boldsymbol{\varepsilon}_t^*$. We shall initially assume that, conditional on $\mathbf{z}_t$ and I_{t-1}, $\boldsymbol{\varepsilon}_t^*$ is independent and identically distributed as some particular member of the spherical family with a well defined density (see Appendix A), or $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0, \boldsymbol{\eta}_0 \sim i.i.d.\ s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0)$ for short, where $\boldsymbol{\eta}$ are some q additional parameters that determine the shape of the distribution of $\varsigma_t = \boldsymbol{\varepsilon}_t^{*\prime}\boldsymbol{\varepsilon}_t^*$. The most prominent example is the spherical normal distribution, which we denote by $\boldsymbol{\eta}_0 = \mathbf{0}$. For illustrative purposes, though, we shall also look in some detail at the special case of a standardised multivariate t with ν_0 degrees of freedom, or $i.i.d.\ t(\mathbf{0}, \mathbf{I}_N, \nu_0)$ for short. As is well known, the multivariate student t approaches the multivariate normal as $\nu_0 \to \infty$, but has generally fatter tails. For that reason, we define η as $1/\nu$, which will always remain in the finite range $[0, 1/2)$ under our assumptions.

2.2 The log-likelihood function, its score, Hessian and information matrix

Let $\boldsymbol{\phi} = (\boldsymbol{\theta}', \boldsymbol{\eta})'$ denote the $p+q$ parameters of interest, which we assume variation free. Ignoring initial conditions, the log-likelihood function of a sample of size T based on a particular parametric spherical assumption will take the form $L_T(\boldsymbol{\phi}) = \sum_{t=1}^T l_t(\boldsymbol{\phi})$, with $l_t(\boldsymbol{\phi}) = d_t(\boldsymbol{\theta}) + c(\boldsymbol{\eta}) + g[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]$, where $d_t(\boldsymbol{\theta}) = -1/2 \ln|\boldsymbol{\Sigma}_t(\boldsymbol{\theta})|$ corresponds to the Jacobian, $c(\boldsymbol{\eta})$ to the constant of integration of the assumed density, and $g[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]$ to its kernel, where $\varsigma_t(\boldsymbol{\theta}) = \boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})$, $\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}) = \boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})$ and $\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}) = \mathbf{y}_t - \boldsymbol{\mu}_t(\boldsymbol{\theta})$. FSC provide expressions for $c(\eta)$ and $g[\varsigma_t(\boldsymbol{\theta}), \eta]$ in the multivariate student t case, which are obviously such that $L_T(\boldsymbol{\theta}, 0)$ collapses to a conditionally Gaussian log-likelihood.

Let $\mathbf{s}_t(\boldsymbol{\phi})$ denote the score function $\partial l_t(\boldsymbol{\phi})/\partial\boldsymbol{\phi}$, and partition it into two blocks, $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi})$ and $s_{\boldsymbol{\eta}t}(\boldsymbol{\phi})$, whose dimensions conform to those of $\boldsymbol{\theta}$ and $\boldsymbol{\eta}$, respectively. Then, it is straightforward to show that if $\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$ has full rank, and $\boldsymbol{\mu}_t(\boldsymbol{\theta})$, $\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$, $c(\boldsymbol{\eta})$ and $g[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]$ are differentiable

$$\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}) = \frac{\partial d_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} + \frac{\partial g[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]}{\partial\varsigma}\frac{\partial\varsigma_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} = [\mathbf{Z}_{lt}(\boldsymbol{\theta}), \mathbf{Z}_{st}(\boldsymbol{\theta})]\left[\begin{array}{c} \mathbf{e}_{lt}(\boldsymbol{\phi}) \\ \mathbf{e}_{st}(\boldsymbol{\phi}) \end{array}\right] = \mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\phi}), \tag{2}$$

$$\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\phi}) = \partial c(\boldsymbol{\eta})/\partial\boldsymbol{\eta} + \partial g[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]/\partial\boldsymbol{\eta} = \mathbf{e}_{rt}(\boldsymbol{\phi}), \tag{3}$$

where

$$\begin{aligned} \partial d_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta} &= -\mathbf{Z}_{st}(\boldsymbol{\theta})vec(\mathbf{I}_N) \\ \partial\varsigma_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta} &= -2\{\mathbf{Z}_{lt}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}) + \mathbf{Z}_{st}(\boldsymbol{\theta})vec[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})]\}, \end{aligned} \tag{4}$$

$$\begin{aligned}
\mathbf{Z}_{lt}(\boldsymbol{\theta}) &= \partial\boldsymbol{\mu}_t'(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\cdot\boldsymbol{\Sigma}_t^{-1/2\prime}(\boldsymbol{\theta}),\\
\mathbf{Z}_{st}(\boldsymbol{\theta}) &= \frac{1}{2}\partial vec'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]/\partial\boldsymbol{\theta}\cdot[\boldsymbol{\Sigma}_t^{-1/2\prime}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1/2\prime}(\boldsymbol{\theta})],\\
\mathbf{e}_{lt}(\boldsymbol{\theta},\boldsymbol{\eta}) &= \delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}]\cdot\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}), &(5)\\
\mathbf{e}_{st}(\boldsymbol{\theta},\boldsymbol{\eta}) &= vec\left\{\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}]\cdot\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})-\mathbf{I}_N\right\}, &(6)\\
\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] &= -2\partial g[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}]/\partial\varsigma, &(7)
\end{aligned}$$

and $\partial\boldsymbol{\mu}_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta}'$ and $\partial vec\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]/\partial\boldsymbol{\theta}'$ depend on the particular specification adopted.[1]

Given that $\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}]$ is equal to $(N\eta+1)/[1-2\eta+\eta\varsigma_t(\boldsymbol{\theta})]$ in the student t case, and to 1 under Gaussianity, it is straightforward to check that $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\eta)$ coincides with the expression in FSC, while $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta},0)$ reduces to the multivariate normal expression in Bollerslev and Wooldridge (1992), in which case:

$$\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0})=\left[\begin{array}{c}\mathbf{e}_{lt}(\boldsymbol{\theta},\mathbf{0})\\ \mathbf{e}_{st}(\boldsymbol{\theta},\mathbf{0})\end{array}\right]=\left\{\begin{array}{c}\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\\ vec\left[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})-\mathbf{I}_N\right]\end{array}\right\}.$$

As for $\mathbf{e}_{rt}(\boldsymbol{\theta},\mathbf{0})$, FSC show that in the multivariate student t case it is proportional to the second generalised Laguerre polynomial:

$$e_{rt}(\boldsymbol{\theta},0)=\varsigma_t^2(\boldsymbol{\theta})/4-(N+2)\varsigma_t(\boldsymbol{\theta})/2+N(N+2)/4.$$

Let $\mathbf{h}_t(\boldsymbol{\phi})$ denote the Hessian function $\partial\mathbf{s}_t(\boldsymbol{\phi})/\partial\boldsymbol{\phi}'=\partial^2 l_t(\boldsymbol{\phi})/\partial\boldsymbol{\phi}\partial\boldsymbol{\phi}'$. Assuming twice differentiability of the different functions involved, we will have

$$\begin{aligned}
\mathbf{h}_{\boldsymbol{\theta\theta}t}(\boldsymbol{\phi}) &= \frac{\partial^2 d_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'}+\frac{\partial^2 g\left[\varsigma_t(\boldsymbol{\theta}),\eta\right]}{(\partial\varsigma)^2}\frac{\partial\varsigma_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\frac{\partial\varsigma_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}+\frac{\partial g\left[\varsigma_t(\boldsymbol{\theta}),\eta\right]}{\partial\varsigma}\frac{\partial^2\varsigma_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'} &(8)\\
\mathbf{h}_{\boldsymbol{\theta\eta}t}(\boldsymbol{\phi}) &= \partial\varsigma_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\cdot\partial^2 g\left[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}\right]/\partial\varsigma\partial\boldsymbol{\eta}', &(9)\\
\mathbf{h}_{\boldsymbol{\eta\eta}t}(\boldsymbol{\phi}) &= \partial^2 c(\boldsymbol{\eta})/\partial\boldsymbol{\eta}\partial\boldsymbol{\eta}'+\partial^2 g\left[\varsigma_t(\boldsymbol{\theta}),\eta\right]/\partial\boldsymbol{\eta}\partial\boldsymbol{\eta}',
\end{aligned}$$

where

$$\partial^2 d_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'=2\mathbf{Z}_{st}(\boldsymbol{\theta})\mathbf{Z}_{st}'(\boldsymbol{\theta})-\frac{1}{2}\left\{vec'\left[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\right]\otimes\mathbf{I}_p\right\}\partial vec\left\{\partial vec'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]/\partial\boldsymbol{\theta}\right\}/\partial\boldsymbol{\theta}',\quad(10)$$

$$\begin{aligned}
\partial^2\varsigma_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}' &= 2\mathbf{Z}_{lt}(\boldsymbol{\theta})\mathbf{Z}_{lt}'(\boldsymbol{\theta})+8\mathbf{Z}_{st}(\boldsymbol{\theta})[\mathbf{I}_N\otimes\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})]\mathbf{Z}_{st}'(\boldsymbol{\theta})+4\mathbf{Z}_{lt}(\boldsymbol{\theta})[\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})\otimes\mathbf{I}_N]\mathbf{Z}_{st}'(\boldsymbol{\theta})\\
&+4\mathbf{Z}_{st}(\boldsymbol{\theta})[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\otimes\mathbf{I}_N]\mathbf{Z}_{lt}'(\boldsymbol{\theta})-2[\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1/2\prime}(\boldsymbol{\theta})\otimes\mathbf{I}_p]\partial vec[\partial\boldsymbol{\mu}_t'(\boldsymbol{\theta})/\partial\boldsymbol{\theta}]\partial\boldsymbol{\theta}'\\
&-\{vec'[\boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1/2\prime}(\boldsymbol{\theta})]\otimes\mathbf{I}_p\}\partial vec\{\partial vec'[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})]/\partial\boldsymbol{\theta}\}/\partial\boldsymbol{\theta}',
\end{aligned}$$

and $\partial^2 g(\varsigma,\eta)/(\partial\varsigma)^2$, $\partial^2 g(\varsigma,\eta)/\partial\varsigma\partial\boldsymbol{\eta}'$ and $\partial g(\varsigma,\eta)/\partial\boldsymbol{\eta}\partial\boldsymbol{\eta}'$ depend on the specific distribution assumed for estimation purposes (see FSC for the multivariate student t).

[1] Note that while both $\mathbf{Z}_t(\boldsymbol{\theta})$ and $\mathbf{e}_{dt}(\boldsymbol{\phi})$ depend on the specific choice of square root matrix $\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})$, $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi})$ does not, a property that inherits from $l_t(\boldsymbol{\phi})$. The same result is not generally true for non-elliptical distributions (see Mencía and Sentana (2005)), in which case one should redefine $\mathbf{Z}_{st}(\boldsymbol{\theta})$ as $\{\partial vec'[\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})]/\partial\boldsymbol{\theta}\}[\mathbf{I}_N\otimes\boldsymbol{\Sigma}_t^{-1/2\prime}(\boldsymbol{\theta})]$, as in the proofs of Propositions 6, 13 and 17, or in Appendix B.2.

Given correct specification, the results in Crowder (1976) imply that $\mathbf{e}_t(\boldsymbol{\phi}) = [\mathbf{e}'_{dt}(\boldsymbol{\phi}), \mathbf{e}_{rt}(\boldsymbol{\phi})]'$ evaluated at $\boldsymbol{\phi}_0$ follows a vector martingale difference, and therefore, the same is true of the score vector $\mathbf{s}_t(\boldsymbol{\phi})$. His results also imply that, under suitable regularity conditions, which in particular require that $\boldsymbol{\phi}_0$ belongs to the interior of the parameter space, the asymptotic distribution of the feasible ML estimator will be $\sqrt{T}(\hat{\boldsymbol{\phi}}_T - \boldsymbol{\phi}_0) \to N\left[\mathbf{0}, \mathcal{I}^{-1}(\boldsymbol{\phi}_0)\right]$, where $\mathcal{I}(\boldsymbol{\phi}_0) = E[\mathcal{I}_t(\boldsymbol{\phi}_0)|\boldsymbol{\phi}_0]$,

$$\begin{aligned}
\mathcal{I}_t(\boldsymbol{\phi}) &= V\left[\mathbf{s}_t(\boldsymbol{\phi})|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}\right] = \mathbf{Z}_t(\boldsymbol{\theta})\mathcal{M}(\boldsymbol{\phi})\mathbf{Z}'_t(\boldsymbol{\theta}) = -E\left[\mathbf{h}_t(\boldsymbol{\phi})|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}\right], \\
\mathbf{Z}_t(\boldsymbol{\theta}) &= \begin{pmatrix} \mathbf{Z}_{dt}(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_q \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_{lt}(\boldsymbol{\theta}) & \mathbf{Z}_{st}(\boldsymbol{\theta}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_q \end{pmatrix},
\end{aligned}$$

and $\mathcal{M}(\boldsymbol{\phi}) = V\left[\mathbf{e}_t(\boldsymbol{\phi})|\boldsymbol{\phi}\right]$.

The following result generalises Propositions 3 in Lange, Little and Taylor (1989), 1 in FSC and 5.2 in Hafner and Rombouts (2007):

Proposition 1 *If* $\boldsymbol{\varepsilon}^*_t|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}$ *is i.i.d.* $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta})$ *with density* $\exp[c(\boldsymbol{\eta}) + g(\varsigma_t, \boldsymbol{\eta})]$, *then*

$$\mathcal{M}(\boldsymbol{\eta}) = \begin{pmatrix} \mathcal{M}_{ll}(\boldsymbol{\eta}) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{M}_{ss}(\boldsymbol{\eta}) & \mathcal{M}_{sr}(\boldsymbol{\eta}) \\ \mathbf{0} & \mathcal{M}'_{sr}(\boldsymbol{\eta}) & \mathcal{M}_{rr}(\boldsymbol{\eta}) \end{pmatrix}, \tag{11}$$

$$\mathcal{M}_{ll}(\boldsymbol{\eta}) = V[\mathbf{e}_{lt}(\boldsymbol{\phi})|\boldsymbol{\phi}] = \mathrm{M}_{ll}(\boldsymbol{\eta})\mathbf{I}_N, \tag{12}$$

$$\mathcal{M}_{ss}(\boldsymbol{\eta}) = V[\mathbf{e}_{st}(\boldsymbol{\phi})|\,\boldsymbol{\phi}] = \mathrm{M}_{ss}(\boldsymbol{\eta})\left(\mathbf{I}_{N^2} + \mathbf{K}_{NN}\right) + [\mathrm{M}_{ss}(\boldsymbol{\eta}) - 1]vec(\mathbf{I}_N)vec'(\mathbf{I}_N), \tag{13}$$

$$\mathcal{M}_{sr}(\boldsymbol{\eta}) = E[\mathbf{e}_{st}(\boldsymbol{\phi})\mathbf{e}'_{rt}(\boldsymbol{\phi})|\,\boldsymbol{\phi}] = -E\left\{\partial\mathbf{e}_{st}(\boldsymbol{\phi})/\partial\boldsymbol{\eta}'|\,\boldsymbol{\phi}\right\} = vec(\mathbf{I}_N)\mathrm{M}_{sr}(\boldsymbol{\eta}), \tag{14}$$

$$\mathcal{M}_{rr}(\boldsymbol{\eta}) = V[\,\mathbf{e}_{rt}(\boldsymbol{\phi})|\,\boldsymbol{\phi}] = -E[\partial\mathbf{e}_{rt}(\boldsymbol{\phi})/\partial\boldsymbol{\eta}'|\,\boldsymbol{\phi}],$$

$$\mathrm{M}_{ll}(\boldsymbol{\eta}) = E\left\{\delta^2[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]\frac{\varsigma_t(\boldsymbol{\theta})}{N}\right|\boldsymbol{\phi}\left.\right\} = E\left\{\frac{2\partial\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]}{\partial\varsigma}\frac{\varsigma_t(\boldsymbol{\theta})}{N} + \delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]\right|\boldsymbol{\phi}\left.\right\},$$

$$\mathrm{M}_{ss}(\boldsymbol{\eta}) = \frac{N}{N+2}\left[1 + V\left\{\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]\frac{\varsigma_t}{N}\right|\boldsymbol{\phi}\left.\right\}\right] = E\left\{\frac{2\partial\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]}{\partial\varsigma}\frac{\varsigma_t^2(\boldsymbol{\theta})}{N(N+2)}\right|\boldsymbol{\phi}\left.\right\} + 1,$$

$$\mathrm{M}_{sr}(\boldsymbol{\eta}) = E\left[\left\{\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]\frac{\varsigma_t(\boldsymbol{\theta})}{N} - 1\right\}\mathbf{e}'_{rt}(\boldsymbol{\phi})\right|\boldsymbol{\phi}\left.\right] = -E\left\{\frac{\varsigma_t(\boldsymbol{\theta})}{N}\frac{\partial\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]}{\partial\boldsymbol{\eta}'}\right|\boldsymbol{\phi}\left.\right\},$$

where $\mathbf{K}_{mn}$ *is the commutation matrix of orders* m *and* n.

In the multivariate standardised student t case, in particular:

$$\mathrm{M}_{ll}(\eta) = \frac{\nu(N+\nu)}{(\nu-2)(N+\nu+2)}, \; \mathrm{M}_{ss}(\eta) = \frac{(N+\nu)}{(N+\nu+2)}, \; \mathrm{M}_{sr}(\eta) = -\frac{2(N+2)\nu^2}{(\nu-2)(N+\nu)(N+\nu+2)},$$

$$\mathcal{M}_{rr}(\eta) = \frac{\nu^4}{4}\left[\psi'\left(\frac{\nu}{2}\right) - \psi'\left(\frac{N+\nu}{2}\right)\right] - \frac{N\nu^4\left[\nu^2 + N(\nu-4) - 8\right]}{2(\nu-2)^2(N+\nu)(N+\nu+2)},$$

where $\psi(.)$ is the di-gamma function (see Abramowitz and Stegun (1964)), which under normality reduce to 1, 1, 0 and $N(N+2)/2$, respectively. In this sense, it is interesting to note that as N increases, $\mathrm{M}_{ll}(\eta)$, $\mathrm{M}_{ss}(\eta)$ and $\mathrm{M}_{sr}(\eta)$ converge to $\nu/(\nu$-2), 1 and 0, respectively. This is due to the fact that the multivariate student t can be written as a scale mixture of normals, with a positive mixing variable that can be filtered out with increasing precision as $N \to \infty$ (see Mencía and Sentana (2005)). Thus, $l_t(\boldsymbol{\phi})$ will become arbitrarily close to the sum of the conditional log-likelihood of $\mathbf{y}_t$ given the mixing variable, which is multivariate Gaussian and only depends on $\boldsymbol{\theta}$, plus the marginal of the mixing variable, which only depends on η. Another point to note in relation to the student t is that $\mathrm{M}_{ll}(\eta)$ increases without bound as $\nu \to 2^+$ while $\mathrm{M}_{ss}(\eta)$ remains bounded. This differential behaviour is also characteristic of other leptokurtic elliptical distributions, such as the normal-gamma mixture, the Kotz distribution, or the Pearson type II.

2.3 Gaussian pseudo maximum likelihood estimators of $\boldsymbol{\theta}$

If the interest of the researcher lied exclusively in $\boldsymbol{\theta}$, which are the parameters characterising the conditional mean and variance functions, then one attractive possibility would be to estimate an equality restricted version of the model in which $\boldsymbol{\eta}$ is set to zero. Let $\tilde{\boldsymbol{\theta}}_T = \arg\max_{\boldsymbol{\theta}} L_T(\boldsymbol{\theta}, \mathbf{0})$ denote such a PML estimator of $\boldsymbol{\theta}$. As we mentioned in the introduction, $\tilde{\boldsymbol{\theta}}_T$ remains root-T consistent for $\boldsymbol{\theta}_0$ under correct specification of $\boldsymbol{\mu}_t(\boldsymbol{\theta})$ and $\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$ even though the conditional distribution of $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is not Gaussian, provided that it has bounded fourth moments. The proof is based on the fact that in those circumstances, the pseudo log-likelihood score, $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}, \mathbf{0})$, is a vector martingale difference sequence when evaluated at $\boldsymbol{\theta}_0$, a property that inherits from $\mathbf{e}_{dt}(\boldsymbol{\theta}, \mathbf{0})$. Importantly, this property is preserved even when the standardised innovations, $\boldsymbol{\varepsilon}_t^*$, are not stochastically independent of $\mathbf{z}_t$ and I_{t-1}. The asymptotic distribution of the PML estimator of $\boldsymbol{\theta}$ is stated in the following result:[2]

Proposition 2 *If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0)$ with $\kappa_0 < \infty$, and the regularity conditions A.1 in Bollerslev and Wooldridge (1992) are satisfied, then $\sqrt{T}(\tilde{\boldsymbol{\theta}}_T - \boldsymbol{\theta}_0) \to N[\mathbf{0}, \mathcal{C}(\boldsymbol{\phi}_0)]$, where*

$$\begin{aligned}
\mathcal{C}(\boldsymbol{\phi}) &= \mathcal{A}^{-1}(\boldsymbol{\phi})\mathcal{B}(\boldsymbol{\phi})\mathcal{A}^{-1}(\boldsymbol{\phi}),\\
\mathcal{A}(\boldsymbol{\phi}) &= -E\left[\mathbf{h}_{\boldsymbol{\theta\theta}t}(\boldsymbol{\theta}, \mathbf{0})|\boldsymbol{\phi}\right] = E\left[\mathcal{A}_t(\boldsymbol{\phi})|\boldsymbol{\phi}\right],\\
\mathcal{A}_t(\boldsymbol{\phi}) &= -E[\mathbf{h}_{\boldsymbol{\theta\theta}t}(\boldsymbol{\theta}; \mathbf{0})|\,\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}] = \mathbf{Z}_{dt}(\boldsymbol{\theta})\mathcal{K}(0)\mathbf{Z}'_{dt}(\boldsymbol{\theta}),\\
\mathcal{B}(\boldsymbol{\phi}) &= V\left[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}, \mathbf{0})|\boldsymbol{\phi}\right] = E\left[\mathcal{B}_t(\boldsymbol{\phi})|\boldsymbol{\phi}\right],\\
\mathcal{B}_t(\boldsymbol{\phi}) &= V[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}; \mathbf{0})|\,\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}] = \mathbf{Z}_{dt}(\boldsymbol{\theta})\mathcal{K}(\kappa)\mathbf{Z}'_{dt}(\boldsymbol{\theta}),
\end{aligned}$$

$$\textit{and } \mathcal{K}(\kappa) = V[\mathbf{e}_{dt}(\boldsymbol{\theta}, \mathbf{0})|\,\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}] = \begin{bmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & (\kappa+1)(\mathbf{I}_{N^2} + \mathbf{K}_{NN}) + \kappa vec(\mathbf{I}_N)vec'(\mathbf{I}_N) \end{bmatrix}, \tag{15}$$

which only depends on $\boldsymbol{\eta}$ through the population coefficient of multivariate excess kurtosis

$$\kappa = E(\varsigma_t^2|\boldsymbol{\eta})/[N(N+2)] - 1. \tag{16}$$

But if κ_0 is infinite then $\mathcal{B}(\boldsymbol{\phi}_0)$ will be unbounded, and the asymptotic distribution of some or all the elements of $\tilde{\boldsymbol{\theta}}_T$ will be non-standard, unlike that of $\hat{\boldsymbol{\theta}}_T$ (see Hall and Yao (2003)).

The following result, which specifies the covariance between the Gaussian pseudo score and the true score, will repeatedly prove useful below:

[2] Throughout this paper, we use the high level regularity conditions in Bollerslev and Wooldridge (1992) because we want to leave unspecified the conditional mean vector and covariance matrix in order to maintain full generality. Primitive conditions for specific multivariate models can be found for instance in Ling and McAleer (2003).

Proposition 3 *If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0, \boldsymbol{\varrho}_0$ is i.i.d. $(\mathbf{0}, \mathbf{I}_N)$ with density function $f(\boldsymbol{\varepsilon}_t^*; \boldsymbol{\varrho})$, where $\boldsymbol{\varrho}$ are some shape parameters and $\boldsymbol{\varrho} = \mathbf{0}$ denotes normality, then*

$$E\left\{\mathbf{e}_{dt}(\boldsymbol{\theta}, \mathbf{0})\left[\mathbf{e}'_{dt}(\boldsymbol{\theta}, \boldsymbol{\varrho}), \mathbf{e}'_{rt}(\boldsymbol{\theta}, \boldsymbol{\varrho})\right]\middle|\, \mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}, \boldsymbol{\varrho}\right\} = [\mathcal{K}(0)\,|\mathbf{0}]. \tag{17}$$

Note that (17) holds regardless of whether or not the conditional distribution of $\boldsymbol{\varepsilon}_t^*$ is spherical, provided we interpret $\mathbf{e}_{rt}(\boldsymbol{\phi})$ as the gradient with respect to the shape parameters $\boldsymbol{\varrho}$.

2.4 Sequential estimators of $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$

In practice, we will often be interested in features of the distribution of asset returns, such as its quantiles, which go beyond its conditional mean and variance. For that purpose, we can use $\tilde{\boldsymbol{\theta}}_T$ to obtain a sequential ML estimator of $\boldsymbol{\eta}$ as $\tilde{\boldsymbol{\eta}}_T = \arg\max_\eta L_T(\tilde{\boldsymbol{\theta}}_T, \boldsymbol{\eta})$, possibly subject to some inequality constraints on $\boldsymbol{\eta}$. In the student t case, for instance, $\tilde{\eta}_T$ will be characterised by the first-order Kuhn-Tucker (KT) conditions

$$\bar{s}_{\eta T}(\tilde{\boldsymbol{\theta}}_T, \tilde{\eta}_T) + \tilde{\lambda}_{\eta T} = 0; \quad \tilde{\eta}_T \geq 0; \quad \tilde{\lambda}_{\eta T} \geq 0; \quad \tilde{\lambda}_{\eta T} \cdot \tilde{\eta}_T = 0,$$

where $\bar{s}_{\eta T}(\boldsymbol{\theta}, \eta)$ is the sample mean of $s_{\eta t}(\boldsymbol{\theta}, \eta)$, and λ_η the KT multiplier associated with the constraint $\eta \geq 0$.

Such a sequential ML estimator of $\boldsymbol{\eta}$ can be given a rather intuitive interpretation. If $\boldsymbol{\theta}_0$ were known, then the squared Euclidean norm of the standardised innovations, $\varsigma_t(\boldsymbol{\theta}_0)$, would be *i.i.d.* over time, with density function $h(\varsigma; \boldsymbol{\eta})$.[3] Therefore, we could obtain the infeasible ML estimator of $\boldsymbol{\eta}$ by maximising with respect to $\boldsymbol{\eta}$ the log-likelihood function of the observed $\varsigma_t(\boldsymbol{\theta}_0)'s$, $\sum_{t=1}^T \ln h\left[\varsigma_t(\boldsymbol{\theta}_0); \boldsymbol{\eta}\right]$. Although in practice the standardised residuals are usually unobservable, it turns out that $\tilde{\boldsymbol{\eta}}_T$ is the estimator so obtained when we treat $\varsigma_t(\tilde{\boldsymbol{\theta}}_T)$ as if they were really observed.

The asymptotic distribution of the sequential ML estimator of $\boldsymbol{\eta}$, which reflects the sample uncertainty in $\tilde{\boldsymbol{\theta}}_T$, is stated in the following result:

Proposition 4 *If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0)$ with $\kappa_0 < \infty$, and the regularity conditions A.1 in Bollerslev and Wooldridge (1992) are satisfied, then $\sqrt{T}(\tilde{\boldsymbol{\eta}}_T - \boldsymbol{\eta}_0) \to N\left[0, \mathcal{F}(\boldsymbol{\phi}_0)\right]$, where*

$$\mathcal{F}(\boldsymbol{\phi}_0) = \mathcal{I}_{\boldsymbol{\eta\eta}}^{-1}(\boldsymbol{\phi}_0) + \mathcal{I}_{\boldsymbol{\eta\eta}}^{-1}(\boldsymbol{\phi}_0)\mathcal{I}'_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0)\mathcal{C}(\boldsymbol{\phi}_0)\mathcal{I}_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0)\mathcal{I}_{\boldsymbol{\eta\eta}}^{-1}(\boldsymbol{\phi}_0).$$

Importantly, since $\mathcal{C}(\boldsymbol{\phi}_0)$ will become unbounded as $\kappa_0 \to \infty$, the asymptotic distribution of $\tilde{\boldsymbol{\eta}}_T$ will also be non-standard in that case, unlike that of the feasible ML estimator $\hat{\boldsymbol{\eta}}_T$.

If we can obtain closed-form expressions for at least q functions of ς_t, $\boldsymbol{\upsilon}(.)$ say, then we can also compute a sequential method of moments (MM) estimator of $\boldsymbol{\eta}$, $\breve{\boldsymbol{\eta}}_T(\boldsymbol{\Omega})$ say, by minimising

[3] For instance, when $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is *i.i.d.* $t(\mathbf{0}, \mathbf{I}_N, \nu_0)$, the distribution of ς_t will be that of either an F variate with N and ν_0 degrees of freedom multiplied by $N(\nu_0 - 2)/\nu_0$ if $\nu_0 < \infty$, or a chi-square random variable with N degrees of freedom under Gaussianity (see e.g. Lemma 1 in FSC).

with respect to $\boldsymbol{\eta}$ the quadratic form $\bar{\mathbf{n}}'_{\boldsymbol{\eta}T}(\tilde{\boldsymbol{\theta}}_T,\boldsymbol{\eta})\boldsymbol{\Omega}\bar{\mathbf{n}}_{\boldsymbol{\eta}T}(\tilde{\boldsymbol{\theta}}_T,\boldsymbol{\eta})$, where $\boldsymbol{\Omega}$ is a positive definite weighting matrix, and $\mathbf{n}_{\boldsymbol{\eta}t}(\boldsymbol{\theta},\boldsymbol{\eta}) = \boldsymbol{\upsilon}[\varsigma_t(\boldsymbol{\theta})] - E\{\boldsymbol{\upsilon}[\varsigma_t(\boldsymbol{\theta})]|\boldsymbol{\phi}\}$. Given that $E[\varsigma_t(\boldsymbol{\theta})|\boldsymbol{\phi}] = N$, the most obvious moment to use is (16), which suffices to identify η in the multivariate student t case through the theoretical relationship $\kappa = 2/(\nu-4)$ (see FSC). In this context, if we define the influence function

$$n_{\eta t}(\boldsymbol{\theta},\eta) = \frac{\varsigma_t^2(\boldsymbol{\theta})}{N(N+2)} - \frac{1-2\eta}{1-4\eta},$$

we obtain

$$\breve{\eta}_T = \frac{\max[0,\bar{\kappa}_T(\tilde{\boldsymbol{\theta}}_T)]}{4\max[0,\bar{\kappa}_T(\tilde{\boldsymbol{\theta}}_T)]+2}, \tag{18}$$

where

$$\bar{\kappa}_T(\tilde{\boldsymbol{\theta}}_T) = \frac{T^{-1}\sum_{t=1}^T \varsigma_t^2(\tilde{\boldsymbol{\theta}}_T)}{N(N+2)} - 1$$

is Mardia's (1970) sample coefficient of multivariate excess kurtosis of the estimated standardised residuals. We can obtain a closely related estimator, $\mathring{\eta}_T$ say, from the modified influence function

$$\mathring{n}_{\eta t}(\boldsymbol{\theta},\eta) = \frac{\varsigma_t^2(\boldsymbol{\theta})}{N(N+2)} - \frac{2(1-2\eta)\varsigma_t(\boldsymbol{\theta})}{N(1-6\eta)} + \frac{(1-2\eta)^2}{(1-4\eta)(1-6\eta)},$$

which is the relevant second-order orthogonal polynomial when ς_t is proportional to an $F_{N,\nu}$ random variable. The asymptotic distributions of these two sequential MM estimators of η are stated in the following result:

Proposition 5 *If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1},\boldsymbol{\phi}_0$ is i.i.d. $t(\mathbf{0},\mathbf{I}_N,\nu_0)$, with $\nu_0 > 8$, then under the regularity conditions A.1 in Bollerslev and Wooldridge (1992) we have that $\sqrt{T}(\breve{\eta}_T-\eta_0)\to N[0,\mathcal{G}(\boldsymbol{\phi}_0)]$ and $\sqrt{T}(\mathring{\eta}_T-\eta_0)\to N[0,\mathcal{J}(\boldsymbol{\phi}_0)]$, where*

$$\begin{aligned}
\mathcal{G}(\boldsymbol{\phi}_0) &= [\mathcal{E}(\boldsymbol{\phi}_0) + \mathcal{R}'(\boldsymbol{\phi}_0)\mathcal{C}(\boldsymbol{\phi}_0)\mathcal{R}(\boldsymbol{\phi}_0) - 2\mathcal{R}'(\boldsymbol{\phi}_0)\mathcal{A}^{-1}(\boldsymbol{\phi}_0)\mathcal{D}(\boldsymbol{\phi}_0)]/\mathcal{N}^2(\boldsymbol{\phi}_0),\\
\mathcal{J}(\boldsymbol{\phi}_0) &= [\mathcal{L}(\boldsymbol{\phi}_0) + \mathcal{Q}'(\boldsymbol{\phi}_0)\mathcal{C}(\boldsymbol{\phi}_0)\mathcal{Q}(\boldsymbol{\phi}_0)]/\mathcal{N}^2(\boldsymbol{\phi}_0),\\
\mathcal{D}(\boldsymbol{\phi}_0) &= cov[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,0), n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\boldsymbol{\phi}_0] = \frac{4(\nu_0-2)(N+\nu_0-2)}{N(\nu_0-4)(\nu_0-6)}\mathbf{W}_s(\boldsymbol{\phi}_0),\\
\mathcal{E}(\boldsymbol{\phi}_0) &= V[n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\boldsymbol{\phi}_0] = \frac{(\nu_0-2)^2}{(\nu_0-4)^2}\left[\frac{(N+6)(N+4)}{N(N+2)}\frac{(\nu_0-2)(\nu_0-4)}{(\nu_0-6)(\nu_0-8)} - 1\right],\\
\mathcal{L}(\boldsymbol{\phi}_0) &= V[\mathring{n}_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\boldsymbol{\phi}_0] = \mathcal{E}(\boldsymbol{\phi}_0) - \frac{8(\nu_0-2)^2(N+\nu_0-2)}{N(\nu_0-6)^2(\nu_0-4)},\\
\mathcal{R}(\boldsymbol{\phi}_0) &= cov[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,\eta_0), n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\boldsymbol{\phi}_0] = \frac{4(\nu_0-2)}{N(\nu_0-4)}\mathbf{W}_s(\boldsymbol{\phi}_0),\\
\mathcal{Q}(\boldsymbol{\phi}_0) &= cov[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,\eta_0), \mathring{n}_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\boldsymbol{\phi}_0] = -\frac{8(\nu_0-2)}{N(\nu_0-4)(\nu_0-6)}\mathbf{W}_s(\boldsymbol{\phi}_0),\\
\mathcal{N}(\boldsymbol{\phi}_0) &= cov[s_{\eta t}(\boldsymbol{\theta}_0,\eta_0), n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\boldsymbol{\phi}_0] = \frac{2\nu_0^2}{(\nu_0-4)^2},
\end{aligned}$$

and

$$\mathbf{W}_s(\boldsymbol{\phi}_0) = \mathbf{Z}_d(\boldsymbol{\phi}_0)[\mathbf{0}', vec'(\mathbf{I}_N)]' = E[\mathbf{Z}_{dt}(\boldsymbol{\theta}_0)|\boldsymbol{\phi}_0][\mathbf{0}', vec'(\mathbf{I}_N)]'$$
$$= E\left\{\frac{1}{2}\partial vec'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta}_0)\right]/\partial\boldsymbol{\theta}\cdot vec[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta}_0)]\right|\boldsymbol{\phi}_0\right\} = E[\mathbf{W}_{st}(\boldsymbol{\theta}_0)|\boldsymbol{\phi}_0] = -E\left\{\partial d_t(\boldsymbol{\theta})/\partial\boldsymbol{\theta}|\,\boldsymbol{\phi}_0\right\}. \tag{19}$$

Note that since both $\mathcal{G}(\boldsymbol{\phi}_0)$ and $\mathcal{J}(\boldsymbol{\phi}_0)$ will diverge to infinity as ν_0 converges to 8 from above, $\breve{\eta}_T$ and $\mathring{\eta}_T$ will not be root-T consistent for $4 \leq \nu_0 \leq 8$. Moreover, since κ is infinite for $2 < \nu_0 \leq 4$, $\breve{\eta}_T$ and $\mathring{\eta}_T$ will not even be consistent in the interior of this range.

More generally, we could consider the higher order moment parameters of spherical random variables introduced by Berkane and Bentler (1986), $\tau_k(\boldsymbol{\eta})$, which Maruyama and Seo (2003) relate to the higher order moments of ς_t as $E(\varsigma_t^k|\boldsymbol{\eta}) = [\tau_k(\boldsymbol{\eta}) + 1]E(\varsigma_t^k|\mathbf{0})$, where

$$E(\varsigma_t^k|\mathbf{0}) = 2^k(N/2)(1 + N/2)\cdots(k-2+N/2)(k-1+N/2),$$

whence we can also obtain the higher-order orthogonal polynomials of ς_t.[4] By using these additional moments, we can in principle improve the efficiency of the sequential MM estimators, although the precision with which we can estimate $\tau_k(\boldsymbol{\eta})$ rapidly decreases with k (see Newey and Powell (1998) for a characterisation of efficient sequential estimators).

Finally, if we were to iterate the sequential ML procedure, and achieved convergence, then we would obtain fully efficient ML estimators of all model parameters. In fact, a single scoring iteration without line searches that started from $\tilde{\boldsymbol{\theta}}_T$ and $\tilde{\boldsymbol{\eta}}_T$ (or any other root-T consistent estimators) would suffice to yield an estimator of $\boldsymbol{\phi}$ that would be asymptotically equivalent to the full-information ML estimator $\hat{\boldsymbol{\phi}}_T$, at least up to terms of order $O_p(T^{-1/2})$. Specifically,

$$\begin{pmatrix} \ddot{\boldsymbol{\theta}}_T - \tilde{\boldsymbol{\theta}}_T \\ \ddot{\boldsymbol{\eta}}_T - \tilde{\boldsymbol{\eta}}_T \end{pmatrix} = \begin{bmatrix} \mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) & \mathcal{I}_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0) \\ \mathcal{I}'_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0) & \mathcal{I}_{\boldsymbol{\eta\eta}}(\boldsymbol{\phi}_0) \end{bmatrix}^{-1} \frac{1}{T}\sum_{t=1}^{T} \begin{bmatrix} \mathbf{s}_{\boldsymbol{\theta}t}(\tilde{\boldsymbol{\theta}}_T, \tilde{\boldsymbol{\eta}}_T) \\ \mathbf{s}_{\boldsymbol{\eta}t}(\tilde{\boldsymbol{\theta}}_T, \tilde{\boldsymbol{\eta}}_T) \end{bmatrix}.$$

If we use the partitioned inverse formula, then it is easy to see that

$$\ddot{\boldsymbol{\theta}}_T - \tilde{\boldsymbol{\theta}}_T = \left[\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) - \mathcal{I}_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0)\mathcal{I}^{-1}_{\boldsymbol{\eta\eta}}(\boldsymbol{\phi}_0)\mathcal{I}'_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0)\right]^{-1}$$
$$\times\frac{1}{T}\sum_{t=1}^{T}\left[\mathbf{s}_{\boldsymbol{\theta}t}(\tilde{\boldsymbol{\theta}}_T, \tilde{\boldsymbol{\eta}}_T) - \mathcal{I}_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_0)\mathcal{I}^{-1}_{\boldsymbol{\eta\eta}}(\boldsymbol{\phi}_0)\mathbf{s}_{\boldsymbol{\theta}t}(\tilde{\boldsymbol{\theta}}_T, \tilde{\boldsymbol{\eta}}_T)\right] = \mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\frac{1}{T}\sum_{t=1}^{T}\mathbf{s}_{\boldsymbol{\theta}|\boldsymbol{\eta}t}(\tilde{\boldsymbol{\theta}}_T, \tilde{\boldsymbol{\eta}}_T),$$

where

$$\mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) = \left[\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) - \mathcal{I}_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0)\mathcal{I}^{-1}_{\boldsymbol{\eta\eta}}(\boldsymbol{\phi}_0)\mathcal{I}'_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0)\right]^{-1},$$

and

$$\begin{aligned} \mathbf{s}_{\boldsymbol{\theta}|\eta t}(\boldsymbol{\theta}_0, \boldsymbol{\eta}_0) &= \mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0, \boldsymbol{\eta}_0) - \mathcal{I}_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0)\mathcal{I}^{-1}_{\boldsymbol{\eta\eta}}(\boldsymbol{\phi}_0)\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\theta}_0, \boldsymbol{\eta}_0) \\ &= \mathbf{Z}_{dt}(\boldsymbol{\theta}_0)\mathbf{e}_{dt}(\boldsymbol{\phi}_0) - \mathbf{W}_s(\boldsymbol{\phi}_0) \cdot \left[\mathrm{M}_{sr}(\boldsymbol{\eta}_0)\mathcal{M}^{-1}_{rr}(\boldsymbol{\eta}_0)\mathbf{e}_{rt}(\boldsymbol{\phi}_0)\right] \end{aligned} \quad (20)$$

is the residual from the unconditional theoretical regression of the score corresponding to $\boldsymbol{\theta}$, $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}_0)$, on the score corresponding to $\boldsymbol{\eta}$, $\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\phi}_0)$. The residual score $\mathbf{s}_{\boldsymbol{\theta}|\boldsymbol{\eta}t}(\boldsymbol{\theta}_0, \boldsymbol{\eta}_0)$ is sometimes

[4] In the standardised multivariate student t, for instance,
$\tau_k(\boldsymbol{\eta}) + 1 = (1-2\eta)^{k-1}/\{(1-2k\eta)[1-2(k-1)\eta]\cdots\ (1-4\eta)\}$ for $2 \leq k < \nu/2$.

called the parametric efficient score of $\boldsymbol{\theta}$, and its variance,

$$\begin{aligned}\mathcal{P}(\boldsymbol{\phi}_0) &= \mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) - \mathcal{I}_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0)\mathcal{I}^{-1}_{\boldsymbol{\eta\eta}}(\boldsymbol{\phi}_0)\mathcal{I}'_{\boldsymbol{\theta\eta}}(\boldsymbol{\phi}_0) \\ &= \mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) - \mathbf{W}_s(\boldsymbol{\phi}_0)\mathbf{W}'_s(\boldsymbol{\phi}_0) \cdot \left[\mathrm{M}_{sr}(\boldsymbol{\eta}_0)\mathcal{M}^{-1}_{rr}(\boldsymbol{\eta}_0)\mathrm{M}'_{sr}(\boldsymbol{\eta}_0)\right],\end{aligned}$$

the marginal information matrix of $\boldsymbol{\theta}$, or the feasible parametric efficiency bound. In this respect, note that $\mathcal{I}^{\theta\theta}(\boldsymbol{\phi}_0)$, which is the inverse of $\mathcal{P}(\boldsymbol{\phi}_0)$, coincides with the first block of $\mathcal{I}^{-1}(\boldsymbol{\phi}_0)$, and therefore it gives us the asymptotic variance of the feasible ML estimator, $\hat{\boldsymbol{\theta}}_T$.

2.5 Semiparametric estimators of $\boldsymbol{\theta}$

It is worth noting that the last summand of (20) coincides with $\mathbf{Z}_d(\boldsymbol{\phi}_0)$ times the theoretical least squares projection of $\mathbf{e}_{dt}(\boldsymbol{\phi}_0)$ on (the linear span of) $\mathbf{e}_{rt}(\boldsymbol{\phi}_0)$, which is conditionally orthogonal to $\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \mathbf{0})$ from Proposition 3. Such an interpretation immediately suggests alternative estimators of $\boldsymbol{\theta}$ that replace our parametric assumption on the shape of the distribution of the standardised innovations $\boldsymbol{\varepsilon}^*_t$ by nonparametric or semiparametric alternatives. In this section, we shall consider two such estimators.

The first one is fully nonparametric, and therefore replaces the linear span of $\mathbf{e}_{rt}(\boldsymbol{\phi}_0)$ by the so-called unrestricted tangent set, which is the Hilbert space generated by all the time-invariant functions of $\boldsymbol{\varepsilon}^*_t$ with bounded second moments that have zero conditional means and are conditionally orthogonal to $\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \mathbf{0})$. The following proposition, which generalises the univariate results of Gonzalez-Rivera and Drost (1999) and Propositions 3 and 4 in Hafner and Rombouts (2007) to multivariate models in which the conditional mean vector is not identically zero, describes the resulting semiparametric efficient score and the corresponding efficiency bound:

Proposition 6 *If $\boldsymbol{\varepsilon}^*_t|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0, \boldsymbol{\varrho}_0$ is i.i.d. $(\mathbf{0}, \mathbf{I}_N)$ with density function $f(\boldsymbol{\varepsilon}^*_t; \boldsymbol{\varrho})$, where $\boldsymbol{\varrho}$ are some shape parameters and $\boldsymbol{\varrho} = \mathbf{0}$ denotes normality, such that both its Fisher information matrix for location and scale*

$$\begin{aligned}\mathcal{M}_{dd}(\boldsymbol{\varrho}) &= V\left[\mathbf{e}_{dt}(\boldsymbol{\theta}, \boldsymbol{\varrho})|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}, \boldsymbol{\varrho}\right] \\ &= V\left\{\left[\begin{array}{c}\mathbf{e}_{lt}(\boldsymbol{\theta}, \boldsymbol{\varrho}) \\ \mathbf{e}_{st}(\boldsymbol{\theta}, \boldsymbol{\varrho})\end{array}\right]\middle|\boldsymbol{\theta}, \boldsymbol{\varrho}\right\} = V\left\{\left[\begin{array}{c}-\partial \ln f[\boldsymbol{\varepsilon}^*_t(\boldsymbol{\theta}); \boldsymbol{\varrho}]/\partial \boldsymbol{\varepsilon}^* \\ -vec\left\{\mathbf{I}_N + \partial \ln f[\boldsymbol{\varepsilon}^*_t(\boldsymbol{\theta}); \boldsymbol{\varrho}]/\partial \boldsymbol{\varepsilon}^* \cdot \boldsymbol{\varepsilon}^{*\prime}_t(\boldsymbol{\theta})\right\}\end{array}\right]\middle|\boldsymbol{\theta}, \boldsymbol{\varrho}\right\}\end{aligned}$$

and the matrix of third and fourth order central moments

$$\mathcal{K}(\boldsymbol{\varrho}) = V[\mathbf{e}_{dt}(\boldsymbol{\theta}, \mathbf{0})|\, \mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}, \boldsymbol{\varrho}] \qquad (21)$$

are bounded, then the semiparametric efficient score will be given by:

$$\mathbf{Z}_{dt}(\boldsymbol{\theta}_0, \boldsymbol{\varrho}_0)\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \boldsymbol{\varrho}_0) - \mathbf{Z}_d(\boldsymbol{\theta}_0, \boldsymbol{\varrho}_0)\left[\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \boldsymbol{\varrho}_0) - \mathcal{K}(0)\,\mathcal{K}^+(\boldsymbol{\varrho}_0)\,\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \mathbf{0})\right], \qquad (22)$$

while the semiparametric efficiency bound is

$$\mathcal{S}(\boldsymbol{\phi}_0) = \mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\theta}_0, \boldsymbol{\varrho}_0) - \mathbf{Z}_d(\boldsymbol{\theta}_0, \boldsymbol{\varrho}_0)\left[\mathcal{M}_{dd}(\boldsymbol{\varrho}_0) - \mathcal{K}(0)\,\mathcal{K}^+(\boldsymbol{\varrho}_0)\,\mathcal{K}(0)\right]\mathbf{Z}'_d(\boldsymbol{\theta}_0, \boldsymbol{\varrho}_0), \qquad (23)$$

where + denotes Moore-Penrose inverses, and $\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\theta}, \boldsymbol{\varrho}) = E\left[\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathcal{M}_{dd}(\boldsymbol{\varrho})\mathbf{Z}'_{dt}(\boldsymbol{\theta})|\boldsymbol{\theta}, \boldsymbol{\varrho}\right]$.

In practice, however, $f(\boldsymbol{\varepsilon}_t^*;\boldsymbol{\varrho})$ has to be replaced by a nonparametric estimator, which suffers from the curse of dimensionality. For this reason, Hodgson and Vorkink (2001), Hafner and Rombouts (2007) and other authors have suggested to limit the admissible distributions to the class of spherically symmetric ones. As a consequence, the restricted tangent set in this case becomes the Hilbert space generated by all time-invariant functions of $\varsigma_t(\boldsymbol{\theta}_0)$ with bounded second moments that have zero conditional means and are conditionally orthogonal to $\mathbf{e}_{dt}(\boldsymbol{\theta}_0,\mathbf{0})$. The following proposition, which corrects and extends Proposition 9 in Hafner and Rombouts (2007), provides the resulting elliptically symmetric semiparametric efficient score and the corresponding efficiency bound:

Proposition 7 *When $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1},\boldsymbol{\phi}_0$ is i.i.d. $s(\mathbf{0},\mathbf{I}_N,\boldsymbol{\eta}_0)$ with $-2/(N+2)<\kappa_0<\infty$, the elliptically symmetric semiparametric efficient score is given by:*

$$\mathring{\mathbf{s}}_{\theta t}(\boldsymbol{\phi}_0)=\mathbf{Z}_{dt}(\boldsymbol{\theta}_0)\mathbf{e}_{dt}(\boldsymbol{\phi}_0)-\mathbf{W}_s(\boldsymbol{\phi}_0)\left\{\left[\delta[\varsigma_t(\boldsymbol{\theta}_0),\boldsymbol{\eta}_0]\frac{\varsigma_t(\boldsymbol{\theta}_0)}{N}-1\right]-\frac{2}{(N+2)\kappa_0+2}\left[\frac{\varsigma_t(\boldsymbol{\theta}_0)}{N}-1\right]\right\}, \quad (24)$$

while the elliptically symmetric semiparametric efficiency bound is

$$\mathring{\mathcal{S}}(\boldsymbol{\phi}_0)=\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)-\mathbf{W}_s(\boldsymbol{\phi}_0)\mathbf{W}_s'(\boldsymbol{\phi}_0)\cdot\left\{\left[\frac{N+2}{N}\mathrm{M}_{ss}(\boldsymbol{\eta}_0)-1\right]-\frac{4}{N\left[(N+2)\kappa_0+2\right]}\right\}. \quad (25)$$

Once again, $\mathbf{e}_{dt}(\boldsymbol{\phi})$ has to be replaced in practice by a semiparametric estimate obtained from the joint density of $\boldsymbol{\varepsilon}_t^*$. However, the elliptical symmetry assumption allows us to obtain such an estimate from a nonparametric estimate of the univariate density of ς_t, $h(\varsigma_t;\boldsymbol{\eta})$, avoiding in this way the curse of dimensionality.

3 The relative efficiency of the different estimators of $\boldsymbol{\theta}$

3.1 General ranking and full efficiency conditions

In the previous section we have effectively considered five different estimators of $\boldsymbol{\theta}$: (1) the infeasible ML estimator, whose computation requires knowledge of $\boldsymbol{\eta}_0$; (2) the feasible ML estimator, which simultaneously estimates $\boldsymbol{\eta}$; (3) the elliptically symmetric semiparametric estimator, which restricts $\boldsymbol{\varepsilon}_t^*$ to have an *i.i.d.* $s(\mathbf{0},\mathbf{I}_N,\boldsymbol{\eta})$ conditional distribution, but does not impose any additional structure on the distribution of ς_t; (4) the unrestricted semiparametric estimator, which only assumes that the conditional distribution of $\boldsymbol{\varepsilon}_t^*$ is *i.i.d.*$(\mathbf{0},\mathbf{I}_N)$; and (5) the Gaussian PML estimator, which imposes $\boldsymbol{\eta}=\mathbf{0}$ even though the true conditional distribution of $\boldsymbol{\varepsilon}_t^*$ may not be normal. The following proposition ranks (in the usual positive semidefinite sense) the "information matrices" of those five estimators:

Proposition 8 *If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1};\boldsymbol{\phi}_0$ is i.i.d. $s(\mathbf{0},\mathbf{I}_N,\boldsymbol{\eta}_0)$ with $\kappa_0<\infty$, then*

$$\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\geq\mathcal{P}(\boldsymbol{\phi}_0)\geq\mathring{\mathcal{S}}(\boldsymbol{\phi}_0)\geq\mathcal{S}(\boldsymbol{\phi}_0)\geq\mathcal{C}^{-1}(\boldsymbol{\phi}_0).$$

In general, the above matrix inequalities are strict, at least in part. However, there is one instance in which all the above inequalities become equalities: when the true conditional distribution is Gaussian. In that case, the PML estimator is obviously fully efficient, which implies that all the other estimators of $\boldsymbol{\theta}$ must also be efficient. Moreover, normality is the only such instance within the spherical family:

Proposition 9 *1. If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $N(\mathbf{0}, \mathbf{I}_N)$, then*

$$\mathcal{I}_t(\boldsymbol{\theta}_0, \mathbf{0}) = V[\mathbf{s}_t(\boldsymbol{\theta}_0, \mathbf{0})|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0, \mathbf{0}] = \begin{bmatrix} V[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0, \mathbf{0})|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0, \mathbf{0}] & \mathbf{0} \\ \mathbf{0}' & \mathcal{M}_{rr}(\mathbf{0}) \end{bmatrix}$$

where

$$V[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0, \mathbf{0})|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0, \mathbf{0}] = -E[\mathbf{h}_{\boldsymbol{\theta}\boldsymbol{\theta}t}(\boldsymbol{\theta}_0, \mathbf{0})|\mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0, \mathbf{0}] = \mathcal{A}_t(\boldsymbol{\theta}_0, \mathbf{0}) = \mathcal{B}_t(\boldsymbol{\theta}_0, \mathbf{0}).$$

2. If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0)$ with $-2/(N+2) < \kappa_0 < \infty$, and $\mathbf{W}_s(\boldsymbol{\phi}_0) \neq \mathbf{0}$, then $\mathring{\mathcal{S}}(\boldsymbol{\phi}_0) = \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_0)$ only if $\varsigma_t|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. Gamma with mean N and variance $N[(N+2)\kappa_0 + 2]$.

3. If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0)$ with $\kappa_0 < \infty$, and $\mathbf{Z}_l(\boldsymbol{\phi}_0) \neq \mathbf{0}$, then $\mathcal{S}(\boldsymbol{\phi}_0) = \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_0)$ only if $\boldsymbol{\eta}_0 = \mathbf{0}$.

The first part of this proposition, which generalises Proposition 2 in FSC, implies that as far as $\boldsymbol{\theta}$ is concerned, there is no asymptotic efficiency loss in estimating $\boldsymbol{\eta}$ when $\boldsymbol{\eta}_0 = \mathbf{0}$.[5] The second part, which generalises the results in Gonzalez-Rivera (1997), implies that the SSP estimator can be fully efficient only if $\boldsymbol{\varepsilon}_t^*$ has a conditional Kotz distribution (see Kotz (1975)), which is a sufficient but not necessary condition for $\text{M}_{sr}(\boldsymbol{\eta}_0) = \mathbf{0}$, which in turn implies $\mathcal{P}(\boldsymbol{\phi}_0) = \mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_0)$. Finally, the last part of Proposition 9 generalises Result 2 in Drost and Gonzalez-Rivera (1999) and Proposition 6 in Hafner and Rombouts (2007).

Unfortunately, it is virtually impossible to obtain closed-form expressions for the different efficiency bounds in dynamic conditionally heteroskedastic non-Gaussian models, as one has to resort to Monte Carlo integration methods to compute the expected values of $\mathbf{Z}_{dt}(\boldsymbol{\theta})$ or $\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathcal{K}(\kappa)\mathbf{Z}'_{dt}(\boldsymbol{\theta})$ (see e.g. Engle and Gonzalez-Rivera (1991) and Gonzalez-Rivera and Drost (1999)). In the next subsection, though, we shall obtain closed-form expressions in two situations of practical interest.

3.2 Examples

Univariate conditionally heteroskedastic autoregressive models:

Consider the following univariate, covariance stationary AR(h)-ARCH(q) model:

[5] In the multivariate student t case, in fact, the feasible ML estimator of $\boldsymbol{\theta}$ will be numerically identical to the PML estimator approximately half the time in large samples because $\eta = 0$ lies at the boundary of the admissible parameter space (see e.g. Andrews (1999)).

$$
\left.\begin{array}{c}
y_t = \mu_t(\pi_0, \boldsymbol{\rho}_0) + \sigma_t(\boldsymbol{\theta}_0)\varepsilon_t^*, \\
\mu_t(\pi, \boldsymbol{\rho}) = \pi(1 - \sum_{j=1}^{h} \rho_j) + \sum_{j=1}^{h} \rho_j y_{t-j}, \\
\sigma_t^2(\boldsymbol{\theta}) = \gamma(1 - \sum_{j=1}^{q} \alpha_j) + \sum_{j=1}^{q} \alpha_j [y_{t-j} - \mu_{t-j}(\pi, \boldsymbol{\rho})]^2, \\
\varepsilon_t^* | \mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0, \boldsymbol{\eta}_0 \sim i.i.d.\ s(0, 1, \boldsymbol{\eta}_0).
\end{array}\right\} \quad (26)
$$

Define $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_h)'$ and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_q)'$, so that $\boldsymbol{\theta} = (\pi, \boldsymbol{\rho}', \gamma, \boldsymbol{\alpha}')'$. We can establish the following result:

Proposition 10 *If in model (26) $\boldsymbol{\alpha}_0 = \mathbf{0}$, and all the roots of $1 - \sum_{j=1}^{h} \rho_{j0} L^j = 0$ are outside the unit circle, then the feasible ML estimators of π, $\boldsymbol{\rho}$ and $\boldsymbol{\alpha}$ are as efficient as the infeasible ML estimators, which require knowledge of $\boldsymbol{\eta}_0$. If in addition $\kappa_0 < \infty$, then the elliptically symmetric semiparametric estimators of π, $\boldsymbol{\rho}$ and $\boldsymbol{\alpha}$ are also fully efficient. The same is true of the semiparametric estimators of $\boldsymbol{\rho}$ and $\boldsymbol{\alpha}$, but not of π. In contrast, the inefficiency ratio of the Gaussian PML estimators is $\text{M}_{ll}^{-1}(\boldsymbol{\eta}_0)$ for π and $\boldsymbol{\rho}$, and $4/\{[3\text{M}_{ss}(\boldsymbol{\eta}_0) - 1](3\kappa_0 + 2)\}$ for $\boldsymbol{\alpha}$.*

Not surprisingly, we can also show that these inefficiency ratios coincide with the ratios of the non-centrality parameters of the corresponding tests of conditional homoskedasticity against local alternatives of the form $\boldsymbol{\alpha}_{0T} = \boldsymbol{\alpha}_0/\sqrt{T}$ in model (26) (see Linton and Steigerwald (2000)).

Multivariate conditionally heteroskedastic autoregressive models:

Consider a single factor version of the conditionally heteroskedastic factor model in Sentana and Fiorentini (2001) augmented with covariance stationary diagonal VAR(1) dynamics:

$$
\left.\begin{array}{c}
\mathbf{y}_t = \boldsymbol{\mu}_t(\boldsymbol{\pi}_0, \boldsymbol{\rho}_0) + \boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta}_0)\varepsilon_t^*, \\
\boldsymbol{\mu}_t(\boldsymbol{\pi}, \boldsymbol{\rho}) = [\mathbf{I}_N - diag(\boldsymbol{\rho})]\boldsymbol{\pi} + diag(\boldsymbol{\rho})\mathbf{y}_{t-1}, \\
\boldsymbol{\Sigma}_t(\boldsymbol{\theta}) = \mathbf{c}\mathbf{c}'\lambda_t(\boldsymbol{\theta}) + \boldsymbol{\Gamma}, \\
\lambda_t(\boldsymbol{\theta}) = 1 + \sum_{j=1}^{q} \alpha_j [f_{kt-j}^2(\boldsymbol{\theta}) + \omega_{t-j}(\boldsymbol{\theta}) - 1], \\
\varepsilon_t^* | \mathbf{z}_t, I_{t-1}; \boldsymbol{\theta}_0, \boldsymbol{\eta}_0 \sim i.i.d.\ s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0),
\end{array}\right\} \quad (27)
$$

where $f_{kt}(\boldsymbol{\theta})$ is the conditionally linear Kalman filter estimator of the underlying common factor, and $\omega_t(\boldsymbol{\theta})$ the corresponding conditional mean square error (see Sentana (2004) for details). Define $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_N)'$, $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_N)'$, $\boldsymbol{\gamma} = vecd(\boldsymbol{\Gamma})$, and $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_q)'$, so that $\boldsymbol{\theta} = (\boldsymbol{\pi}', \boldsymbol{\rho}', \mathbf{c}', \boldsymbol{\gamma}', \boldsymbol{\alpha}')'$. We can establish the following result:

Proposition 11 *If in model (27) $\boldsymbol{\alpha}_0 = 0$, $\gamma_{i0} > 0\ \forall i$, and $|\rho_{i0}| < 1\ \forall i$, then the feasible ML estimators of $\boldsymbol{\pi}$, $\boldsymbol{\rho}$ and $\boldsymbol{\alpha}$ are as efficient as the infeasible ML estimators, which require $\boldsymbol{\eta}_0$ to be known. If in addition $\kappa_0 < \infty$, then the elliptically symmetric semiparametric estimators of $\boldsymbol{\pi}$, $\boldsymbol{\rho}$ and $\boldsymbol{\alpha}$ are also fully efficient. The same is also true of the semiparametric estimators of $\boldsymbol{\rho}$ and $\boldsymbol{\alpha}$, but not of $\boldsymbol{\pi}$. In contrast, the inefficiency ratio of the Gaussian PML estimators is $\text{M}_{ll}^{-1}(\boldsymbol{\eta}_0)$ for $\boldsymbol{\pi}$ and $\boldsymbol{\rho}$, and $4/\{[3\text{M}_{ss}(\boldsymbol{\eta}_0) - 1](3\kappa_0 + 2)\}$ for $\boldsymbol{\alpha}$.*

These inefficiency ratios coincide with the corresponding ratios in the univariate example of Proposition 10. In the multivariate student t case with $\nu_0 > 4$, in particular, they become $(\nu_0 - 2)(\nu_0 + N + 2)/[\nu_0(\nu_0 + N)]$ and $(\nu_0 + N + 2)\,(\nu_0 - 4)/[(\nu_0 - 1)(\nu_0 + N - 1)]$, respectively. For any given N, these ratios are monotonically increasing in ν_0, and approach 1 from below as

$\nu_0 \to \infty$ in accordance to Proposition 9, and 0 from above as $\nu_0 \to 2^+$ or $\nu_0 \to 4^+$. For instance, for $N = 1$ and $\nu_0 = 9$, they take the value of $.9\mathring{3}$ and $.8\mathring{3}$, respectively, while for $\nu_0 = 5$, their values are only .8 and .4. At the same time, these ratios are decreasing in N for a given ν_0, which reflects the fact that the information matrix is "increasing" in N, as discussed after Proposition 1. For $\nu_0 = 9$ and $N = 3$, for instance, they take the value of .907 and .795, respectively, while for $\nu_0 = 5$, their values are only .75 and .357.

Furthermore, we can also show that these inefficiency ratios coincide with the ratios of the non-centrality parameters of the corresponding tests of conditional homoskedasticity against local alternatives of the form $\boldsymbol{\alpha}_{0T} = \boldsymbol{\alpha}_0/\sqrt{T}$ in model (27) (see Sentana and Fiorentini (2001)).

3.3 General results on partial adaptivity

We have just studied two situations in which some, but not all elements of $\boldsymbol{\theta}$ can be estimated as efficiently as if $\boldsymbol{\eta}_0$ were known (see also Lange, Little and Taylor (1989)), a fact that would be described in the semiparametric literature as partial adaptivity. Effectively, this requires that some elements of $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}_0)$ be orthogonal to the relevant tangent set after partiallying out the effects of the remaining elements of $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}_0)$ by regressing the former on the latter. Partial adaptivity, though, often depends on the model parametrisation. The following reparametrisation provides a general sufficient condition in multivariate dynamic models:

Reparametrisation 1 *A homeomorphic transformation $\mathbf{r}_s(.) = [\mathbf{r}'_{1s}(.), r'_{2s}(.)]'$ of the conditional mean and variance parameters $\boldsymbol{\theta}$ into an alternative set of parameters $\boldsymbol{\vartheta} = (\boldsymbol{\vartheta}'_1, \vartheta'_2)'$, where ϑ_2 is a scalar, and $\mathbf{r}_s(\boldsymbol{\theta})$ is twice continuously differentiable with rank$[\partial \mathbf{r}'_s(\boldsymbol{\theta})/\partial \boldsymbol{\theta}] = p$ in a neighbourhood of $\boldsymbol{\theta}_0$, such that*

$$\left.\begin{array}{c} \boldsymbol{\mu}_t(\boldsymbol{\theta}) = \boldsymbol{\mu}_t(\boldsymbol{\vartheta}_1) \\ \boldsymbol{\Sigma}_t(\boldsymbol{\theta}) = \vartheta_2 \boldsymbol{\Sigma}_t^\circ(\boldsymbol{\vartheta}_1) \end{array}\right\} \quad \forall t. \tag{28}$$

Such a reparametrisation is not unique, since we can always multiply the overall scale parameter ϑ_2 by some scalar positive smooth function of $\boldsymbol{\vartheta}_1$, $k(\boldsymbol{\vartheta}_1)$ say, and divide $\boldsymbol{\Sigma}_t^\circ(\boldsymbol{\vartheta}_1)$ by the same function without violating (28). As we shall see, a particularly convenient function would be $k(\boldsymbol{\vartheta}_1) = \exp\{N^{-1}E[\ln|\boldsymbol{\Sigma}_t^\circ(\boldsymbol{\vartheta}_1)|\}|\boldsymbol{\phi}_0]$, so that after re-scaling

$$E[\ln|\boldsymbol{\Sigma}_t^\circ(\boldsymbol{\vartheta}_1)||\boldsymbol{\phi}_0] = 1 \quad \forall \boldsymbol{\vartheta}_1. \tag{29}$$

The following proposition generalises and extends earlier results by Bickel (1982), Linton (1993), Drost, Klaassen and Werker (1997) and Hodgson and Vorkink (2003):

Proposition 12 *1. If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0)$, and (28) holds, then:*

(a) the elliptically symmetric semiparametric estimator of $\boldsymbol{\vartheta}_1$ is ϑ_2-adaptive,

(b) *If $\mathring{\boldsymbol{\vartheta}}_T$ denotes the iterated elliptically symmetric semiparametric estimator of $\boldsymbol{\vartheta}$, then $\mathring{\vartheta}_{2T} = \vartheta_{2T}(\mathring{\boldsymbol{\vartheta}}_{1T})$, where*

$$\begin{aligned} \vartheta_{2T}(\boldsymbol{\vartheta}_1) &= \frac{1}{N}\frac{1}{T}\sum_{t=1}^{T}\varsigma_t^{\circ}(\boldsymbol{\vartheta}_1), & (30)\\ \varsigma_t^{\circ}(\boldsymbol{\vartheta}_1) &= [\mathbf{x}_t - \boldsymbol{\mu}_t(\boldsymbol{\vartheta}_1)]'\boldsymbol{\Sigma}_t^{\circ -1}(\boldsymbol{\vartheta}_1)[\mathbf{x}_t - \boldsymbol{\mu}_t(\boldsymbol{\vartheta}_1)], & (31)\end{aligned}$$

(c) *rank* $\left[\mathring{\mathcal{S}}(\boldsymbol{\phi}_0) - \mathcal{C}^{-1}(\boldsymbol{\phi}_0)\right] \leq \dim(\boldsymbol{\vartheta}_1) = p-1$.

2. *If in addition condition (29) holds, then:*

(a) $\mathcal{I}_{\boldsymbol{\vartheta}\boldsymbol{\vartheta}}(\boldsymbol{\phi}_0), \mathcal{P}(\boldsymbol{\phi}_0), \mathring{\mathcal{S}}(\boldsymbol{\phi}_0), \mathcal{S}(\boldsymbol{\phi}_0)$ *and* $\mathcal{C}(\boldsymbol{\phi}_0)$ *are block-diagonal between* $\boldsymbol{\vartheta}_1$ *and* ϑ_2,

(b) $\sqrt{T}(\mathring{\vartheta}_{2T} - \tilde{\vartheta}_{2T}) = o_p(1)$, *where* $\tilde{\boldsymbol{\vartheta}}_T' = (\tilde{\boldsymbol{\vartheta}}_{1T}', \tilde{\vartheta}_{2T})$ *is the PMLE of* $\boldsymbol{\vartheta}$, *with* $\tilde{\vartheta}_{2T} = \vartheta_{2T}(\tilde{\boldsymbol{\vartheta}}_{1T})$.

This proposition provides a saddle point characterisation of the asymptotic efficiency of the elliptically symmetric semiparametric estimator of $\boldsymbol{\theta}$, in the sense that in principle it can estimate $p-1$ "parameters" as efficiently as if we fully knew the true conditional distribution of the data, while for the remaining scalar "parameter" it only achieves the efficiency of the PMLE. Obviously, the feasible ML estimator of $\boldsymbol{\vartheta}_1$ will also be ϑ_2-adaptive when the assumed parametric conditional distribution of $\boldsymbol{\varepsilon}_t^*$ is correct in view of Proposition 8.

At first sight, it may seem that the two examples discussed in the previous sections cannot be rationalised in terms of Proposition 12 because their parametrisations do not satisfy condition (28). In particular, the ARCH parameters $\boldsymbol{\alpha}$ are not generally scale-invariant. However, as explained by Linton and Steigerwald (2000) in the context of model (26), condition (28) will be effectively satisfied under the maintained hypothesis of $\boldsymbol{\alpha}_0 = \mathbf{0}$.

It is also possible to find an analogous result for the unrestricted semiparametric estimator, but at the cost of restricting further the set of parameters that can be estimated in a partially adaptive manner

Reparametrisation 2 *A homeomorphic transformation $\mathbf{r}_g(.) = [\mathbf{r}_{1g}'(.), \mathbf{r}_{2g}'(.), \mathbf{r}_{3g}'(.)]'$ of the conditional mean and variance parameters $\boldsymbol{\theta}$ into an alternative parameter set $\boldsymbol{\psi} = (\boldsymbol{\psi}_1', \boldsymbol{\psi}_2', \boldsymbol{\psi}_3')'$, where $\boldsymbol{\psi}_2 = vech(\boldsymbol{\Psi}_2)$, $\boldsymbol{\Psi}_2$ is an unrestricted positive (semi)definite matrix of order N, $\boldsymbol{\psi}_3$ is $N \times 1$, and $\mathbf{r}_g(\boldsymbol{\theta})$ is twice continuously differentiable with rank$\left[\partial\mathbf{r}_g'(\boldsymbol{\theta})/\partial\boldsymbol{\theta}\right] = p$ in a neighbourhood of θ_0, such that*

$$\left.\begin{aligned} \boldsymbol{\mu}_t(\boldsymbol{\theta}) &= \boldsymbol{\mu}_t^{\diamond}(\boldsymbol{\psi}_1) + \boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_1)\boldsymbol{\psi}_3 \\ \boldsymbol{\Sigma}_t(\boldsymbol{\theta}) &= \boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_1)\boldsymbol{\Psi}_2\boldsymbol{\Sigma}_t^{\diamond 1/2\prime}(\boldsymbol{\psi}_1) \end{aligned}\right\} \quad \forall t. \tag{32}$$

This parametrisations simply requires the pseudo-standardised residuals

$$\boldsymbol{\varepsilon}_t^{\diamond}(\boldsymbol{\psi}_1) = \boldsymbol{\Sigma}_t^{\diamond -1/2}(\boldsymbol{\psi}_1)[\mathbf{y}_t - \boldsymbol{\mu}_t^{\diamond}(\boldsymbol{\psi}_1)] \tag{33}$$

to be *i.i.d.* $(\boldsymbol{\psi}_3, \boldsymbol{\Psi}_2)$. Again, (32) is not unique, since it continues to hold if we replace $\boldsymbol{\Psi}_2$ by $\mathbf{K}^{-1/2}(\boldsymbol{\psi}_1)\boldsymbol{\Psi}_2\mathbf{K}^{-1/2\prime}(\boldsymbol{\psi}_1)$ and $\boldsymbol{\psi}_3$ by $\mathbf{K}^{-1/2}(\boldsymbol{\psi}_1)\boldsymbol{\psi}_3 - \mathbf{l}(\boldsymbol{\psi}_1)$, and adjust $\boldsymbol{\mu}_t^\diamond(\boldsymbol{\psi}_1)$ and $\boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_1)$ accordingly, where $\mathbf{l}(\boldsymbol{\psi}_1)$ and $\mathbf{K}(\boldsymbol{\psi}_1)$ are a $N \times 1$ vector and a $N \times N$ positive definite matrix of smooth functions of $\boldsymbol{\psi}_1$, respectively. Particularly convenient forms for these functions would be those for which the Jacobian matrix of $vech[\mathbf{K}^{-1/2}(\boldsymbol{\psi}_1)\boldsymbol{\Psi}_2\mathbf{K}^{-1/2\prime}(\boldsymbol{\psi}_1)]$ and $\mathbf{K}^{-1/2}(\boldsymbol{\psi}_1)\boldsymbol{\psi}_3 - \mathbf{l}(\boldsymbol{\psi}_1)$ with respect to $\boldsymbol{\psi}$ evaluated at the true values is equal to:

$$\left\{ -V^{-1} \left[\begin{array}{c} \mathbf{s}_{\boldsymbol{\psi}_{2t}}(\boldsymbol{\psi}_0) \\ \mathbf{s}_{\boldsymbol{\psi}_{3t}}(\boldsymbol{\psi}_0) \end{array} \middle| \boldsymbol{\phi}_0 \right] E \left[\begin{array}{c} \mathbf{s}_{\boldsymbol{\psi}_{2t}}(\boldsymbol{\psi}_0)\mathbf{s}'_{\boldsymbol{\psi}_{1t}}(\boldsymbol{\psi}_0) \\ \mathbf{s}_{\boldsymbol{\psi}_{3t}}(\boldsymbol{\psi}_0)\mathbf{s}'_{\boldsymbol{\psi}_{1t}}(\boldsymbol{\psi}_0) \end{array} \middle| \boldsymbol{\phi}_0 \right] \middle| \begin{array}{c|c} \mathbf{I}_{N(N+1)/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_N \end{array} \right\}. \tag{34}$$

The following proposition, which does not require sphericity, generalises and extends Theorems 3.1 in Drost and Klaassen (1997) and 3.2 in Sun and Stengos (2006):

Proposition 13 *1. If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $(\mathbf{0}, \mathbf{I}_N)$, and (32) holds, then*

(a) the semiparametric estimator of $\boldsymbol{\psi}_1$, $\breve{\boldsymbol{\psi}}_{1T}$, is $(\boldsymbol{\psi}_2, \boldsymbol{\psi}_3)$-adaptive,

(b) If $\breve{\boldsymbol{\psi}}_T$ denotes the iterated semiparametric estimator of $\boldsymbol{\psi}$, then $\breve{\boldsymbol{\psi}}_{2T} = \boldsymbol{\psi}_{2T}(\breve{\boldsymbol{\psi}}_{1T})$ and $\breve{\boldsymbol{\psi}}_{3T} = \boldsymbol{\psi}_{3T}(\breve{\boldsymbol{\psi}}_{1T})$, where

$$\boldsymbol{\psi}_{2T}(\boldsymbol{\psi}_1) = vech\left\{ \frac{1}{T}\sum_{t=1}^{T} [\boldsymbol{\varepsilon}_t^\diamond(\boldsymbol{\psi}_1) - \boldsymbol{\psi}_{3T}(\boldsymbol{\psi}_1)] [\boldsymbol{\varepsilon}_t^\diamond(\boldsymbol{\psi}_1) - \boldsymbol{\psi}_{3T}(\boldsymbol{\psi}_1)]' \right\}, \tag{35}$$

$$\boldsymbol{\psi}_{3T}(\boldsymbol{\psi}_1) = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{\varepsilon}_t^\diamond(\boldsymbol{\psi}_1) \tag{36}$$

(c) rank $\left[\mathcal{S}(\boldsymbol{\phi}_0) - \mathcal{C}^{-1}(\boldsymbol{\phi}_0)\right] \leq \dim(\boldsymbol{\psi}_1) = p - N - N(N+1)/2$.

2. If in addition condition (34) holds, then

(a) $\mathcal{I}_{\boldsymbol{\psi}\boldsymbol{\psi}}(\boldsymbol{\phi}_0), \mathcal{P}(\boldsymbol{\phi}_0), \mathring{\mathcal{S}}(\boldsymbol{\phi}_0), \mathcal{S}(\boldsymbol{\phi}_0)$ and $\mathcal{C}(\boldsymbol{\phi}_0)$ are block diagonal between $\boldsymbol{\psi}_1$ and $(\boldsymbol{\psi}_2, \boldsymbol{\psi}_3)$.

(b) $\sqrt{T}[(\breve{\boldsymbol{\psi}}'_{2T} - \tilde{\boldsymbol{\psi}}'_{2T}), (\breve{\boldsymbol{\psi}}'_{3T} - \tilde{\boldsymbol{\psi}}'_{3T})]' = o_p(1)$, where $\tilde{\boldsymbol{\psi}}'_T = (\tilde{\boldsymbol{\psi}}'_{1T}, \tilde{\boldsymbol{\psi}}'_{2T}, \tilde{\boldsymbol{\psi}}'_{3T})$ is the PMLE of $\boldsymbol{\psi}$, with $\tilde{\boldsymbol{\psi}}_{2T} = \boldsymbol{\psi}_{2T}(\tilde{\boldsymbol{\psi}}'_{1T})$ and $\tilde{\boldsymbol{\psi}}_{3T} = \boldsymbol{\psi}_{3T}(\tilde{\boldsymbol{\psi}}'_{1T})$.

This proposition provides a saddle point characterisation of the asymptotic efficiency of the semiparametric estimator of $\boldsymbol{\theta}$, in the sense that in principle it can estimate $p - N(N+3)/2$ "parameters" as efficiently as if we fully knew the true conditional distribution of the data, while for the remaining "parameters" it only achieves the efficiency of the PMLE.

Unfortunately, the constant conditional correlation model of Bollerslev (1990), which assumes that $\boldsymbol{\Sigma}_t(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \mathbf{D}_t(\boldsymbol{\theta}_1)\mathbf{R}\mathbf{D}_t(\boldsymbol{\theta}_1)$, where $\mathbf{D}_t$ is a positive diagonal matrix, $\boldsymbol{\theta}_2 = vecl(\mathbf{R})$ and $\mathbf{R}$ a correlation matrix, seems to be the only multivariate GARCH specification proposed so far that can be parametrised as (32) *if* we additionally assume that $\boldsymbol{\mu}_t(\boldsymbol{\theta}) = \mathbf{0}\ \forall t$, in which case $\boldsymbol{\psi}_3$ is unnecessary. And even in that case, we could only adaptively estimate the parameters of $\boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_1) = \mathbf{D}_t(\boldsymbol{\theta}_1)\{E[\mathbf{D}_t(\boldsymbol{\theta}_1)]|\boldsymbol{\phi}_0\}^{-1}$, which will typically correspond to the relative scale parameters of the N univariate ARCH models for the elements of $\mathbf{y}_t$, although Ling and McAleer (2003) consider a more general specification. In most other models, we may need to artificially augment the original parametrisation with $\boldsymbol{\psi}_2$ and $\boldsymbol{\psi}_3$ even though we know that $\boldsymbol{\psi}_{20} = vech(\mathbf{I}_N)$ and $\boldsymbol{\psi}_{30} = \mathbf{0}$, which could be associated with a substantial efficiency cost. Furthermore, in doing so, we must guarantee that the parameters $\boldsymbol{\psi}_1$ remain identified (see Newey and Steigerwald (1997) for a detailed discussion of these issues in univariate models). In this sense, the main difference between Propositions 12 and 13 is that in the elliptically symmetric case we can restrict $\boldsymbol{\Psi}_2$ to be a scalar matrix, and $\boldsymbol{\psi}_3$ to $\mathbf{0}$ regardless of the mean specification, which reduces the number of parameters by a factor of $N(N+3)/2$.

4 The relative efficiency of ML and sequential estimators of $\boldsymbol{\eta}$

The asymptotic variance of the feasible ML estimator of $\boldsymbol{\eta}$, $\hat{\boldsymbol{\eta}}_T$, is

$$\mathcal{I}^{\boldsymbol{\eta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0) = \left[\mathcal{I}_{\boldsymbol{\eta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0) - \mathcal{I}'_{\boldsymbol{\theta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0)\mathcal{I}^{-1}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\phi}_0)\mathcal{I}'_{\boldsymbol{\theta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0)\right]^{-1},$$

which coincides with the inverse of the variance of the efficient parametric score of $\boldsymbol{\eta}$, $\mathbf{s}_{\boldsymbol{\eta}|\boldsymbol{\theta}}(\boldsymbol{\phi}_0)$, which is the residual in the theoretical regression of $\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\phi}_0)$ on $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}_0)$. As a result, this residual variance, or marginal information matrix, will generally be smaller than $\mathcal{I}_{\boldsymbol{\eta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0)$, which corresponds to the infeasible ML estimator of $\boldsymbol{\eta}$ that we could compute if the $\varsigma_t(\boldsymbol{\theta}_0)'s$ were directly observed. The following proposition characterises the ranking of the asymptotic covariance matrices of the five estimators of $\boldsymbol{\eta}$ that we have considered:

Proposition 14 *1. If $\boldsymbol{\varepsilon}^*_t|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta}_0)$ with $\kappa_0 < \infty$, then $\mathcal{I}^{-1}_{\boldsymbol{\eta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0) \leq \mathcal{I}^{\boldsymbol{\eta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0) \leq \mathcal{F}(\boldsymbol{\phi}_0)$.*

2. If $\boldsymbol{\varepsilon}^*_t|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0$ is i.i.d. $t(\mathbf{0}, \mathbf{I}_N, \nu_0)$ with $\nu_0 > 8$, then $\mathcal{F}(\boldsymbol{\phi}_0) \leq \mathcal{J}(\boldsymbol{\phi}_0)$. If in addition

$$\mathcal{A}^{-1}(\boldsymbol{\phi}_0)\mathbf{W}_s(\boldsymbol{\phi}_0) = \frac{(N+\nu_0-2)}{(\nu_0-4)}\mathcal{B}^{-1}(\boldsymbol{\phi}_0)\mathbf{W}_s(\boldsymbol{\phi}_0), \qquad (37)$$

then $\mathcal{J}(\boldsymbol{\phi}_0) \leq \mathcal{G}(\boldsymbol{\phi}_0)$, with equality if and only if

$$\left[\frac{\varsigma_t(\boldsymbol{\theta}_0)}{N} - 1\right] - \frac{2(N+\nu_0-2)}{N(\nu_0-4)}\mathbf{W}'_s(\boldsymbol{\phi}_0)\mathcal{B}^{-1}(\boldsymbol{\phi}_0)\mathbf{s}_{\boldsymbol{\theta}\mathbf{t}}(\boldsymbol{\theta}_0, 0) = 0 \ \forall t. \qquad (38)$$

Condition (37) is trivially satisfied in Gaussian models, and in dynamic univariate models with no mean. Also, it is worth mentioning that (38), which in turn implies (37), is satisfied by most dynamic univariate GARCH-M models (see Fiorentini, Sentana and Calzolari (2004)).

Given that $\mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\eta}}(\boldsymbol{\phi}_0) = \mathbf{0}$ under normality from Proposition 9, it is clear that $\tilde{\boldsymbol{\eta}}_T$ will be as asymptotically efficient as the feasible ML estimator $\hat{\boldsymbol{\eta}}_T$ when $\boldsymbol{\eta}_0 = \mathbf{0}$, which in turn is as efficient as the infeasible ML estimator in that case. Moreover, if we use a multivariate student t

log-likelihood function, these estimators will share the same half normal asymptotic distribution under conditional normality, although they would not necessarily be equal when they are not zero. Similarly, the asymptotic distributions of $\breve{\eta}_T$ and $\mathring{\eta}_T$ will also tend to be half normal as the sample size increases when $\eta_0 = 0$, since $\bar{\kappa}_T(\tilde{\boldsymbol{\theta}}_T)$ is root-T consistent for κ, which is 0 in the Gaussian case. However, while $\mathring{\eta}_T$ will always be as efficient as $\hat{\eta}_T$ under normality because $\mathring{n}_{\eta t}(\boldsymbol{\theta}_0, 0)$ is proportional to $s_{\eta t}(\boldsymbol{\theta}_0, 0)$, $\breve{\eta}_T$ will be less efficient unless condition (38) is satisfied.

5 Distributional misspecification and parameter consistency

5.1 Parameter estimation

So far, we have maintained the assumption that the conditional distribution of the standardised innovations $\boldsymbol{\varepsilon}_t^*$ is either *i.i.d.* $s(\mathbf{0}, \mathbf{I}_N, \boldsymbol{\eta})$ or sometimes $t(\mathbf{0}, \mathbf{I}_N, \nu_0)$. However, one of the most important reasons for the popularity of the Gaussian pseudo-ML estimator of $\boldsymbol{\theta}$ despite its inefficiency is that it remains root-T consistent and asymptotically normally distributed under fairly weak distributional assumptions provided that (1) is true. In contrast, the efficient spherically-based ML estimator may become inconsistent if the true distribution of $\boldsymbol{\varepsilon}_t^*$ given $\mathbf{z}_t$ and I_{t-1} does not coincide with the assumed one, even though (1) holds, as forcefully argued by Newey and Steigerwald (1997) in the univariate case. To focus our discussion, in the remaining of this section we shall assume that (1) is true, and that we specifically decide to use the student t log-likelihood function for estimation purposes. Nevertheless, our results can be trivially extended to any other spherically-based likelihood estimators, as the only advantage of the student t likelihood four our purposes is the fact that its limiting relationship to the Gaussian distribution can be made explicit. For simplicity, we shall also define the pseudo-true values of $\boldsymbol{\theta}$ and η as consistent roots of the expected t pseudo log-likelihood score, which under appropriate regularity conditions will maximise the expected value of the t pseudo log-likelihood function.

Two important points to bear in mind in studying the potential inconsistencies in $\hat{\boldsymbol{\theta}}_T$ are (i) that the spherical distribution assumed for estimation purposes will often nest the Gaussian distribution as a limiting case, and (ii) that $\hat{\boldsymbol{\theta}}_T = \tilde{\boldsymbol{\theta}}_T$ whenever $\hat{\boldsymbol{\eta}}_T = \mathbf{0}$. For instance, the t distribution is estimated subject to the inequality constraint $\eta \geq 0$. The following proposition explains the consequences of this inequality restriction:

Proposition 15 *1. Let $\boldsymbol{\phi}_\infty$ denote the pseudo-true values of the parameters $\boldsymbol{\theta}$ and η implied by a multivariate student t log-likelihood function. If the unconditional coefficient of multivariate excess kurtosis of $\boldsymbol{\varepsilon}_t^*$ is not positive, where the expectation in (16) is taken with respect to the true unconditional distribution of the data, then $\boldsymbol{\theta}_\infty = \boldsymbol{\theta}_0$ and $\eta_\infty = 0$.*

2. If the unconditional coefficient of multivariate excess kurtosis of $\boldsymbol{\varepsilon}_t^*$ is strictly negative, and the regularity conditions A.1 in Bollerslev and Wooldridge (1992) are satisfied, then $\sqrt{T}\hat{\eta}_T = o_p(1)$ and $\sqrt{T}(\tilde{\boldsymbol{\theta}}_T - \hat{\boldsymbol{\theta}}_T) = o_p(1)$.

3. *If the unconditional coefficient of multivariate excess kurtosis of $\boldsymbol{\varepsilon}_t^*$ is exactly 0, and the regularity conditions A.1 in Bollerslev and Wooldridge (1992) are satisfied, then $\sqrt{T}\hat{\eta}_T$ will have an asymptotic normal distribution censored from below at 0, and $\tilde{\boldsymbol{\theta}}_T$ will be identical to $\hat{\boldsymbol{\theta}}_T$ with probability approaching 1/2. If in addition*

$$\mathcal{H}_{\boldsymbol{\theta}\boldsymbol{\eta}}(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0) = E[[N+2-\varsigma_t(\boldsymbol{\theta}_0)]\{\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)|vec'[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)]\}\mathbf{Z}_{dt}'(\boldsymbol{\theta}_0)|\boldsymbol{\varphi}_0] = \mathbf{0}, \quad (39)$$

where $\boldsymbol{\varphi}_0 = (\boldsymbol{\theta}_0, \boldsymbol{\varrho}_0)$, then $\sqrt{T}(\tilde{\boldsymbol{\theta}}_T - \hat{\boldsymbol{\theta}}_T) = o_p(1)$ the rest of the time.

In the rest of this section we will concentrate on those distributions for which the condition $\kappa_0 \leq 0$ in Proposition 15 is violated. The first part of the following proposition extends the first part of Theorem 1 in Newey and Steigerwald (1997) to a broad class of multivariate dynamic models, while the rest does the same thing for Proposition 4 in Amengual and Sentana (2007).

Proposition 16 *If $\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t, I_{t-1};\boldsymbol{\varphi}_0$ is i.i.d. $s(\mathbf{0},\mathbf{I}_N,\boldsymbol{\varrho}_0)$ but not t with $\kappa_0 > 0$, where $\boldsymbol{\varphi}_0 = (\boldsymbol{\vartheta}_{10}', \vartheta_{20}, \boldsymbol{\varrho}_0)$, and (28) holds, then:*

1. *The pseudo-true value of feasible student-t based ML estimator of $\boldsymbol{\phi} = (\boldsymbol{\vartheta}_1', \vartheta_2, \eta)'$, $\boldsymbol{\phi}_\infty$, is such that $\boldsymbol{\vartheta}_{1\infty}$ is equal to the true value $\boldsymbol{\vartheta}_{10}$.*

2. *$\mathcal{O}_t(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0) = V[\mathbf{s}_t(\boldsymbol{\phi}_\infty)|\mathbf{z}_t, I_{t-1};\boldsymbol{\varphi}_0] = \mathbf{Z}_t(\boldsymbol{\vartheta}_\infty)\mathcal{M}^O(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)\mathbf{Z}_t(\boldsymbol{\vartheta}_\infty)$, while $\mathcal{H}_t(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0) = -E[\mathbf{h}_t(\boldsymbol{\phi}_\infty)|\mathbf{z}_t, I_{t-1};\boldsymbol{\varphi}_0] = \mathbf{Z}_t(\boldsymbol{\vartheta}_\infty)\mathcal{M}^H(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)\mathbf{Z}_t(\boldsymbol{\vartheta}_\infty)$, where both $\mathcal{M}^O(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)$ and $\mathcal{M}^H(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)$ share the structure of (11), (12), (13) and (14), with*

$$\begin{aligned}
\mathrm{M}_{ll}^O(\boldsymbol{\phi};\boldsymbol{\varphi}) &= E\left\{\delta^2[\varsigma_t(\boldsymbol{\vartheta}),\eta]\cdot[\varsigma_t(\boldsymbol{\vartheta})/N]\right|\boldsymbol{\varphi}\right\}\\
\mathrm{M}_{ss}^O(\boldsymbol{\phi};\boldsymbol{\varphi}) &= N(N+2)^{-1}\left[1+V\left\{\delta[\varsigma_t(\boldsymbol{\vartheta}),\eta]\cdot[\varsigma_t(\boldsymbol{\vartheta})/N]|\,\boldsymbol{\varphi}\right\}\right],\\
\mathrm{M}_{sr}^O(\boldsymbol{\phi};\boldsymbol{\varphi}) &= E\left[\left\{\delta[\varsigma_t(\boldsymbol{\vartheta}),\eta]\cdot[\varsigma_t(\boldsymbol{\vartheta})/N]-1\right\}\mathbf{e}_{rt}'(\boldsymbol{\phi})\right|\boldsymbol{\varphi}\right],\\
\mathcal{M}_{rr}^O(\boldsymbol{\phi};\boldsymbol{\varphi}) &= V[\,\mathbf{e}_{rt}(\boldsymbol{\phi})|\,\boldsymbol{\varphi}],\\
\mathrm{M}_{ll}^H(\boldsymbol{\phi};\boldsymbol{\varphi}) &= E\left\{2\partial\delta[\varsigma_t(\boldsymbol{\vartheta}),\eta]/\partial\varsigma\cdot[\varsigma_t(\boldsymbol{\vartheta})/N]+\delta[\varsigma_t(\boldsymbol{\theta}),\eta]|\,\boldsymbol{\varphi}\right\},\\
\mathrm{M}_{ss}^H(\boldsymbol{\phi};\boldsymbol{\varphi}) &= E\left\{2\partial\delta[\varsigma_t(\boldsymbol{\vartheta}),\eta]/\partial\varsigma\cdot\varsigma_t^2(\boldsymbol{\vartheta})/[N(N+2)]\right|\boldsymbol{\varphi}\right\}+1,\\
\mathrm{M}_{sr}^H(\boldsymbol{\phi};\boldsymbol{\varphi}) &= -E\left\{[\varsigma_t(\boldsymbol{\vartheta})/N]\cdot\partial\delta[\varsigma_t(\boldsymbol{\vartheta}),\eta]/\partial\eta|\boldsymbol{\varphi}\right\},\\
\mathcal{M}_{rr}^H(\boldsymbol{\phi};\boldsymbol{\varphi}) &= -E[\,\partial\mathbf{e}_{rt}(\boldsymbol{\phi})/\partial\eta'\right|\boldsymbol{\varphi}].
\end{aligned}$$

3. *If in addition (29) holds, then $E[\mathcal{O}_t(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)|\boldsymbol{\varphi}_0]$ and $E[\mathcal{H}_t(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)|\boldsymbol{\varphi}_0]$ will be block diagonal between $\boldsymbol{\vartheta}_1$ and (ϑ_2,η).*

Part 1 says that the t-based MLE can estimate consistently all the parameters except the expected value of $\varsigma_t^\circ(\boldsymbol{\vartheta}_{10})$ in (31), while Part 2 allows us to obtain the asymptotic variance of the t-based ML estimators with the usual sandwich formula. It should also be straightforward to consistently estimate the overall scale parameter ϑ_2 by combining $\hat{\boldsymbol{\vartheta}}_{1T}$ with the expression for the concentrated PML and iterated SSP estimators in (30).

Importantly, note that the transformed parameters that we can estimate in a partially adaptive manner by means of the SSP estimator coincide with the parameters that we continue to estimate consistently with a misspecified student t-based pseudo-ML estimator.

If $\varepsilon_t^*|\mathbf{z}_t, I_{t-1}, \phi_0$ is not *i.i.d. spherical*, and $\kappa_0 > 0$, then in general the feasible student t-based ML estimator will be inconsistent, and the same applies to the SSP estimator.[6] However, it may still be possible to estimate consistently some parameters:

Proposition 17 *If $\varepsilon_t^*|\mathbf{z}_t, I_{t-1}$ is i.i.d. $(\mathbf{0}, \mathbf{I}_N)$ but not spherical, with $\kappa_0 > 0$, and (32) holds, then the pseudo-true value of feasible student-t based ML estimator of ψ_1, $\psi_{1\infty}$, is equal to the true value ψ_{10}.*

This proposition is the multivariate generalisation of Theorem 2 in Newey and Steigerwald (1997).[7] In simple terms, it says that the t-based MLE cannot estimate consistently either the mean or the covariance matrix of the *i.i.d.* pseudo-standardised residuals $\varepsilon_t^\diamond(\psi_{10})$ in (33). However, it should be straightforward to consistently estimate ψ_2 and ψ_3 by combining $\breve{\psi}_{1T}$ with the expressions for the concentrated PML and SP estimators in (35) and (36). As discussed at the end of section 3.3, though, we may only be able to write the conditional mean and covariance functions as in (32) at the cost of augmenting the model with a large number of additional parameters, which will generally lead to either inefficiency loss or even lack of identification.

Importantly, note that the transformed parameters that we can estimate in a partially adaptive manner by means of the unrestricted semiparametric estimator coincide with the parameters that we continue to estimate consistently with a misspecified student-t based ML estimator.

However, the semiparametric estimator may also become inconsistent if the *i.i.d.* assumption does not hold. In this sense, one should bear in mind that in non-elliptical models the conditional distribution of $\mathbf{y}_t$ is not invariant to the specific choice of $\mathbf{\Sigma}_t^{1/2}(\boldsymbol{\theta})$ assumed to generate the data (see Mencía and Sentana (2005)), a choice that could conceivably change over time.

5.2 Hausman tests

There are several ways in which we can test the validity of the multivariate t assumption. One possibility is to nest that distribution within a more flexible parametric family, which allows us to conduct an LM test of the nesting restrictions. This is the approach in Mencía and Sentana (2005), who use the generalised hyperbolic family as the nesting distribution. An alternative procedure would be an information matrix test that compares some or all the elements of $\mathcal{M}^O(\phi_\infty; \varphi_0)$ and $\mathcal{M}^H(\phi_\infty; \varphi_0)$ in Proposition 16 by means of an unconditional moment test. But we can also consider a Hausman specification test. The rationale is that the feasible elliptical ML estimator $\hat{\boldsymbol{\theta}}_T$ is efficient under correct specification of the conditional distribution of $\mathbf{y}_t$. In

[6] Hodgson (2000) shows that the consistency of the conditional mean parameters is preserved in non-linear univariate regression models when the innovations are conditionally symmetric but not *i.i.d.* if certain conditions are satisfied. See also Proposition 5 in Amengual and Sentana (2007) for a multivariate example.

[7] It is also possible to generalise the second part of their Theorem 1, in the sense that if the true conditional mean of $\mathbf{y}_t$ is $\mathbf{0}$, and we impose this restriction in estimation, then ψ_3 is unnecessary.

contrast, if the conditional mean and variance of $\mathbf{y}_t$ are correctly specified, but the conditional distribution of $\boldsymbol{\varepsilon}_t^*$ is not *i.i.d.* $t(\mathbf{0}, \mathbf{I}_N, \eta)$, then $\tilde{\boldsymbol{\theta}}_T$ will remain root-T consistent as long as κ_0 is bounded, while $\hat{\boldsymbol{\theta}}_T$ will probably not, as Propositions 16 and 17 illustrate. More formally

Proposition 18 *Let*

$$H^W_{\hat{\boldsymbol{\theta}}_T} = T(\tilde{\boldsymbol{\theta}}_T - \hat{\boldsymbol{\theta}}_T)' \left[\mathcal{C}(\boldsymbol{\phi}_0) - \mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\right]^+ (\tilde{\boldsymbol{\theta}}_T - \hat{\boldsymbol{\theta}}_T),$$

and

$$H^s_{\hat{\boldsymbol{\theta}}_T} = T\bar{\mathbf{s}}'_{\boldsymbol{\theta}T}(\hat{\boldsymbol{\theta}}_T, \mathbf{0}) \left[\mathcal{B}(\boldsymbol{\phi}_0) - \mathcal{A}(\boldsymbol{\phi}_0)\mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\mathcal{A}(\boldsymbol{\phi}_0)\right]^+ \bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\hat{\boldsymbol{\theta}}_T, \mathbf{0}),$$

where $\bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\hat{\boldsymbol{\theta}}_T, \mathbf{0})$ *is the sample average of the Gaussian PML score evaluated at the feasible ML estimator* $\hat{\boldsymbol{\theta}}_T$. *If the regularity conditions A.1 in Bollerslev and Wooldridge (1992) are satisfied and* $\kappa_0 < \infty$, *then* $H^W_{\hat{\boldsymbol{\theta}}_T} \xrightarrow{d} \chi^2_s$ *and* $H^W_{\hat{\boldsymbol{\theta}}_T} - H^s_{\hat{\boldsymbol{\theta}}_T} = o_p(1)$ *under correct specification of the conditional distribution of* $\mathbf{y}_t$, *where* $s = rank\left[\mathcal{C}(\boldsymbol{\phi}_0) - \mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\right]$.

In practice, we must replace $\mathcal{A}(\boldsymbol{\phi}_0)$, $\mathcal{B}(\boldsymbol{\phi}_0)$ and $\mathcal{I}(\boldsymbol{\phi}_0)$ by consistent estimators to make $H^W_{\hat{\boldsymbol{\theta}}_T}$ and $H^s_{\hat{\boldsymbol{\theta}}_T}$ operational. In order to guarantee the positive semidefiniteness of their weighting matrices, it is convenient to estimate all these matrices as the sample averages of the corresponding conditional expressions in Propositions 1 and 2 evaluated at a common estimator of $\boldsymbol{\phi}$, such as $\hat{\boldsymbol{\phi}}_T$, $(\tilde{\boldsymbol{\theta}}_T, \tilde{\eta}_T)$ or $(\tilde{\boldsymbol{\theta}}_T, \breve{\eta}_T)$, the latter being such that $\mathcal{B}(\tilde{\boldsymbol{\theta}}_T, \breve{\eta}_T)$ is always bounded.

In view of Proposition 9, though, such feasible Hausman tests will become numerically unstable when $\hat{\eta}_T > 0$ but $\eta_0 = 0$ even though in theory they should be identically 0 because $\left[\mathcal{C}(\boldsymbol{\phi}_0) - \mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\right] = \mathbf{0}$ in that case. Similarly, the Hausman tests will not work properly when $\eta_0 \geq \frac{1}{4}$ because κ_0 becomes unbounded, although its sample counterpart will obviously remain bounded, which violates one of the assumptions of Proposition 2. Moreover, it may also have poor finite sample properties for $\eta_0 \geq 1/8$ because the asymptotic distribution of $\breve{\eta}_T$ will not be root-T consistent in that case.

Given that the power of these Hausman tests depends on the asymptotic biases of $\hat{\boldsymbol{\theta}}_T$ under misspecification of the conditional distribution of the standardised innovations, it may be convenient to concentrate on those parameters that may be more affected by such distributional misspecification. For instance, in the situation discussed in Proposition 16 power would be maximised if we based our Hausman test on the overall scale parameter ϑ_2 exclusively, and the same will be true in the context of Proposition 17 if we look at $\boldsymbol{\psi}_2$ and $\boldsymbol{\psi}_3$, which are the variance and mean parameters of the pseudo standardised residuals $\boldsymbol{\varepsilon}_t^\diamond(\boldsymbol{\psi}_1)$ in (33).

Given that the SSP estimator is also efficient relative to the PML estimator under sphericity, but it may lose its consistency otherwise, we can consider alternative specification tests as follows:

Proposition 19 *Let*

$$H^W_{\mathring{\boldsymbol{\theta}}_T} = T(\tilde{\boldsymbol{\theta}}_T - \mathring{\boldsymbol{\theta}}_T)'[\mathcal{C}(\boldsymbol{\phi}_0) - \mathring{\mathcal{S}}^{-1}(\boldsymbol{\phi}_0)]^+(\tilde{\boldsymbol{\theta}}_T - \mathring{\boldsymbol{\theta}}_T),$$

and

$$H^s_{\mathring{\boldsymbol{\theta}}_T} = T\bar{\mathbf{s}}'_{\boldsymbol{\theta}T}(\mathring{\boldsymbol{\theta}}_T, \mathbf{0}) \left[\mathcal{B}(\boldsymbol{\phi}_0) - \mathcal{A}(\boldsymbol{\phi}_0)\mathring{\mathcal{S}}^{-1}(\boldsymbol{\phi}_0)\mathcal{A}(\boldsymbol{\phi}_0)\right]^+ \bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\mathring{\boldsymbol{\theta}}_T, \mathbf{0}),$$

where $\bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\mathring{\boldsymbol{\theta}}_T, \mathbf{0})$ *is the sample average of the Gaussian PML score evaluated at the SSP estimator* $\mathring{\boldsymbol{\theta}}_T$. *If the regularity conditions A.1 in Bollerslev and Wooldridge (1992) are satisfied, then* $H^W_{\mathring{\boldsymbol{\theta}}_T} \xrightarrow{d} \chi^2_s$ *and* $H^W_{\mathring{\boldsymbol{\theta}}_T} - H^s_{\mathring{\boldsymbol{\theta}}_T} = o_p(1)$ *under correct specification of the conditional distribution of* $\mathbf{y}_t$, *where* $s = rank[\mathcal{C}(\boldsymbol{\phi}_0) - \mathring{\mathcal{S}}^{-1}(\boldsymbol{\phi}_0)] \leq p - 1$.

Once again, it may be convenient to concentrate on the parameters that are more likely to reflect the distributional misspecification, such as $\boldsymbol{\psi}_2$ and $\boldsymbol{\psi}_3$.

Finally, the difference between $\tilde{\eta}_T$ and $\hat{\eta}_T$ suggests yet another Hausman specification test of the model, which will be given by the following expression:

$$H^W_{\eta T} = T(\tilde{\eta}_T - \hat{\eta}_T)^2 \left[\mathcal{F}(\boldsymbol{\phi}_0) - \mathcal{I}^{\eta\eta}(\boldsymbol{\phi}_0)\right]^+,$$

where the Moore-Penrose generalised inverse in this scalar case is simply the reciprocal of $\mathcal{F}(\boldsymbol{\phi}_0) - \mathcal{I}^{\eta\eta}(\boldsymbol{\phi}_0)$ if $\mathcal{F}(\boldsymbol{\phi}_0) - \mathcal{I}^{\eta\eta}(\boldsymbol{\phi}_0)$ is positive, and 0 otherwise. Under correct specification of the conditional distribution of $\boldsymbol{\varepsilon}^*_t$, $H^W_{\eta T}$ will be asymptotically distributed as a chi-square with one degree of freedom when $\eta_0 > 0$. But again, feasible versions of $H^W_{\eta T}$ may become numerically unstable when $\hat{\eta}_T > 0$ or $\tilde{\eta}_T > 0$ but $\eta_0 = 0$, even though the infeasible version would be identically 0 because $[\mathcal{F}(\boldsymbol{\phi}_0) - \mathcal{I}^{\eta\eta}(\boldsymbol{\phi}_0)] = 0$ in that case. Note that the power of this third Hausman test depends on the difference between the pseudo true values of $\tilde{\eta}_T$ and $\hat{\eta}_T$ when the conditional distribution of $\boldsymbol{\varepsilon}^*_t$ is not multivariate t, which will depend in turn on the asymptotic bias in $\hat{\boldsymbol{\theta}}_T$.

6 Monte Carlo Evidence

6.1 Design and estimation details

In this section, we assess the finite sample performance of the different estimators and testing procedures discussed above by means of an extensive Monte Carlo exercise, with an experimental design that augments (27) with GARCH dynamics. Specifically, we simulate and estimate a model in which $N = 6$, $\boldsymbol{\pi}_0 = .1 \cdot \boldsymbol{\iota}_6$, $\boldsymbol{\rho}_0 = .1 \cdot \boldsymbol{\iota}_6$, $\mathbf{c}_0 = \boldsymbol{\iota}_6$, $\boldsymbol{\gamma}_0 = 2 \cdot \boldsymbol{\iota}_6$, $\boldsymbol{\iota}_6 = (1, 1, 1, 1, 1, 1)'$, and

$$\lambda_t(\boldsymbol{\theta}) - \lambda = \alpha[f^2_{kt-1}(\boldsymbol{\theta}) + \omega_{t-1}(\boldsymbol{\theta}) - \lambda] + \beta[\lambda_{t-1}(\boldsymbol{\theta}) - \lambda], \tag{40}$$

with $\lambda_0 = 1$, $\alpha_0 = .1$ and $\beta_0 = .85$. As for $\boldsymbol{\varepsilon}^*_t$, we consider a Gaussian distribution, and two multivariate student t's with 8 and 4 degrees of freedom respectively. In order to assess the effects of distributional misspecification, we also consider an *i.i.d.* normal-gamma mixture with the same coefficient of multivariate excess kurtosis as the t_8, an *i.i.d.* asymmetric student t such that the marginal distribution of an equally-weighted average of the six series has the maximum negative skewness possible for the kurtosis of the t_8, and a symmetric student t distribution with time-varying kurtosis, in which the degrees of freedom parameter evolves according to the following stochastic difference equation

$$\nu_t = .8 + .8(f_{kt-1}^2 + \omega_{t-1})\lambda_{t-1}^{-1} + .8\nu_{t-1},$$

which can be regarded as a multivariate version of expression (7) in Demos and Sentana (1998).[8] We exploit the results in Mencía and Sentana (2005) to simulate standardised versions of all these distributions by appropriately mixing a 6-dimensional spherical normal vector with a univariate gamma random variable, which we obtain from the NAG Fortran 77 Mark 19 library routines G05DDF and G05FFF, respectively (see Numerical Algorithm Group (2001) for details). With the objective of speeding up the computations, we systematically resort to Cholesky decompositions to factorise Σ_t. As explained at the end of section 5.1, this choice is inconsequential for all simulated distributions except the asymmetric t, and all estimators except the SP one. Although we have considered other sample sizes, for the sake of brevity we only report the results for $T = 1,000$ observations (plus another 100 for initialisation) based on 10,000 Monte Carlo replications. This sample size corresponds roughly to 20 years of weekly data, or 4 years of daily data.

Our ML estimation procedure employs the following numerical strategy. First, we estimate the conditional mean and variance parameters $\boldsymbol{\theta}$ under normality with a scoring algorithm that combines the E04LBF routine with the analytical expressions for the score in Appendix B and the $\mathcal{A}(\boldsymbol{\phi}_0)$ matrix in Proposition 2. Then, we compute the sequential MM estimator $\breve{\eta}_T$ in (18), which we use as initial value for a univariate optimisation procedure that obtains the sequential ML estimator $\tilde{\eta}_T$ in Proposition 4 with the E04ABF routine. This estimator, together with the PML of $\boldsymbol{\theta}$, become the initial values for the t-based ML estimators, which are obtained with the same scoring algorithm as the PML estimator, but this time using the analytical expressions for the information matrix $\mathcal{I}(\boldsymbol{\phi}_0)$ in Proposition 1. We rule out numerically problematic solutions by imposing the inequality constraints $|\rho_i| \leq .999$ and $\gamma_i \geq 10^{-10}$ for $i = 1, \ldots, N$, $\alpha \geq 10^{-4}$, $\beta \geq 0$, $\alpha + \beta \leq .999$ and $0 \leq \eta \leq .499$.[9] Given that the scale of the common factor is free, we set $\lambda = 1$ in estimation for computational convenience but report results for the alternative normalisation $c_1 = 1$.

[8] A direct application of the formulas in Demos and Sentana (1998, sect.3.1) yields $\inf_t \nu_t = 4$ and $E(\nu_t) = 8$.

[9] We implicitly impose the restrictions on α and β by numerically maximising the Gaussian and t log-likelihood functions with respect to θ_I^* and θ_{II}^* subject to the restrictions $10^{-4} \leq \theta_I^* \leq .999$ and $0 \leq \theta_{II}^* \leq .999$, where $\beta = \theta_I^*\theta_{II}^*$ and $\alpha = \theta_I^*(1 - \theta_{II}^*)$. Nevertheless, we always compute scores and information bounds in terms of α and β, using the chain rule for derivatives whenever necessary.

Computational details for the two semiparametric procedures can be found in Appendix B. Given that a proper cross-validation procedure is extremely costly to implement in a Monte Carlo exercise with $N = 6$, we have done some experimentation to choose "optimal" bandwidths by scaling up and down the automatic choices given in Silverman (1986).[10]

6.2 Sampling distributions of estimators

Figures 1A-1F display box-plots with the sampling distributions of the Gaussian- and t-based ML estimators, and the two semiparametric ones. In the case of vector parameters, we report the values corresponding to the third series. As usual, the central boxes describe the first and third quartiles of the sampling distributions, as well as their median. The maximum length of the whiskers is one interquartile range. Finally, we also report the fraction of estimates outside those whiskers to complement the information on the tails of the distributions.

As expected from Proposition 9.1, the distribution of the four estimators is essentially identical under normality across all the parameters, with the only exception of the SP estimator of γ_3, which is not very surprising given that the ML and PML are numerically identical over half the time. However, they progressively differ under correct student t specification as the degrees of freedom decrease.

Another thing to note is that the sampling distributions of the Gaussian PML estimators of π_3 and ρ_3 do not seem to be affected much by the true conditional distribution of the data, which suggests that the different information bounds of the simulated model are almost block diagonal between the conditional mean parameters $(\boldsymbol{\pi}, \boldsymbol{\rho})$ and the rest. The same seems to be true for the SP estimator of π_3, which is in line with Proposition 11, and essentially reflects the fact that there is no SP adjustment for unconditional means. In contrast, the behaviour of the SP estimator of the autoregressive coefficient ρ_3 described in Figure 1B is very much at odds with the same proposition, probably as a result of the fact that the adjustment of this parameter described in (22) becomes very noisy once we replace the unknown score by the one obtained with the multivariate kernel estimator.

On the other hand, the sampling distributions of the SSP and t-based ML estimators of π_3 and ρ_3 are quite sensitive to the nature of the underlying distribution. In particular, when the true distribution is elliptical, the sampling distributions of those estimators are narrower than the distributions of the PML and SP estimators. This is particularly noticeable in the t_4 case, but also in the normal-gamma case, for which the ML estimator should lose its asymptotic

[10] We considered .3, .5, .8, 1, 1.25, 1.5, 2, 2.5, 3 and 4 times the bandwidth $[4/(N+2)]^{1/(N+4)} \cdot s \cdot T^{-1/(N+4)}$ recommended by Silverman (1986) for multivariate density estimation under normality, where s^2 is the second sample moment of $\varepsilon^*_{it}(\tilde{\boldsymbol{\theta}}_T)$ averaged across t and i in the case of the SP estimator, and the sample variance of $\sqrt[3]{\varsigma_t(\tilde{\boldsymbol{\theta}}_T)}$ in the case of the SSP estimator. The reported results use scaling factors of 1.25 (SSP) and 2.5 (SP).

efficiency but not its consistency according to Proposition 16. At the same time, an asymmetric distribution introduces substantial positive biases in the ML and SSP estimators of π_3. Intuitively, since the true distribution of the standardised innovations is negatively skewed, those estimators are re-centring their estimated distributions so as to make them more symmetric. Somewhat surprisingly, though, the biases in the unconditional mean seem to go a long way in mopping up the biases in the autocorrelation coefficients. As for time-varying kurtosis, it seems to have little effect on the estimators of the two conditional mean parameters that we analyse, with results that broadly resemble the ones obtained for the t_8.

Unlike what happens with the conditional mean parameters, the sampling distributions of the PML estimators of both the static variance parameters c_3 and γ_3, and the dynamic variance parameters α and β are quite sensitive to the distribution of the innovations. In this sense, the first thing to note is that those sampling distributions deteriorate as the distribution of the standardised innovations becomes more leptokurtic. In fact, when $\nu_0 = 4$ the shape of the distribution of the PML estimators of the Arch and Garch parameters is clearly non-standard, as discussed after Proposition 2. On the other hand, the PML estimators of α and β are the least affected by the existence of time-varying higher order moments. The SP estimators of the conditional variance parameters also suffer when κ_0 increases, becoming substantially downward biased in the case of γ_3, as well as in the case of α when the innovations are t_4.

In contrast, the ML estimators of the conditional variance parameters behave very much as expected: there are substantial efficiency gains when the distribution of the innovations coincides with the assumed one, and some noticeable biases when it does not. However, it is interesting to note that those biases only affect γ_3 and α in the normal-gamma case, and α and β in the time-varying leptokurtic case. The unbiasedness results that we obtain with the asymmetric t are somewhat remarkable, and suggest once again that the biases in the unconditional mean that we observe in Figure 1A adequately re-centre the estimated distribution of the innovations.

The behaviour of the SSP estimators of the conditional variance parameters is mixed. When the distribution is elliptical, this estimator does a reasonably good job, although by no means does it achieve the efficiency of the ML estimator. This is especially true in the case of t_4 innovations, when it also shares a downward bias for α with the SP estimator. Like the ML estimators, though, the SSP estimators also seem somewhat resilient to misspecification, since the only noticeable biases correspond to γ_3 for the asymmetric student t, and α and β for the t distribution with time-varying degrees of freedom.

Model (27) can be easily reparametrised as in (28) if we ignore the small adjustment term $\omega_{t-j}(\boldsymbol{\theta})$ in (40). For instance, we can choose ϑ_2 to be the cross-sectional average of the idiosyncratic variances ($=\boldsymbol{\gamma}'\boldsymbol{\iota}_N/N$), and then re-scale λ, α and the elements of $\boldsymbol{\gamma}$ accordingly. Figures 1G and 1H display box-plots of γ_3/ϑ_2 and α/ϑ_2. As can be seen, the t-based ML estimators of these two derived parameters become consistent when the true distribution is normal-gamma, which confirms Proposition 16.a (see also Thm.1 in Newey and Steigerwald (1997)). But contrary to the asymptotic results in Proposition 12.a, they seem to be at least as efficient as the SSP estimator in that case. Similarly, the SSP estimators also seem to be consistent in the case of the asymmetric student t, but the downward bias that affects α when the distribution is t_4 continues to contaminate α/ϑ_2.

Finally, Figure 2 displays box-plots of the sampling distributions of the ML, sequential ML and sequential MM estimators of η centred around their true values when $\nu_0 = \infty$, 8 or 4, or around the pseudo-true values implied by the sequential ML procedure when the *i.i.d.* t assumption is incorrect. The first thing to note is that the proportions of zero estimates of η exceed the theoretical value of 1/2 when $\eta_0 = 0$. Although the three estimators behave similarly under Gaussianity, they are radically different in the other two correctly specified cases. As explained in Section 4, while $\hat{\eta}_T$ is asymptotically normally distributed in those two cases, $\breve{\eta}_T$ has a non-standard asymptotic distribution when $\nu_0 = 8$ or $\nu_0 = 4$, and the same applies to $\tilde{\eta}_T$ in the latter case. The sampling distributions are also very different in the case of the normal-gamma, but less so in the case of the asymmetric student t or the t with time-varying degrees of freedom. In this sense, the main effects of ν_t moving around its average value of 8 (see footnote 8) seem to be small increases in the medians and dispersions of the estimated tail thickness parameters relative to the *i.i.d.* t_8. case, probably due to the increase in higher order moments that a time-varying kurtosis entails.

6.3 Hausman tests

Following our discussion on power in section 5.2, we focus our attention on two parameters only: the cross-sectional mean of the unconditional mean parameters $\pi's$ and the cross-sectional mean of the idiosyncratic variances $\gamma's$. In the remaining of this section, we shall refer to those two average parameters as $\bar{\pi}$ and $\bar{\gamma}$. The Wald version of single coefficient tests is straightforward. The LM version is also easy to obtain if we use the results in the proofs of Propositions 18 and 19 to show that

$$\begin{aligned}\sqrt{T}(\tilde{\boldsymbol{\theta}}_T - \hat{\boldsymbol{\theta}}_T) - \mathcal{A}^{-1}(\boldsymbol{\phi}_0)\sqrt{T}\bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\hat{\boldsymbol{\theta}}_T, 0) &= o_p(1),\\ \sqrt{T}(\tilde{\boldsymbol{\theta}}_T - \mathring{\boldsymbol{\theta}}_T) - \mathcal{A}^{-1}(\boldsymbol{\phi}_0)\sqrt{T}\bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\mathring{\boldsymbol{\theta}}_T, 0) &= o_p(1).\end{aligned}$$

To simplify the comparisons between parametric and semiparametric testing procedures, we systematically use the PML estimator of $\boldsymbol{\theta}$ in computing the different information bounds. We also use the sequential MM estimator of η in (18), which amounts to replacing κ_0 by its sample analogue when it is positive. We provide further details on how we compute the SSP bound $\mathring{\mathcal{S}}(\boldsymbol{\phi}_0)$ in Appendix B.

The first two panels of Table 1 report the fraction of simulations in which the parametric and SSP Hausman tests in Propositions 18 and 19, respectively, exceed the 1, 5 and 10% critical values of a χ_1^2 when the true distribution is a student t_8, while the last panel reports the corresponding fractions for the SSP test in the normal-gamma case. All tests tend to overreject, but the size distortions of the parametric tests are typically small, especially if compared to the huge distortions shown by the SSP Hausman procedures based on $\bar{\gamma}$. Although the estimators of $\mathring{\mathcal{S}}(\boldsymbol{\phi}_0)$ are noisier than the estimators of $\mathcal{I}(\boldsymbol{\phi}_0)$ or $\mathcal{C}(\boldsymbol{\phi}_0)$, the main problem with the SSP tests is that the difference between the Monte Carlo variances of the PML estimators of $\bar{\pi}$ and $\bar{\gamma}$ and its asymptotically efficient SSP counterparts is smaller than the Monte Carlo variance of the difference between those two estimators, which violates the principle underlying Hausman tests. In fact, the Monte Carlo variance of the SSP estimator of $\bar{\gamma}$ turns out to be higher than that of the PML estimator both in the case of the student t_8 and the normal-gamma mixture, despite the fact that the Monte Carlo variances of the estimators of the individual $\gamma_i's$ are in the correct order, which suggests that the SSP estimators of the $\gamma_i's$ have a more positive cross-sectional correlation. Monte Carlo experiments with $T = 10,000$ indicate, though, that those problems are mitigated as the first-order asymptotic results become more representative.

Table 2 contains the fraction of simulations in which the parametric (upper panels) and SSP (lower panels) Hausman tests exceed the 1, 5 and 10% empirical critical values obtained by simulation when the true distribution is a student t_8 (see Table 1).

As expected, the parametric test based on $\bar{\pi}$ has little power when the true distribution is normal-gamma, which is not surprising given that the ML estimators of the conditional mean parameters are consistent, but no longer efficient, in that case. In contrast, the power is essentially 1 if we base the test on the idiosyncratic variance parameter $\bar{\gamma}$. In the case of the asymmetric t, though, the parametric Hausman tests based on the unconditional means have substantially more power than the tests based on the unconditional idiosyncratic variances, which is also in line with the Monte Carlo distributions presented in the previous section. Finally, neither of those parameters is useful to detect a t distribution with time-varying degrees of freedom.

On its part, the SSP Hausman test based on $\bar{\pi}$ and $\bar{\gamma}$ have a lot of power to detect departures in the asymmetric direction, but again no power against time-varying kurtosis. The odd size-adjusted power results observed at the 1% level simply reflect the imprecision of the estimated Monte Carlo critical values.

7 Conclusions

In the context of a general multivariate dynamic regression model with time-varying variances and covariances, we compare the efficiency of the feasible ML procedure that jointly estimates the shape parameters with the efficiency of the infeasible ML, SSP, SP and Gaussian PML estimators of the conditional mean and variance parameters considered in the existing literature. In this respect, we show that if the distribution of the standardised innovations is *i.i.d.* spherical, the ranking is infeasible ML, feasible ML, SSP, SP and PML, with equality if and only if the spherical distribution is in fact Gaussian, in which case there is no efficiency loss in simultaneously estimating the shape parameters. In this respect, our results generalise earlier findings by Gonzalez-Rivera and Drost (1999), FSC and Hafner and Rombouts (2007).

Furthermore, we study in detail two popular examples of conditionally heteroskedastic models, one univariate and the other one multivariate, and obtain closed-formed expressions for the inefficiency ratios of different subsets of parameters under the assumption of constant variances. Not surprisingly, those inefficiency ratios coincide with the ratios of the non-centrality parameters of the tests of conditional homoskedasticity associated with the different estimators.

More generally, we show that the SSP estimator is adaptive for all but one global scale parameter in an appropriate reparametrisation of the model. This result directly generalises the one obtained for univariate GARCH models by Linton (1993), as well as the results in Hodgson and Vorkink (2003) for a specific multivariate GARCH-M model. We also show that the general SP estimator is adaptive for a much more restricted set of parameters in an alternative reparametrisation that only seems to fit the constant conditional correlation model of Bollerslev (1987) when the conditional mean is 0. This second result generalises the ones obtained for specific univariate GARCH models by Drost and Klaassen (1997) and Sun and Stengos (2006), which seem overly simple from a multivariate perspective. Importantly, we prove that both semiparametric estimators share a saddle point efficiency property, in that they are as inefficient as the Gaussian PMLE for the parameters that they cannot estimate adaptively.

We also thoroughly analyse the effects of distributional misspecification on the consistency of the conditional mean and variance parameters. In particular, we initially show that when the conditional distribution is platykurtic, so that the coefficient of multivariate excess kurtosis is negative, the feasible ML estimators based on the multivariate student distribution converge to the Gaussian PML estimators. On the other hand, we show that when the conditional distribution is spherical and leptokurtic, but neither t nor Gaussian, the feasible student t-based ML estimator is consistent for exactly the same parameters for which the SSP estimator is adaptive, which are effectively all but a global scale factor. This result generalises Theorem 1 in

Newey and Steigerwald (1997), which applies to univariate models. Furthermore, we show that when the conditional distribution is leptokurtic but not spherical, the feasible ML estimator is consistent for exactly the same restricted subset of parameters for which the general SP estimator is adaptive, which excludes both the mean and the covariance matrix of the *i.i.d.* pseudo-standardised innovations. This second result also generalises Theorem 2 in Newey and Steigerwald (1997), which again looks misleadingly simple from a multivariate perspective. We would also like to emphasise that our inconsistency results apply not only to the multivariate student t log-likelihood, but also to any other spherically-based likelihood estimators. The main advantage of the student t for our purposes is that we can make explicit its limiting relationship to the Gaussian distribution. In any case, we provide closed-form expressions for consistent estimators of the parameters that the feasible ML estimator cannot estimate consistently.

In view of the importance of the distributional assumptions, we propose simple Hausman tests that compare the feasible ML and SSP estimators to the Gaussian PML ones.

Finally, we also consider sequential estimators of the shape parameters, which can be easily obtained from the standardised innovations evaluated at the Gaussian PML estimators. In particular, we consider a sequential ML estimator, as well as sequential MM estimators based on the coefficient of multivariate excess kurtosis. The main advantage of such estimators is that they preserve the consistency of the conditional mean and variance functions, but at the same time allow for a more realistic conditional distribution. We show that the usual efficiency ranking of the estimators of the shape parameters is infeasible ML, feasible ML, sequential ML and sequential MM. These results are important in practice because empirical researchers often want to go beyond the first two conditional moments, which implies that one cannot simply treat the shape parameters as if they were nuisance parameters. We also propose an alternative Hausman test that compares the feasible and sequential ML estimator of the shape parameters.

In a detailed Monte Carlo experiment we find that there is a substantial difference between the estimation of the following four groups of parameters: (a) the unconditional mean parameters, (b) the unconditional variance parameters, (c) the dynamic mean parameters, and (d) the dynamic variance parameters. We also find that the finite sample performance of the semiparametric procedures is not well approximated by the first-order asymptotic theory that justifies them. This is particularly true of the SP estimators of the dynamic mean and variance parameters, but also affects the SSP estimators of the latter. As for the feasible ML estimators based on the student t, we find that they offer substantial efficiency gains relative to the PML estimators when the true distribution coincides with the one assumed for estimation purposes, but they be biased otherwise. Nevertheless, we find that the biases seem to be limited to the unconditional mean parameters when the true distribution is asymmetric, and the variance parameters when it is elliptical but not t. In this second case, our simulation results also confirm that we can obtain consistent estimators of all parameters but one by using one of the reparametrisations previously discussed.

As for the Hausman tests, we find that the one based on the feasible ML estimator works quite well, both in terms of size and power, while the one based on the SSP estimator suffers from substantial size distortions when we base it on the unconditional variance parameters. In this sense, it would be useful to explore bootstrap procedures that exploit the fact that elliptical distributions are parametric in $N-1$ dimensions, and non-parametric in only one.

Further work is required in at least four other directions. First, from a modelling point of view, the assumption of *i.i.d.* innovations in non-spherical multivariate models seems rather strong, for it forces the conditional distribution of the observed variables to depend on the choice of square root matrix used to obtain the underlying innovations from the observations. Secondly, from an estimation point of view, the development of semiparametric estimators that do not require the assumption of *i.i.d.* innovations remains an important unresolved issue that merits further investigation. Thirdly, the availability of analytical finite sample results would probably make the choice between bias and efficiency look more balanced than what standard root-T asymptotics suggests. Finally, the existing literature, including our paper, places too much emphasis on parameter estimation, while practitioners are often more interested in functionals of the conditional distribution, such as the forecasting intervals required in value at risk calculations. An evaluation of the consequences that the different estimation procedures that we have considered have for such objects constitutes a fruitful avenue for future research.

Appendix

A Proofs and auxiliary results

Some useful distribution results

A spherically symmetric random vector of dimension N, $\boldsymbol{\varepsilon}_t^\circ$, is fully characterised in Theorem 2.5 (iii) of Fang, Kotz and Ng (1990) as $\boldsymbol{\varepsilon}_t^\circ = e_t \mathbf{u}_t$, where $\mathbf{u}_t$ is uniformly distributed on the unit sphere surface in $\mathbb{R}^N$, and e_t is a non-negative random variable independent of $\mathbf{u}_t$, whose distribution determines the distribution of $\boldsymbol{\varepsilon}_t^\circ$. The variables e_t and $\mathbf{u}_t$ are referred to as the generating variate and the uniform base of the spherical distribution. Assuming that $E(e_t^2) < \infty$, we can standardise $\boldsymbol{\varepsilon}_t^\circ$ by setting $E(e_t^2) = N$, so that $E(\boldsymbol{\varepsilon}_t^\circ) = \mathbf{0}$, $V(\boldsymbol{\varepsilon}_t^\circ) = \mathbf{I}_N$. Specifically, if $\boldsymbol{\varepsilon}_t^\circ$ is distributed as a standardised multivariate student t random vector of dimension N with ν_0 degrees of freedom, then $e_t = \sqrt{(\nu_0 - 2)\zeta_t/\xi_t}$, where ζ_t is a chi-square random variable with N degrees of freedom, and ξ_t is an independent Gamma variate with mean $\nu_0 > 2$ and variance $2\nu_0$. If we further assume that $E(e_t^4) < \infty$, then the coefficient of multivariate excess kurtosis κ_0, which is given by $E(e_t^4)/[N(N+2)] - 1$, will also be bounded. For instance, $\kappa_0 = 2/(\nu_0 - 4)$ in the student t case with $\nu_0 > 4$, and $\kappa_0 = 0$ under normality. In this respect, note that since $E(e_t^4) \geq E^2(e_t^2) = N^2$ by the Cauchy-Schwarz inequality, with equality if and only if $e_t = \sqrt{N}$ so that $\boldsymbol{\varepsilon}_t^\circ$ is proportional to $\mathbf{u}_t$, then $\kappa_0 \geq -2/(N+2)$, the minimum value being achieved in the uniformly distributed case.

Then, it is easy to combine the representation of elliptical distributions above with the higher order moments of a multivariate normal vector in Balestra and Holly (1990) to prove that the third and fourth moments of a spherically symmetric distribution with $V(\boldsymbol{\varepsilon}_t^\circ) = \mathbf{I}_N$ are given by

$$E(\boldsymbol{\varepsilon}_t^\circ \boldsymbol{\varepsilon}_t^{\circ\prime} \otimes \boldsymbol{\varepsilon}_t^\circ) = \mathbf{0}, \tag{A1}$$

$$E(\boldsymbol{\varepsilon}_t^\circ \boldsymbol{\varepsilon}_t^{\circ\prime} \otimes \boldsymbol{\varepsilon}_t^\circ \boldsymbol{\varepsilon}_t^{\circ\prime}) = E[vec(\boldsymbol{\varepsilon}_t^\circ \boldsymbol{\varepsilon}_t^{\circ\prime}) vec'(\boldsymbol{\varepsilon}_t^\circ \boldsymbol{\varepsilon}_t^\circ)] = (\kappa_0 + 1)[(\mathbf{I}_{N^2} + \mathbf{K}_{NN}) + vec(\mathbf{I}_N)\, vec'(\mathbf{I}_N)]. \tag{A2}$$

We shall also make use of the fact that in the student t case $\zeta_t/(\xi_t + \zeta_t)$ has a beta distribution with parameters $N/2$ and $\nu_0/2$, which is independent of $\mathbf{u}_t$. As is well known, if a random variable X defined over $[0, 1]$ has a beta distribution with parameters (a, b), where $a > 0$, $b > 0$, then its density function is

$$f_X(x; a, b) = \frac{1}{B(a, b)} x^{a-1}(1 - x)^{b-1},$$

where

$$B(a, b) = \int_0^1 x^{a-1}(1 - x)^{b-1} dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}$$

is the usual beta function. Fortunately, it is often trivial to find apparently complex moments of a beta random variable from first principles. For instance,

$$E[X^p(1-X)^q|a,b] = \frac{1}{B(a,b)}\int_0^1 x^p(1-x)^q x^{a-1}(1-x)^{b-1}dx = \frac{B(a+p,b+q)}{B(a,b)}$$

for any real values of p and q such that $a+p>0$ and $b+q>0$. Similarly, since

$$\int_0^1 \ln(1-x)x^{a+p-1}(1-x)^{b-1}dx = \frac{\partial}{\partial b}\int_0^1 x^{a+p-1}(1-x)^{b-1}dx = \frac{\partial}{\partial b}B(a+p,b),$$

we can also write

$$\begin{aligned} E[X^p(1-X)^q\ln(1-X)|a,b] &= \frac{B(a+p,b+q)}{B(a,b)}\frac{\partial \ln B(a+p,b+q)}{\partial b} \\ &= \frac{B(a+p,b+q)}{B(a,b)}\left[\psi(b+q)-\psi(a+p+b+q)\right], \end{aligned}$$

thanks to the definition of the beta function in terms of the gamma function above.

Lemmata

Lemma 1 *Let ς denote a scalar random variable with continuously differentiable density function $h(\varsigma;\boldsymbol{\eta})$ over the possibly infinite domain $[a,b]$, and let $m(\varsigma)$ denote a continuously differentiable function over the same domain such that $E[m(\varsigma)|\boldsymbol{\eta}] = k(\boldsymbol{\eta}) < \infty$. Then*

$$E\left[\partial m(\varsigma)/\partial\varsigma|\,\boldsymbol{\eta}\right] = -E\left[m(\varsigma)\partial\ln h(\varsigma;\boldsymbol{\eta})/\partial\varsigma|\,\boldsymbol{\eta}\right],$$

as long as the required expectations are defined and bounded.

Proof. If we differentiate

$$k(\boldsymbol{\eta}) = E\left[m(\varsigma)|\,\boldsymbol{\eta}\right] = \int_a^b m(\varsigma)h(\varsigma;\boldsymbol{\eta})d\varsigma$$

with respect to ς, we get

$$0 = \int_a^b \frac{\partial m(\varsigma)}{\partial\varsigma}h(\varsigma;\boldsymbol{\eta})d\varsigma + \int_a^b m(\varsigma)\frac{\partial h(\varsigma;\boldsymbol{\eta})}{\partial\varsigma}d\varsigma = \int_a^b \frac{\partial m(\varsigma)}{\partial\varsigma}h(\varsigma;\boldsymbol{\eta})d\varsigma + \int_a^b m(\varsigma)h(\varsigma;\boldsymbol{\eta})\frac{\partial\ln h(\varsigma;\boldsymbol{\eta})}{\partial\varsigma}d\varsigma,$$

as required. □

Proposition 1

For our purposes it is convenient to rewrite $\mathbf{e}_{dt}(\boldsymbol{\phi}_0)$ as

$$\begin{aligned} \mathbf{e}_{lt}(\boldsymbol{\phi}_0) &= \delta[\varsigma_t(\boldsymbol{\theta}_0),\boldsymbol{\eta}_0]\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0) = \delta(\varsigma_t,\boldsymbol{\eta}_0)\sqrt{\varsigma_t}\mathbf{u}_t, \\ \mathbf{e}_{st}(\boldsymbol{\phi}_0) &= vec\left\{\delta[\varsigma_t(\boldsymbol{\theta}_0),\boldsymbol{\eta}_0]\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0) - \mathbf{I}_N\right\} = vec\left[\delta(\varsigma_t,\boldsymbol{\eta}_0)\varsigma_t\mathbf{u}_t\mathbf{u}_t' - \mathbf{I}_N\right], \end{aligned}$$

where ς_t and $\mathbf{u}_t$ are mutually independent for any standardised spherical distribution, with $E(\mathbf{u}_t) = \mathbf{0}$, $E(\mathbf{u}_t\mathbf{u}_t') = N^{-1}\mathbf{I}_N$, $E(\varsigma_t) = N$ and $E(\varsigma_t^2) = N(N+2)(\kappa_0+1)$. Importantly, we only need to compute unconditional moments because ς_t and $\mathbf{u}_t$ are independent of $\mathbf{z}_t$ and I_{t-1} by assumption. Then, it easy to see that

$$E[\mathbf{e}_{lt}(\boldsymbol{\phi}_0)] = E[\delta(\varsigma_t, \boldsymbol{\eta}_0)\sqrt{\varsigma_t}] \cdot E(\mathbf{u}_t) = \mathbf{0},$$

and that

$$E[\mathbf{e}_{st}(\boldsymbol{\phi}_0)] = vec\left\{E\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)\varsigma_t\right] \cdot E(\mathbf{u}_t\mathbf{u}_t') - \mathbf{I}_N\right\} = vec(\mathbf{I}_N)\left\{E\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)(\varsigma_t/N)\right] - 1\right\}.$$

In this context, we can use expression (2.21) in Fang, Kotz and Ng (1990) to write the density function of ς_t as

$$h(\varsigma_t; \boldsymbol{\eta}) = \frac{\pi^{N/2}}{\Gamma(N/2)}\varsigma_t^{N/2-1}\exp[c(\boldsymbol{\eta}) + g(\varsigma_t, \boldsymbol{\eta})], \tag{A3}$$

whence

$$[\delta(\varsigma_t, \boldsymbol{\eta}_0)(\varsigma_t/N) - 1] = -\frac{2}{N}\left[1 + \varsigma_t \cdot \partial \ln h(\varsigma_t; \boldsymbol{\eta})/\partial\varsigma\right]. \tag{A4}$$

On this basis, we can use Lemma 1 to show that $E(\varsigma_t) = N < \infty$ implies

$$E\left[\varsigma_t \cdot \partial \ln h(\varsigma_t; \boldsymbol{\eta})/\partial\varsigma\right] = -E\left[1\right] = -1,$$

which in turn implies that

$$E\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)(\varsigma_t/N) - 1\right] = 0 \tag{A5}$$

in view of (A4). Consequently, $E[\mathbf{e}_{st}(\boldsymbol{\phi}_0)] = \mathbf{0}$, as required.

Similarly, we can also show that

$$\begin{aligned} E[\mathbf{e}_{lt}(\boldsymbol{\phi}_0)\mathbf{e}_{lt}'(\boldsymbol{\phi}_0)] &= E\left\{\delta^2(\varsigma_t, \boldsymbol{\eta}_0)\varsigma_t\mathbf{u}_t\mathbf{u}_t'\right\} = \mathbf{I}_N \cdot E[\delta^2(\varsigma_t, \boldsymbol{\eta}_0)(\varsigma_t/N)], \\ E[\mathbf{e}_{lt}(\boldsymbol{\phi}_0)\mathbf{e}_{st}'(\boldsymbol{\phi}_0)] &= E\left\{\delta(\varsigma_t, \boldsymbol{\eta}_0)\sqrt{\varsigma_t}\mathbf{u}_t vec'\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)\varsigma_t\mathbf{u}_t\mathbf{u}_t' - \mathbf{I}_N\right]\right\} = \mathbf{0} \end{aligned}$$

by virtue of (A1), and

$$\begin{aligned} E[\mathbf{e}_{st}(\boldsymbol{\phi}_0)\mathbf{e}_{st}'(\boldsymbol{\phi}_0)] &= E\left\{vec\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)\varsigma_t\mathbf{u}_t\mathbf{u}_t' - \mathbf{I}_N\right] vec'\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)\varsigma_t\mathbf{u}_t\mathbf{u}_t' - \mathbf{I}_N\right]\right\} \\ &= E\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)\varsigma_t\right]^2 \frac{1}{N(N+2)}[(\mathbf{I}_{N^2} + \mathbf{K}_{NN}) + vec\left(\mathbf{I}_N\right) vec'\left(\mathbf{I}_N\right)] \\ &\quad -2E\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)(\varsigma_t/N)\right] vec\left(\mathbf{I}_N\right) vec'\left(\mathbf{I}_N\right) + vec\left(\mathbf{I}_N\right) vec'\left(\mathbf{I}_N\right) \\ &= \frac{N}{(N+2)}E\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)(\varsigma_t/N)\right]^2(\mathbf{I}_{N^2} + \mathbf{K}_{NN}) \\ &\quad + \left\{\frac{N}{(N+2)}E\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)(\varsigma_t/N)\right]^2 - 1\right\} vec\left(\mathbf{I}_N\right) vec'\left(\mathbf{I}_N\right)] \end{aligned}$$

by virtue of (A2), (A4) and (A5).

Finally, it is clear from (3) that $\mathbf{e}_{rt}(\boldsymbol{\phi}_0)$ will be a function of ς_t but not of $\mathbf{u}_t$, which immediately implies that $E[\mathbf{e}_{lt}(\boldsymbol{\phi}_0)\mathbf{e}_{rt}'(\boldsymbol{\phi}_0)] = \mathbf{0}$, and that

$$\begin{aligned} E[\mathbf{e}_{st}(\boldsymbol{\phi}_0)\mathbf{e}_{rt}'(\boldsymbol{\phi}_0)] &= E\left\{vec\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)\varsigma_t \cdot \mathbf{u}_t\mathbf{u}_t' - \mathbf{I}_N\right]\mathbf{e}_{rt}'(\boldsymbol{\phi}_0)\right\} \\ &= vec(\mathbf{I}_N)E\left\{\left[\delta(\varsigma_t, \boldsymbol{\eta}_0)(\varsigma_t/N) - 1\right]\mathbf{e}_{rt}'(\boldsymbol{\phi}_0)\right\}. \end{aligned}$$

To obtain the expected value of the Hessian, it is also convenient to write $\mathbf{h}_{\boldsymbol{\theta\theta}t}(\boldsymbol{\phi}_0)$ in (8) as

$$\begin{gathered}
-4\mathbf{Z}_{st}(\boldsymbol{\theta}_0)[\mathbf{I}_N \otimes \{\delta[\varsigma_t(\boldsymbol{\theta}_0), \eta_0]\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0) - \mathbf{I}_N\}]\mathbf{Z}_{st}'(\boldsymbol{\theta}_0) \\
+[\mathbf{e}_{lt}'(\boldsymbol{\theta}_0, \boldsymbol{\eta}_0)\boldsymbol{\Sigma}_t^{-1/2\prime}(\boldsymbol{\theta}) \otimes \mathbf{I}_p]\frac{\partial vec}{\partial \boldsymbol{\theta}'}\left[\frac{\partial \boldsymbol{\mu}_t'(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right] \\
+\frac{1}{2}\{\mathbf{e}_{st}'(\boldsymbol{\theta}_0, \boldsymbol{\eta}_0)[\boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\theta}_0) \otimes \boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\theta}_0)] \otimes \mathbf{I}_p\}\frac{\partial vec}{\partial \boldsymbol{\theta}'}\left\{\frac{\partial vec'[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})]}{\partial \boldsymbol{\theta}}\right\} \\
-2\mathbf{Z}_{lt}(\boldsymbol{\theta}_0)[\mathbf{e}_{lt}'(\boldsymbol{\theta}_0, \boldsymbol{\eta}_0) \otimes \mathbf{I}_N]\mathbf{Z}_{st}'(\boldsymbol{\theta}_0) - 2\mathbf{Z}_{st}(\boldsymbol{\theta}_0)[\mathbf{e}_{lt}(\boldsymbol{\theta}_0, \boldsymbol{\eta}_0) \otimes \mathbf{I}_N]\mathbf{Z}_{lt}'(\boldsymbol{\theta}_0) \\
-\delta[\varsigma_t(\boldsymbol{\theta}_0), \boldsymbol{\eta}_0]\mathbf{Z}_{lt}(\boldsymbol{\theta}_0)\mathbf{Z}_{lt}'(\boldsymbol{\theta}_0) - 2\mathbf{Z}_{st}(\boldsymbol{\theta}_0)\mathbf{Z}_{st}'(\boldsymbol{\theta}_0) - \frac{2\partial\delta[\varsigma_t(\boldsymbol{\theta}_0), \boldsymbol{\eta}_0]}{\partial \varsigma}\{\mathbf{Z}_{lt}(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)\mathbf{Z}_{lt}'(\boldsymbol{\theta}_0) \\
+\mathbf{Z}_{lt}(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)vec'[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)]\mathbf{Z}_{st}'(\boldsymbol{\theta}_0) + \mathbf{Z}_{st}(\boldsymbol{\theta}_0)vec[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)]\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\mathbf{Z}_{lt}'(\boldsymbol{\theta}_0) \\
+ \mathbf{Z}_{st}(\boldsymbol{\theta}_0)vec[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)]vec'[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)]\mathbf{Z}_{st}'(\boldsymbol{\theta}_0)\}.
\end{gathered}$$

Clearly, the first four lines have zero conditional expectation, and the same is true of the sixth line by virtue of (A1). As for the remaining terms, we can write them as

$$\begin{aligned}
&-\delta(\varsigma_t, \boldsymbol{\eta}_0)\mathbf{Z}_{lt}(\boldsymbol{\theta}_0)\mathbf{Z}_{lt}'(\boldsymbol{\theta}_0) - 2\partial\delta(\varsigma_t, \boldsymbol{\eta}_0)/\partial\varsigma \cdot \mathbf{Z}_{lt}(\boldsymbol{\theta}_0)\varsigma_t\mathbf{u}_t\mathbf{u}_t'\mathbf{Z}_{lt}'(\boldsymbol{\theta}_0) \\
&-2\mathbf{Z}_{st}(\boldsymbol{\theta}_0)\mathbf{Z}_{st}'(\boldsymbol{\theta}_0) - 2\partial\delta(\varsigma_t, \boldsymbol{\eta}_0)/\partial\varsigma \cdot \varsigma_t^2\mathbf{Z}_{st}(\boldsymbol{\theta}_0)vec(\mathbf{u}_t\mathbf{u}_t')vec'(\mathbf{u}_t\mathbf{u}_t')\mathbf{Z}_{st}'(\boldsymbol{\theta}_0),
\end{aligned}$$

whose conditional expectation will be

$$\begin{aligned}
&-\mathbf{Z}_{lt}(\boldsymbol{\theta}_0)\mathbf{Z}_{lt}'(\boldsymbol{\theta}_0)E[\delta(\varsigma_t; \boldsymbol{\eta}_0) + 2(\varsigma_t/N) \cdot \partial\delta(\varsigma_t, \boldsymbol{\eta}_0)/\partial\varsigma|\boldsymbol{\eta}_0] - 2\mathbf{Z}_{st}(\boldsymbol{\theta}_0)\mathbf{Z}_{st}'(\boldsymbol{\theta}_0) \\
&-\mathbf{Z}_{st}(\boldsymbol{\theta}_0)\frac{2E[\varsigma_t^2 \cdot \partial\delta(\varsigma_t, \boldsymbol{\eta}_0)/\partial\varsigma|\boldsymbol{\eta}_0]}{N(N+2)}[(\mathbf{I}_{N^2} \otimes \mathbf{K}_{NN}) + vec(\mathbf{I}_N)vec'(\mathbf{I}_N)]\mathbf{Z}_{st}'(\boldsymbol{\theta}_0).
\end{aligned}$$

As for $\mathbf{h}_{\boldsymbol{\theta\eta}t}(\boldsymbol{\phi}_0)$, it follows from (9) and (4) that we can write it as

$$\begin{aligned}
&\{\mathbf{Z}_{lt}(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0) + \mathbf{Z}_{st}(\boldsymbol{\theta}_0)vec\left[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)\right]\} \cdot \partial\delta\left[\varsigma_t(\boldsymbol{\theta}_0), \boldsymbol{\eta}_0\right]/\partial\boldsymbol{\eta}' \\
= \; &[\mathbf{Z}_{lt}(\boldsymbol{\theta})\mathbf{u}_t\sqrt{\varsigma_t} + \mathbf{Z}_{st}(\boldsymbol{\theta})vec(\mathbf{u}_t\mathbf{u}_t')\varsigma_t] \cdot \partial\delta(\varsigma_t, \boldsymbol{\eta})/\partial\boldsymbol{\eta}',
\end{aligned}$$

whose conditional expected value will be $\mathbf{Z}_{st}(\boldsymbol{\theta}_0)vec(\mathbf{I}_N)E[(\varsigma_t/N) \cdot \partial\delta(\varsigma_t, \boldsymbol{\eta}_0)/\partial\boldsymbol{\eta}'|\boldsymbol{\eta}]$. □

Proposition 2

The proof is based on a straightforward application of Proposition 1 in Bollerslev and Wooldridge (1992) to the spherically symmetric case. Since $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0, \mathbf{0}) = \mathbf{Z}_{dt}(\boldsymbol{\theta}_0)\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \mathbf{0})$, and $\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \mathbf{0})$ is a vector martingale difference sequence, then to obtain $\mathcal{B}_t(\boldsymbol{\phi}_0)$ we only need to compute $V[\mathbf{e}_{dt}(\boldsymbol{\theta}_0, \mathbf{0})|\mathbf{z}_t, I_{t-1}; \boldsymbol{\phi}_0]$. But since

$$\left[\begin{array}{c} \mathbf{e}_{lt}(\boldsymbol{\theta}_0, \mathbf{0}) \\ \mathbf{e}_{st}(\boldsymbol{\theta}_0, \mathbf{0}) \end{array}\right] = \left(\begin{array}{c} \boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0) \\ vec\left[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0) - \mathbf{I}_N\right] \end{array}\right) = \left[\begin{array}{c} \sqrt{\varsigma_t}\mathbf{u}_t \\ vec(\varsigma_t\mathbf{u}_t\mathbf{u}_t' - \mathbf{I}_N) \end{array}\right]$$

for any spherical distribution, with ς_t and $\mathbf{u}_t$ both mutually and serially independent, then (15) follows from (A1) and (A2). As for $\mathcal{A}_t(\boldsymbol{\phi}_0)$, we know that its formula, which is valid regardless of the exact nature of the true conditional distribution, coincides with $\mathcal{B}_t(\boldsymbol{\phi}_0)$ when $\kappa_0 = 0$ by the (conditional) information matrix equality. □

Proposition 3

We can use the conditional analogue to the generalised information matrix equality (see e.g. Newey and McFadden (1994)) to show that

$$E\left\{\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0})\left[\mathbf{s}'_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\boldsymbol{\varrho}),\mathbf{s}'_{\boldsymbol{\varrho}t}(\boldsymbol{\theta},\boldsymbol{\varrho})\right]\right|\mathbf{z}_t,I_{t-1};\boldsymbol{\theta},\boldsymbol{\varrho}\Big\} = -E\left\{\left[\frac{\partial\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0})}{\partial\boldsymbol{\theta}'}\left|\frac{\partial\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0})}{\partial\boldsymbol{\varrho}'}\right.\right]\right|\mathbf{z}_t,I_{t-1};\boldsymbol{\theta},\boldsymbol{\varrho}\right\}$$
$$= -E\left\{[\mathbf{h}_{\boldsymbol{\theta\theta}t}(\boldsymbol{\theta};\mathbf{0})|\mathbf{0}]|\,\mathbf{z}_t,I_{t-1};\boldsymbol{\theta},\boldsymbol{\varrho}\right\} = [\mathcal{A}_t(\boldsymbol{\phi})|\mathbf{0}]$$

irrespective of the conditional distribution of $\boldsymbol{\varepsilon}_t^*$, where we have used the fact that $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0})$ does not vary with $\boldsymbol{\varrho}$ when regarded as the influence function for $\tilde{\boldsymbol{\theta}}_T$. Then, the required result follows from the martingale difference nature of both $\mathbf{e}_{dt}(\boldsymbol{\theta}_0,\mathbf{0})$ and $\mathbf{e}_t(\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0)$. □

Proposition 4

We can use standard arguments (see e.g. Newey and McFadden (1994)) to show that the sequential ML estimator of $\boldsymbol{\eta}$ is asymptotically equivalent to a MM estimator based on the linearised influence function

$$s_{\eta t}(\boldsymbol{\theta}_0,\boldsymbol{\eta}) - \mathcal{I}'_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_0)\mathcal{A}^{-1}(\boldsymbol{\phi}_0)\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,\mathbf{0}).$$

Then, the expression for $\mathcal{F}(\boldsymbol{\phi}_0)$ follows from the definitions of $\mathcal{B}(\boldsymbol{\phi}_0)$, $\mathcal{C}(\boldsymbol{\phi}_0)$ and $\mathcal{I}_{\boldsymbol{\eta\eta}}(\boldsymbol{\phi}_0)$ in Propositions 1 and 2, together with the martingale difference nature of $\mathbf{e}_{dt}(\boldsymbol{\theta}_0,\mathbf{0})$ and $\mathbf{e}_t(\boldsymbol{\phi}_0)$.□

Proposition 5

In this case, the linearised influence functions corresponding to $\breve{\eta}_T$ and $\mathring{\eta}_T$ are

$$n_{\eta t}(\boldsymbol{\theta}_0,\eta) - \mathcal{R}'(\boldsymbol{\phi}_0)\mathcal{A}^{-1}(\boldsymbol{\phi}_0)\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,0),$$

and

$$\mathring{n}_{\eta t}(\boldsymbol{\theta}_0,\eta) - \mathcal{Q}'(\boldsymbol{\phi}_0)\mathcal{A}^{-1}(\boldsymbol{\phi}_0)\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,0),$$

respectively, whence we can directly obtain the formulae for $\mathcal{G}(\boldsymbol{\phi}_0)$ and $\mathcal{J}(\boldsymbol{\phi}_0)$. Therefore, the only remaining task is to obtain closed-form expressions for the required moments. In this respect, we can use the law of iterated expectations to show that

$$\begin{aligned} cov[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,0),n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\boldsymbol{\phi}_0] &= \mathbf{Z}_d(\boldsymbol{\phi}_0)\cdot E[\mathbf{e}_{dt}(\boldsymbol{\theta}_0,0)\cdot n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\varsigma_t;\boldsymbol{\phi}_0] \\ &= \mathbf{W}_s(\boldsymbol{\phi}_0)E\left[\left(\frac{\varsigma_t}{N}-1\right)n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)\right|\boldsymbol{\phi}_0\bigg], \end{aligned}$$

and

$$cov[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,\eta_0),n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\boldsymbol{\phi}_0] = \mathbf{Z}_d(\boldsymbol{\phi}_0)\cdot E[\mathbf{e}_{dt}(\boldsymbol{\theta}_0,\eta_0)\cdot n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)|\varsigma_t;\boldsymbol{\phi}_0]$$
$$= \mathbf{W}_s(\boldsymbol{\phi}_0)E\left[\left(\frac{N+\nu_0}{\nu_0-2+\varsigma_t}\frac{\varsigma_t}{N}-1\right)n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)\right|\boldsymbol{\phi}_0\bigg].$$

Then, we can use the properties of the beta distribution discussed before to show that

$$E\left[\left(\frac{\varsigma_t^2}{N(N+2)}-\frac{\nu_0-2}{\nu_0-4}\right)^2\right] = \frac{(\nu_0-2)^2}{(\nu_0-4)^2}\left[\frac{(N+6)(N+4)}{N(N+2)}\frac{(\nu_0-2)(\nu_0-4)}{(\nu_0-6)(\nu_0-8)}-1\right],$$

$$E\left[\left(\frac{\varsigma_t}{N}-1\right)\left(\frac{\varsigma_t^2}{N(N+2)}-\frac{\nu_0-2}{\nu_0-4}\right)\right] = \frac{4(\nu_0-2)(N+\nu_0-2)}{N(\nu_0-4)(\nu_0-6)},$$

and

$$E\left[\left(\frac{N+\nu_0}{\nu_0-2+\varsigma_t}\frac{\varsigma_t}{N}-1\right)\left(\frac{\varsigma_t^2}{N(N+2)}-\frac{\nu_0-2}{\nu_0-4}\right)\right]=\frac{4(\nu_0-2)}{N(\nu_0-4)}.$$

On the other hand, since $\mathring{n}_{\eta t}(\boldsymbol{\theta}_0,\eta_0)$ is the residual from the least squares projection of $n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)$ on $\varsigma_t/N-1$, we can obtain the relevant expressions for $\mathring{n}_{\eta t}(\boldsymbol{\theta}_0,\eta_0)$ from those of $n_{\eta t}(\boldsymbol{\theta}_0,\eta_0)$ by using the fact that

$$E\left[\left(\frac{\varsigma_t}{N}-1\right)^2\right]=\frac{2(N+\nu_0-2)}{N(\nu_0-4)}$$

and

$$E\left[\left(\frac{N+\nu_0}{\nu_0-2+\varsigma_t}\frac{\varsigma_t}{N}-1\right)\left(\frac{\varsigma_t}{N}-1\right)\right]=\frac{2}{N}.$$

□

Proposition 6

It trivially follows from (21) and (17) that

$$E\left\{\left[\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathcal{K}(0)\,\mathcal{K}^{+}(\boldsymbol{\varrho})\,\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]\mathbf{e}'_{dt}(\boldsymbol{\theta},\mathbf{0})\,|\mathbf{z}_t,I_{t-1};\boldsymbol{\theta},\boldsymbol{\varrho}\right\}=\mathbf{0}$$

for any distribution. In addition, we also know that

$$E\left\{\left[\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathcal{K}(0)\,\mathcal{K}^{+}(\boldsymbol{\varrho})\,\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]|\mathbf{z}_t,I_{t-1};\boldsymbol{\theta},\boldsymbol{\varrho}\right\}=\mathbf{0}.$$

Hence, the second summand of (22), which can be interpreted as $\mathbf{Z}_d(\boldsymbol{\phi}_0)$ times the residual from the theoretical regression of $\mathbf{e}_{dt}(\boldsymbol{\phi}_0)$ on a constant and $\mathbf{e}_{dt}(\boldsymbol{\theta}_0,\mathbf{0})$, belongs to the unrestricted tangent set, which is the Hilbert space spanned by all the time-invariant functions of $\boldsymbol{\varepsilon}_t^*$ with zero conditional means and bounded second moments that are conditionally orthogonal to $\mathbf{e}_{dt}(\boldsymbol{\theta}_0,\mathbf{0})$.

Now, if we write (22) as

$$[\mathbf{Z}_{dt}(\boldsymbol{\theta})-\mathbf{Z}_d(\boldsymbol{\theta},\boldsymbol{\varrho})]\,\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})+\mathbf{Z}_d(\boldsymbol{\theta},\boldsymbol{\varrho})\mathcal{K}(0)\,\mathcal{K}^{+}(\boldsymbol{\varrho})\,\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0}),$$

then we can use the law of iterated expectations to show that the semiparametric efficient score (22) evaluated at the true parameter values will be unconditionally orthogonal to the unrestricted tangent set because so is $\mathbf{e}_{dt}(\boldsymbol{\theta}_0,\mathbf{0})$, and $E\left[\mathbf{Z}_{dt}(\boldsymbol{\theta})-\mathbf{Z}_d(\boldsymbol{\theta},\boldsymbol{\varrho})|\boldsymbol{\theta},\boldsymbol{\varrho}\right]=\mathbf{0}$.

Finally, the expression for the semiparametric efficiency bound will be

$$E\left[\left.\begin{array}{c}\{\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathbf{Z}_{d}(\boldsymbol{\theta},\boldsymbol{\varrho})\left[\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathcal{K}\left(0\right)\mathcal{K}^{+}\left(\boldsymbol{\varrho}\right)\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]\}\\ \times\{\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})'\mathbf{Z}_{dt}'(\boldsymbol{\theta})-\left[\mathbf{e}_{dt}'(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathbf{e}_{dt}'(\boldsymbol{\theta},0)\mathcal{K}^{+}\left(\boldsymbol{\varrho}\right)\mathcal{K}\left(0\right)\right]\mathbf{Z}_{d}'(\boldsymbol{\theta},\boldsymbol{\varrho})\}\end{array}\right|\boldsymbol{\theta},\boldsymbol{\varrho}\right]$$

$$=E\left[\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})\mathbf{e}_{dt}'(\boldsymbol{\theta},\boldsymbol{\varrho})\mathbf{Z}_{dt}(\boldsymbol{\theta})|\boldsymbol{\theta},\boldsymbol{\varrho}\right]$$

$$-E\left\{\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})\left[\mathbf{e}_{dt}'(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathbf{e}_{dt}'(\boldsymbol{\theta},\mathbf{0})\mathcal{K}^{+}\left(\boldsymbol{\varrho}\right)\mathcal{K}\left(0\right)\right]\mathbf{Z}_{d}'(\boldsymbol{\theta},\boldsymbol{\varrho})|\boldsymbol{\theta},\boldsymbol{\varrho}\right\}$$

$$-E\left\{\mathbf{Z}_{d}(\boldsymbol{\theta},\boldsymbol{\varrho})\left[\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathcal{K}\left(0\right)\mathcal{K}^{+}\left(\boldsymbol{\varrho}\right)\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]\mathbf{e}_{dt}(\boldsymbol{\phi})'\mathbf{Z}_{dt}'(\boldsymbol{\theta})|\boldsymbol{\theta},\boldsymbol{\varrho}\right\}$$

$$+E\left\{\mathbf{Z}_{d}(\boldsymbol{\theta},\boldsymbol{\varrho})\left[\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathcal{K}\left(0\right)\mathcal{K}^{+}\left(\boldsymbol{\varrho}\right)\mathbf{e}_{dt}(\boldsymbol{\theta},0)\right]\left[\mathbf{e}_{dt}'(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathbf{e}_{dt}'(\boldsymbol{\theta},0)\mathcal{K}^{+}\left(\boldsymbol{\varrho}\right)\mathcal{K}\left(0\right)\right]\mathbf{Z}_{d}'(\boldsymbol{\theta},\boldsymbol{\varrho})|\boldsymbol{\theta},\boldsymbol{\varrho}\right\}$$

$$=\mathcal{I}_{\boldsymbol{\theta}\boldsymbol{\theta}}(\boldsymbol{\theta},\boldsymbol{\varrho})-\mathbf{Z}_{d}(\boldsymbol{\theta},\boldsymbol{\varrho})\left[\mathcal{M}_{dd}\left(\boldsymbol{\varrho}\right)-\mathcal{K}\left(0\right)\mathcal{K}^{+}\left(\boldsymbol{\varrho}\right)\mathcal{K}\left(0\right)\right]\mathbf{Z}_{d}'(\boldsymbol{\theta},\boldsymbol{\varrho})$$

by virtue of (21), (17) and the law of iterated expectations. □

Proposition 7

First of all, it is easy to show that for any spherical distribution

$$\begin{aligned}\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta}_0,\mathbf{0}) &= E\left[\left.\begin{array}{c}\mathbf{e}_{lt}(\boldsymbol{\theta}_0,\mathbf{0})\\ \mathbf{e}_{st}(\boldsymbol{\theta}_0,\mathbf{0})\end{array}\right|\varsigma_t(\boldsymbol{\theta}_0);\boldsymbol{\phi}_0\right]=E\left\{\left.\begin{array}{c}\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\\ vec\left[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)-\mathbf{I}_N\right]\end{array}\right|\varsigma_t(\boldsymbol{\theta}_0);\boldsymbol{\phi}_0\right\}\\ &= E\left[\left.\begin{array}{c}\sqrt{\varsigma_t}\mathbf{u}_t\\ vec(\varsigma_t\mathbf{u}_t\mathbf{u}_t'-\mathbf{I}_N)\end{array}\right|\varsigma_t\right]=\left(\frac{\varsigma_t}{N}-1\right)\left[\begin{array}{c}\mathbf{0}\\ vec(\mathbf{I}_N)\end{array}\right],\end{aligned}\qquad\text{(A6)}$$

and

$$\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi}_0)=E\left[\left.\begin{array}{c}\mathbf{e}_{lt}(\boldsymbol{\phi}_0)\\ \mathbf{e}_{st}(\boldsymbol{\phi}_0)\end{array}\right|\varsigma_t(\boldsymbol{\theta}_0);\boldsymbol{\phi}_0\right]$$

$$=E\left\{\left.\begin{array}{c}\delta[\varsigma_t(\boldsymbol{\theta}_0),\boldsymbol{\eta}_0]\cdot\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\\ vec\left[\delta[\varsigma_t(\boldsymbol{\theta}_0),\boldsymbol{\eta}_0]\cdot\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}_0)-\mathbf{I}_N\right]\end{array}\right|\varsigma_t(\boldsymbol{\theta}_0);\boldsymbol{\phi}_0\right\}$$

$$=E\left\{\left.\begin{array}{c}\delta(\varsigma_t,\boldsymbol{\eta}_0)\sqrt{\varsigma_t}\mathbf{u}_t\\ vec[\delta(\varsigma_t,\boldsymbol{\eta}_0)\varsigma_t\mathbf{u}_t\mathbf{u}_t'-\mathbf{I}_N]\end{array}\right|\varsigma_t\right\}=\left[\delta(\varsigma_t,\boldsymbol{\eta}_0)\frac{\varsigma_t}{N}\text{-}1\right]\left[\begin{array}{c}\mathbf{0}\\ vec(\mathbf{I}_N)\end{array}\right],\qquad\text{(A7)}$$

where we have used again the fact that $E(\mathbf{u}_t)=\mathbf{0}$, $E(\mathbf{u}_t\mathbf{u}_t')=N^{-1}\mathbf{I}_N$, and ς_t and $\mathbf{u}_t$ are stochastically independent.

In addition, we can use the law of iterated expectations to show that

$$E\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi})\mathbf{e}_{dt}'(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right]=E\left[\mathbf{e}_{dt}(\boldsymbol{\phi})\mathring{\mathbf{e}}_{dt}'(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right]=E\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi})\mathring{\mathbf{e}}_{dt}'(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right]$$

and

$$E\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\mathbf{e}_{dt}'(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right]=E\left[\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0})\mathring{\mathbf{e}}_{dt}'(\boldsymbol{\theta},0)|\boldsymbol{\phi}\right]=E\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\mathring{\mathbf{e}}_{dt}'(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right].$$

Hence, to compute these matrices we simply need to obtain the scalar moments

$$E\left\{\left(\frac{\varsigma_t}{N}-1\right)\left[\delta(\varsigma_t,\boldsymbol{\eta}_0)\frac{\varsigma_t}{N}-1\right]\middle|\boldsymbol{\eta}\right\}$$

and

$$E\left[\left(\frac{\varsigma_t}{N}-1\right)^2\middle|\boldsymbol{\eta}\right].$$

In this respect, we can use (16) to show that the latter is simply $[(N+2)\kappa+2]/N$, so that

$$E\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\mathbf{e}'_{dt}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right] = \frac{(N+2)\kappa+2}{N}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & vec(\mathbf{I}_N)vec'(\mathbf{I}_N) \end{pmatrix} = \mathring{\mathcal{K}}(\kappa).$$

As for the former, we can use Lemma 1 to show that $E(\varsigma_t^2) = N(N+2)(\kappa+1) < \infty$ implies

$$E\left[\varsigma_t^2 \cdot \partial \ln h(\varsigma_t;\boldsymbol{\eta})/\partial\varsigma \middle| \boldsymbol{\eta}\right] = -E\left[2\varsigma_t|\boldsymbol{\eta}\right] = -2N.$$

If we then combine this result with (A4) and (A5), we will have that for any spherically symmetric distribution

$$E\left\{\left(\frac{\varsigma_t}{N}-1\right)\left[\delta(\varsigma_t,\boldsymbol{\eta}_0)\frac{\varsigma_t}{N}-1\right]\middle|\boldsymbol{\eta}\right\} = \frac{2}{N},$$

so that

$$E\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi})\mathbf{e}'_{dt}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right] = \mathring{\mathcal{K}}(0),$$

which coincides with the value of $E\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\mathbf{e}'_{dt}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}\right]$ under normality.

Therefore, it trivially follows from the expressions for $\mathring{\mathcal{K}}(0)$ and $\mathring{\mathcal{K}}(\kappa_0)$ above that

$$\begin{aligned} &E\left\{\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi}) - \mathring{\mathcal{K}}(0)\,\mathring{\mathcal{K}}^+(\kappa)\,\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]\mathbf{e}'_{dt}(\boldsymbol{\theta},\mathbf{0})\,|\mathbf{z}_t,I_{t-1};\boldsymbol{\phi}\right\} \\ &= E\left\{\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi}) - \mathring{\mathcal{K}}(0)\,\mathring{\mathcal{K}}^+(\kappa)\,\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]\mathring{\mathbf{e}}'_{dt}(\boldsymbol{\theta},\mathbf{0})\,|\mathbf{z}_t,I_{t-1};\boldsymbol{\phi}\right\} = \mathbf{0} \end{aligned}$$

for any spherically symmetric distribution. In addition, we also know that

$$E\left\{\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi}) - \mathring{\mathcal{K}}(0)\,\mathring{\mathcal{K}}^+(\kappa)\,\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]|\mathbf{z}_t,I_{t-1};\boldsymbol{\phi}\right\} = \mathbf{0}.$$

Thus, even though $\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi}_0) - \mathring{\mathcal{K}}(0)\,\mathring{\mathcal{K}}^+(\kappa_0)\,\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta}_0,\mathbf{0})\right]$ is the residual from the theoretical regression of $\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi})$ on a constant and $\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})$, it turns out that the second summand of (24) belongs to the restricted tangent set, which is the Hilbert space spanned by all the time-invariant functions of $\boldsymbol{\varsigma}_t(\boldsymbol{\theta}_0)$ with bounded second moments that have zero conditional means and are conditionally orthogonal to $\mathbf{e}_{dt}(\boldsymbol{\theta}_0,\mathbf{0})$.

Now, if write (24) as

$$\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\phi}) - \mathbf{Z}_d(\boldsymbol{\phi})\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi}) + \mathbf{Z}_d(\boldsymbol{\phi})\mathring{\mathcal{K}}(0)\,\mathring{\mathcal{K}}^+(\kappa)\,\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0}),$$

then we can use the law of iterated expectations to show that the elliptically symmetric semiparametric efficient score is indeed unconditionally orthogonal to the restricted tangent set.

Finally, the expression for the semiparametric efficiency bound will be

$$E[\mathring{\mathbf{s}}_{\boldsymbol{\theta}t}(\boldsymbol{\phi})\mathring{\mathbf{s}}'_{\boldsymbol{\theta}t}(\boldsymbol{\phi})|\boldsymbol{\phi}] = E\left[\begin{array}{c} \left\{\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\phi}) - \mathbf{Z}_d(\boldsymbol{\phi})\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi}) - \mathring{\mathcal{K}}(0)\mathring{\mathcal{K}}^+(\kappa)\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]\right\} \\ \times\left\{\mathbf{e}_{dt}(\boldsymbol{\phi})'\mathbf{Z}'_{dt}(\boldsymbol{\theta}) - \left[\mathring{\mathbf{e}}'_{dt}(\boldsymbol{\phi}) - \mathring{\mathbf{e}}'_{dt}(\boldsymbol{\theta},\mathbf{0})\mathring{\mathcal{K}}^+(\kappa)\mathring{\mathcal{K}}(0)\right]\mathbf{Z}'_d(\boldsymbol{\phi})\right\} \end{array}\middle|\boldsymbol{\phi}\right]$$

$$= E\left[\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\phi})\mathbf{e}'_{dt}(\boldsymbol{\phi})\mathbf{Z}_{dt}(\boldsymbol{\theta})|\boldsymbol{\phi}\right]$$

$$-E\left\{\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\phi})\left[\mathring{\mathbf{e}}'_{dt}(\boldsymbol{\phi}) - \mathring{\mathbf{e}}'_{dt}(\boldsymbol{\theta},\mathbf{0})\mathring{\mathcal{K}}^+(\kappa)\mathring{\mathcal{K}}(0)\right]\mathbf{Z}'_d(\boldsymbol{\phi})|\boldsymbol{\phi}\right\}$$

$$-E\left\{\mathbf{Z}_d(\boldsymbol{\phi})\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi}) - \mathring{\mathcal{K}}(0)\mathring{\mathcal{K}}^+(\kappa)\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]\mathbf{e}_{dt}(\boldsymbol{\phi})'\mathbf{Z}'_d(\boldsymbol{\phi})|\boldsymbol{\phi}\right\}$$

$$+E\left\{\mathbf{Z}_d(\boldsymbol{\phi})\left[\mathring{\mathbf{e}}_{dt}(\boldsymbol{\phi}) - \mathring{\mathcal{K}}(0)\mathring{\mathcal{K}}^+(\kappa)\mathring{\mathbf{e}}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]\left[\mathring{\mathbf{e}}'_{dt}(\boldsymbol{\phi}) - \mathring{\mathbf{e}}'_{dt}(\boldsymbol{\theta},\mathbf{0})\mathring{\mathcal{K}}^+(\kappa)\mathring{\mathcal{K}}(0)\right]\mathbf{Z}'_d(\boldsymbol{\phi})|\boldsymbol{\phi}\right\}$$

$$= \mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) - \mathbf{W}_s(\boldsymbol{\phi}_0)\mathbf{W}'_s(\boldsymbol{\phi}_0)\cdot\left\{\left[\frac{N+2}{N}\mathrm{M}_{ss}(\boldsymbol{\eta}) - 1\right] - \frac{4}{N\left[(N+2)\kappa+2\right]}\right\}$$

by virtue of the law of iterated expectations. □

Proposition 8

The proof that $\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)$ is at least as large as $\mathcal{P}(\boldsymbol{\phi}_0)$ in the positive semidefinite matrix sense follows trivially from the fact that the latter is the residual variance in the multivariate theoretical regression of $\mathbf{s}_{\theta t}(\boldsymbol{\phi}_0)$ on $\mathbf{s}_{\boldsymbol{\eta}t}(\boldsymbol{\phi}_0)$, while the former is the unconditional variance of $\mathbf{s}_{\theta t}(\boldsymbol{\phi}_0)$. The fact that the residual variance of a multivariate regression cannot increase as we increase the number of regressors also explains why $\mathcal{P}(\boldsymbol{\phi}_0)$ is at least as large (in the positive semidefinite matrix sense) as $\mathring{\mathcal{S}}(\boldsymbol{\phi}_0)$, and why the latter is at least as large as $\mathcal{S}(\boldsymbol{\phi}_0)$, reflecting the fact that the relevant tangent sets become increasing larger. Finally, the positive semidefiniteness of $\mathcal{S}(\boldsymbol{\phi}_0) - \mathcal{A}(\boldsymbol{\theta})\mathcal{B}^{-1}(\boldsymbol{\phi})\mathcal{A}(\boldsymbol{\theta})$ follows from the fact that it coincides with the residual covariance matrix in the theoretical regression of the semiparametric efficient score on the Gaussian pseudo-score since

$$E[\{\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho}) - \mathbf{Z}_d(\boldsymbol{\theta},\boldsymbol{\varrho})\left[\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho}) - \mathcal{K}(0)\mathcal{K}^+(\boldsymbol{\varrho})\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0})\right]\}\mathbf{e}'_{dt}(\boldsymbol{\theta},\mathbf{0})\mathbf{Z}'_{dt}(\boldsymbol{\theta})|\boldsymbol{\phi}] = \mathcal{A}(\boldsymbol{\theta})$$

because $\mathbf{e}'_{dt}(\boldsymbol{\theta},\mathbf{0})$ is conditionally orthogonal to $[\mathbf{e}_{dt}(\boldsymbol{\theta},\boldsymbol{\varrho}) - \mathcal{K}(0)\mathcal{K}^+(\boldsymbol{\varrho})\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0})]$ by construction. □

Proposition 9

The proof of the first part is trivial, except perhaps for the fact that $\mathrm{M}_{sr}(\mathbf{0}) = \mathbf{0}$, which follows from Proposition 3 because $\mathbf{e}_{st}(\boldsymbol{\theta}_0,\mathbf{0})$ coincides with $\mathbf{e}_{st}(\boldsymbol{\phi}_0)$ under normality.

To prove the second part, note that $\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}) - \mathring{\mathcal{S}}(\boldsymbol{\phi})$ is $\mathbf{W}_d(\boldsymbol{\phi})\mathbf{W}'_d(\boldsymbol{\phi})$ times the residual variance in the theoretical regression of $\delta(\varsigma_t,\boldsymbol{\eta}_0)\varsigma_t/N - 1$ on $(\varsigma_t/N) - 1$, which given that $\mathbf{W}_d(\boldsymbol{\phi}) \neq \mathbf{0}$ can only be 0 if the regression residual is identically 0 for all t. The solution to the resulting differential equation is

$$g(\varsigma_t,\boldsymbol{\eta}) = -\frac{N(N+2)\kappa}{2[(N+2)\kappa+2]}\ln\varsigma_t - \frac{1}{[(N+2)\kappa+2]}\varsigma_t + C,$$

which in view of (A3) implies that

$$h(\varsigma_t;\boldsymbol{\eta}) \propto \varsigma_t^{\frac{N}{(N+2)\kappa+2}-1} \exp\left\{-\frac{1}{[(N+2)\kappa+2]}\varsigma_t\right\},$$

i.e. the density of Gamma random variable with mean N and variance $N[(N+2)\kappa_0+2]$. In this sense, it is worth recalling that $\kappa \geq -2/(N+2)$ for all elliptical distributions, with the lower limit corresponding to the uniform.

Finally, to prove the third part we use the fact that after some tedious algebraic manipulations we can write $\mathcal{M}_{dd}(\boldsymbol{\eta}) - \mathcal{K}(0)\mathcal{K}^+(\kappa)\mathcal{K}(0)$ as

$$\left\{\begin{array}{cc} [\mathrm{M}_{ll}(\boldsymbol{\eta})\text{-}1]\mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & \left[\mathrm{M}_{ss}(\boldsymbol{\eta})\text{-}\frac{1}{\kappa+1}\right](\mathbf{I}_{N^2}+\mathbf{K}_{NN}) + \left[\mathrm{M}_{ss}(\boldsymbol{\eta}_0)\text{-}1+\frac{2\kappa}{(\kappa+1)[(N+2)\kappa+2]}\right] vec(\mathbf{I}_N)vec'(\mathbf{I}_N) \end{array}\right\}.$$

Therefore, given that $\mathbf{Z}_l(\boldsymbol{\phi}_0) \neq \mathbf{0}$, $\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}) - \mathcal{S}(\boldsymbol{\phi})$ will be zero only if $\mathrm{M}_{ll}(\boldsymbol{\eta}) = 1$, which in turn requires that the residual variance in the multivariate regression of $\delta(\varsigma_t,\boldsymbol{\eta}_0)\boldsymbol{\varepsilon}_t^*$ on $\boldsymbol{\varepsilon}_t^*$ is zero for all t, or equivalently, that $\delta(\varsigma_t,\boldsymbol{\eta}_0) = 1$. But since the solution to this differential equation is $g(\varsigma_t,\boldsymbol{\eta}) = -.5\varsigma_t + C$, then the result follows from (A3). □

Proposition 10

It is tedious but otherwise straightforward to prove that when $\boldsymbol{\alpha}_0 = \mathbf{0}$

$$\mathbf{Z}_{dt}(\boldsymbol{\theta}_0) = \left[\begin{array}{cc} \gamma_0^{-1/2}(1-\sum_{j=1}^h \rho_{j0}) & 0 \\ \gamma_0^{-1/2}(y_{t-1}-\pi_0,\ldots,y_{t-h}-\pi_0)' & \mathbf{0} \\ 0 & \frac{1}{2}\gamma_0^{-1} \\ \mathbf{0} & \frac{1}{2}(\varepsilon_{t-1}^{*2}-1,\ldots,\varepsilon_{t-q}^{*2}-1)' \end{array}\right],$$

so that

$$\mathbf{Z}_d(\boldsymbol{\phi}_0) = \left[\begin{array}{cc} \gamma_0^{-1/2}(1-\sum_{j=1}^h \rho_{j0}) & 0 \\ \mathbf{0} & \mathbf{0} \\ 0 & \frac{1}{2}\gamma_0^{-1} \\ \mathbf{0} & \mathbf{0} \end{array}\right].$$

Proposition 1 then implies that the information matrix will have only four non-zero blocks along its diagonal, which correspond to π, $\boldsymbol{\rho}$, $\boldsymbol{\alpha}$ and $(\gamma,\boldsymbol{\eta})$. The same proposition also implies that

$$\begin{aligned} \mathcal{I}_{\pi\pi}(\boldsymbol{\phi}_0) &= \mathrm{M}_{ll}(\boldsymbol{\eta}_0)\gamma_0^{-1}(1-\sum_{j=1}^h \rho_{j0})^2, \\ \mathcal{I}_{\rho\rho}(\boldsymbol{\phi}_0) &= \mathrm{M}_{ll}(\boldsymbol{\eta}_0)\gamma_0^{-1}\boldsymbol{\Sigma}_0, \end{aligned}$$

where $\boldsymbol{\Sigma}_0$ is the $h \times h$ autocovariance matrix of $(y_{t-1},\ldots,y_{t-h})'$, and

$$\begin{aligned} E\left[\mathcal{I}_{\alpha\alpha t}(\boldsymbol{\phi}_0)|\,\boldsymbol{\phi}_0\right] &= [3\mathrm{M}_{ss}(\boldsymbol{\eta}_0)-1]\cdot E\left[\frac{1}{4}(\varepsilon_{t-1}^{*2}-1,\ldots,\varepsilon_{t-q}^{*2}-1)'(\varepsilon_{t-1}^{*2}-1,\ldots,\varepsilon_{t-q}^{*2}-1)\right] \\ &= \frac{3\mathrm{M}_{ss}(\boldsymbol{\eta}_0)-1}{4}\cdot E[(\varepsilon_t^{*2}-1)^2|\boldsymbol{\eta}_0]\cdot\mathbf{I}_q = \frac{[3\mathrm{M}_{ss}(\boldsymbol{\eta}_0)-1](3\kappa_0+2)}{4}\mathbf{I}_q, \end{aligned}$$

in view of the fact that ε_t^* is serially independent when $\boldsymbol{\alpha}_0 = \mathbf{0}$, and $E(\varepsilon_t^{*2} - 1)^2 = V(\varepsilon_t^{*2}) = (3\kappa_0 + 2)$.

Given that $\mathbf{Z}_d(\boldsymbol{\phi}_0)$ has a block-structure, the block-diagonality of $\mathring{\mathcal{S}}(\boldsymbol{\phi}_0)$ and $\mathcal{S}(\boldsymbol{\phi}_0)$ follows from expressions (25) and (23), respectively. Finally, we can use Proposition 1 in Demos and Sentana (1998) to show that $\mathcal{C}(\boldsymbol{\phi}_0)$ is also block-diagonal, with $\mathcal{C}_{\pi\pi}(\boldsymbol{\phi}_0) = \gamma(1 - \sum_{j=1}^{h} \rho_{j0})^{-2}$, $\mathcal{C}_{\boldsymbol{\rho}\boldsymbol{\rho}}(\boldsymbol{\phi}_0) = \gamma\boldsymbol{\Sigma}_0^{-1}$, and $\mathcal{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\phi}_0) = \mathbf{I}_q$, although note that there is a missing scalar term in front of their expression for $\mathcal{C}_{\gamma\gamma}(\boldsymbol{\phi}_0)$. □

Proposition 11

Using the results in appendix A.5 of Sentana and Fiorentini (2001), and appendices C and D in Sentana (2004), it is tedious but otherwise straightforward to prove that when $\boldsymbol{\alpha}_0 = \mathbf{0}$

$$\mathbf{Z}_{dt}(\boldsymbol{\theta}_0) = \left\{ \begin{array}{cc} [\mathbf{I}_N - diag(\boldsymbol{\rho}_0)]\boldsymbol{\Sigma}_0^{-1/2\prime} & \mathbf{0} \\ diag[\mathbf{y}_{t-1} - \boldsymbol{\pi}]\boldsymbol{\Sigma}_0^{-1/2\prime} & \mathbf{0} \\ \mathbf{0} & (\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1/2\prime} \otimes \boldsymbol{\Sigma}_0^{-1/2\prime}) \\ \mathbf{0} & \frac{1}{2}\mathbf{E}_N'(\boldsymbol{\Sigma}_0^{-1/2\prime} \otimes \boldsymbol{\Sigma}_0^{-1/2\prime}) \\ \mathbf{0} & \frac{1}{2}\left[\begin{array}{c} f_{kt-1}^2(\boldsymbol{\theta}_0) + \omega(\boldsymbol{\theta}_0) - 1 \\ \vdots \\ f_{kt-q}^2(\boldsymbol{\theta}_0) + \omega(\boldsymbol{\theta}_0) - 1 \end{array}\right](\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1/2\prime} \otimes \mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1/2\prime}) \end{array} \right\}$$

where $\mathbf{E}_N'$ is the unique matrix that transforms $vec(\mathbf{A})$ in $vecd(\mathbf{A})$ as $vecd(\mathbf{A}) = \mathbf{E}_N' vec(\mathbf{A})$, and $\boldsymbol{\Sigma}$ is shorthand for $\mathbf{cc}' + \Gamma$. As a result,

$$\mathbf{Z}_d(\boldsymbol{\phi}_0) = \left[\begin{array}{cc} [\mathbf{I}_N - diag(\boldsymbol{\rho}_0)]\boldsymbol{\Sigma}_0^{-1/2\prime} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1/2\prime} \otimes \boldsymbol{\Sigma}_0^{-1/2\prime}) \\ \mathbf{0} & \frac{1}{2}\mathbf{E}_N'(\boldsymbol{\Sigma}_0^{-1/2\prime} \otimes \boldsymbol{\Sigma}_0^{-1/2\prime}) \\ \mathbf{0} & \mathbf{0} \end{array} \right],$$

where we have used the fact that $f_{kt}(\boldsymbol{\theta}_0) = \mathbf{c}_0'\Sigma_0^{-1}\boldsymbol{\varepsilon}_t$ and $\omega_t(\boldsymbol{\theta}_0) = 1 - \mathbf{c}_0'\Sigma_0^{-1}\mathbf{c}_0 = (1 + \mathbf{c}_0'\Gamma_0^{-1}\mathbf{c}_0)^{-1} = \omega(\boldsymbol{\theta}_0)\ \forall t$ when $\boldsymbol{\alpha}_0 = \mathbf{0}$ (see Sentana and Fiorentini (2001)), so that $E[f_{kt}^2(\boldsymbol{\theta}_0) + \omega(\boldsymbol{\theta}_0) - 1|\boldsymbol{\phi}_0] = E(\mathbf{c}_0'\Sigma_0^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\Sigma_0^{-1}\mathbf{c}_0 - \mathbf{c}_0'\Sigma_0^{-1}\mathbf{c}_0|\boldsymbol{\phi}_0) = \mathbf{0}$.

Proposition 1 then implies that the information matrix will have only four non-zero blocks along its diagonal, which correspond to $\boldsymbol{\pi}$, $\boldsymbol{\rho}$, $\boldsymbol{\alpha}$ and $(\mathbf{c}, \boldsymbol{\gamma}, \boldsymbol{\eta})$. The same proposition also implies that

$$\begin{aligned} \mathcal{I}_{\boldsymbol{\pi}\boldsymbol{\pi}}(\boldsymbol{\phi}_0) &= \mathrm{M}_{ll}(\boldsymbol{\eta}_0)[\mathbf{I}_N - diag(\boldsymbol{\rho}_0)]\boldsymbol{\Sigma}_0^{-1}[\mathbf{I}_N - diag(\boldsymbol{\rho}_0)], \\ \mathcal{I}_{\boldsymbol{\rho}\boldsymbol{\rho}}(\boldsymbol{\phi}_0) &= \mathrm{M}_{ll}(\boldsymbol{\eta}_0)E[diag(\mathbf{y}_t - \boldsymbol{\pi}_0)\boldsymbol{\Sigma}_0^{-1}diag(\mathbf{y}_t - \boldsymbol{\pi}_0)|\boldsymbol{\phi}_0], \end{aligned}$$

and

$$E\left[\mathcal{I}_{\alpha\alpha t}(\boldsymbol{\phi}_0)\right|\boldsymbol{\phi}_0]=\frac{1}{4}E\left\{\left(\begin{array}{c} f_{kt-1}^2(\boldsymbol{\theta}_0)+\omega(\boldsymbol{\theta}_0)-1 \\ \vdots \\ f_{kt-q}^2(\boldsymbol{\theta}_0)+\omega(\boldsymbol{\theta}_0)-1 \end{array}\right)(\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1/2\prime}\otimes\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1/2\prime})\left[\mathrm{M}_{ss}(\boldsymbol{\eta}_0)\left(\mathbf{I}_{N^2}+\mathbf{K}_{NN}\right)\right.\right.$$

$$\left.\left.+\left[\mathrm{M}_{ss}(\boldsymbol{\eta}_0)-1\right]vec(\mathbf{I}_N)vec'(\mathbf{I}_N)\right](\boldsymbol{\Sigma}_0^{-1/2}\mathbf{c}_0\otimes\boldsymbol{\Sigma}_0^{-1/2}\mathbf{c}_0)\left(\begin{array}{c} f_{kt-1}^2(\boldsymbol{\theta}_0)+\omega(\boldsymbol{\theta}_0)-1 \\ \vdots \\ f_{kt-q}^2(\boldsymbol{\theta}_0)+\omega(\boldsymbol{\theta}_0)-1 \end{array}\right)'\right\}$$

$$=\frac{3\mathrm{M}_{ss}(\boldsymbol{\eta}_0)-1}{4}(\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1}\mathbf{c}_0)^2E\{[f_{kt}^2(\boldsymbol{\theta}_0)+\omega(\boldsymbol{\theta}_0)-1]^2|\boldsymbol{\phi}_0\}\mathbf{I}_q$$

in view of the fact that f_{kt} is serially independent when $\boldsymbol{\alpha}_0=\mathbf{0}$. In this respect, we can show that

$$E\{[f_{kt}^2(\boldsymbol{\theta}_0)+\omega(\boldsymbol{\theta}_0)-1]^2|\boldsymbol{\phi}_0\}=E(\mathbf{c}_0'\Sigma_0^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\Sigma_0^{-1}\mathbf{c}_0)^2-(\mathbf{c}_0'\Sigma_0^{-1}\mathbf{c}_0)^2$$
$$=E[vec(\mathbf{c}_0'\Sigma_0^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\Sigma_0^{-1}\mathbf{c}_0)vec'(\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t'\boldsymbol{\Sigma}_0^{-1}\mathbf{c}_0)|\boldsymbol{\phi}_0]-(\mathbf{c}_0'\Sigma_0^{-1}\mathbf{c}_0)^2$$
$$=(\mathbf{c}_0'\Sigma_0^{-1/2\prime}\otimes\mathbf{c}_0'\Sigma_0^{-1/2\prime})E[vec(\boldsymbol{\varepsilon}_t^*\boldsymbol{\varepsilon}_t^{*\prime})vec'(\boldsymbol{\varepsilon}_t^*\boldsymbol{\varepsilon}_t^{*\prime})|\boldsymbol{\phi}_0](\Sigma_0^{-1/2}\mathbf{c}_0\otimes\Sigma_0^{-1/2}\mathbf{c}_0)-(\mathbf{c}_0'\Sigma_0^{-1}\mathbf{c}_0)^2$$
$$=(\mathbf{c}_0'\Sigma_0^{-1/2\prime}\otimes\mathbf{c}_0'\Sigma_0^{-1/2\prime})(\kappa_0+1)[(\mathbf{I}_{N^2}+\mathbf{K}_{NN})+vec\,(\mathbf{I}_N)\,vec'\,(\mathbf{I}_N)](\Sigma_0^{-1/2}\mathbf{c}_0\otimes\Sigma_0^{-1/2}\mathbf{c}_0)$$
$$-(\mathbf{c}_0'\Sigma_0^{-1}\mathbf{c}_0)^2=(3\kappa_0+2)(\mathbf{c}_0'\Sigma_0^{-1}\mathbf{c}_0)^2=(3\kappa_0+2)(\mathbf{c}_0'\Gamma_0^{-1}\mathbf{c}_0)^2(1+\mathbf{c}_0'\Gamma_0^{-1}\mathbf{c}_0)^{-2}.$$

Given that $\mathbf{Z}_d(\boldsymbol{\phi}_0)$ has a block-structure, the block-diagonality of $\mathring{\mathcal{S}}(\boldsymbol{\phi}_0)$ and $\mathcal{S}(\boldsymbol{\phi}_0)$ follows from expressions (25) and (23), respectively. Finally, it follows directly from Proposition 6 that $\mathcal{C}(\boldsymbol{\phi}_0)$ will also be block-diagonal, with

$$\begin{aligned}
\mathcal{A}_{\boldsymbol{\pi}\boldsymbol{\pi}}(\boldsymbol{\phi}_0) &= \mathcal{B}_{\boldsymbol{\pi}\boldsymbol{\pi}}(\boldsymbol{\phi}_0)=\mathcal{C}_{\boldsymbol{\pi}\boldsymbol{\pi}}^{-1}(\boldsymbol{\phi}_0)=[\mathbf{I}_N-diag(\boldsymbol{\rho}_0)]\boldsymbol{\Sigma}_0^{-1}[\mathbf{I}_N-diag(\boldsymbol{\rho}_0)],\\
\mathcal{A}_{\boldsymbol{\rho}\boldsymbol{\rho}}(\boldsymbol{\phi}_0) &= \mathcal{B}_{\boldsymbol{\rho}\boldsymbol{\rho}}(\boldsymbol{\phi}_0)=\mathcal{C}_{\boldsymbol{\rho}\boldsymbol{\rho}}^{-1}(\boldsymbol{\phi}_0)=E[diag(\mathbf{y}_t-\boldsymbol{\pi}_0)\boldsymbol{\Sigma}_0^{-1}diag(\mathbf{y}_t-\boldsymbol{\pi}_0)|\boldsymbol{\phi}_0],\\
\mathcal{A}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\phi}_0) &= .5(\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1}\mathbf{c}_0)^2E\{[f_{kt}^2(\boldsymbol{\theta}_0)+\omega(\boldsymbol{\theta}_0)-1]^2|\boldsymbol{\phi}_0\}\mathbf{I}_q,\\
\mathcal{B}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}(\boldsymbol{\phi}_0) &= .25(3\kappa_0+2)(\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1}\mathbf{c}_0)^2E\{[f_{kt}^2(\boldsymbol{\theta}_0)+\omega(\boldsymbol{\theta}_0)-1]^2|\boldsymbol{\phi}_0\}\mathbf{I}_q,
\end{aligned}$$

so that

$$\mathcal{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}^{-1}(\boldsymbol{\phi}_0)=\frac{(\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1}\mathbf{c}_0)^2}{3\kappa_0+2}E\{[f_{kt}^2(\boldsymbol{\theta}_0)+\omega(\boldsymbol{\theta}_0)-1]^2|\boldsymbol{\phi}_0\}\mathbf{I}_q=(\mathbf{c}_0'\boldsymbol{\Sigma}_0^{-1}\mathbf{c}_0)^4=\frac{(\mathbf{c}_0'\Gamma_0^{-1}\mathbf{c}_0)^4}{(1+\mathbf{c}_0'\Gamma_0^{-1}\mathbf{c}_0)^4}.$$

□

Proposition 12

Given our assumptions on the mapping $\mathbf{r}_s(.)$, we can directly work in terms of the $\boldsymbol{\vartheta}$ parameters. In this sense, since the conditional covariance matrix of $\mathbf{y}_t$ is of the form $\vartheta_2\boldsymbol{\Sigma}_t^\circ(\boldsymbol{\vartheta}_1)$, it is straightforward to show that

$$\mathbf{Z}_{dt}(\boldsymbol{\vartheta}) = \left\{ \begin{array}{c} \vartheta_2^{-1/2} \left[\partial\boldsymbol{\mu}_t'(\boldsymbol{\vartheta}_1)/\partial\boldsymbol{\vartheta}_1\right] \boldsymbol{\Sigma}_t^{\circ-1/2\prime}(\boldsymbol{\vartheta}_1) \\ 0 \end{array} \right.$$

$$\left. \begin{array}{c} \frac{1}{2}\{\partial vec'[\Sigma_t^{\circ}(\boldsymbol{\vartheta}_1)]/\partial\boldsymbol{\vartheta}_1\}[\boldsymbol{\Sigma}_t^{\circ-1/2\prime}(\boldsymbol{\vartheta}_1) \otimes \boldsymbol{\Sigma}_t^{\circ-1/2\prime}(\boldsymbol{\vartheta}_1)] \\ \frac{1}{2}\vartheta_2^{-1} vec'(\mathbf{I}_N) \end{array} \right\} = \left[\begin{array}{cc} \mathbf{Z}_{\boldsymbol{\vartheta}_1 lt}(\boldsymbol{\vartheta}) & \mathbf{Z}_{\boldsymbol{\vartheta}_1 st}(\boldsymbol{\vartheta}) \\ 0 & \mathbf{Z}_{\vartheta_2 st}(\boldsymbol{\vartheta}) \end{array} \right]. \tag{A8}$$

Thus, the score vector for $\boldsymbol{\vartheta}$ will be

$$\left[\begin{array}{c} \mathbf{s}_{\boldsymbol{\vartheta}_1 t}(\boldsymbol{\vartheta},\boldsymbol{\eta}) \\ s_{\vartheta_2 t}(\boldsymbol{\vartheta},\boldsymbol{\eta}) \end{array} \right] = \left[\begin{array}{c} \mathbf{Z}_{\boldsymbol{\vartheta}_1 lt}(\boldsymbol{\vartheta})\mathbf{e}_{lt}(\boldsymbol{\vartheta},\boldsymbol{\eta}) + \mathbf{Z}_{\boldsymbol{\vartheta}_1 st}(\boldsymbol{\vartheta})\mathbf{e}_{st}(\boldsymbol{\vartheta},\boldsymbol{\eta}) \\ \mathbf{Z}_{\vartheta_2 st}(\boldsymbol{\vartheta})\mathbf{e}_{st}(\boldsymbol{\vartheta},\boldsymbol{\eta}) \end{array} \right], \tag{A9}$$

where $\mathbf{e}_{lt}(\boldsymbol{\vartheta},\boldsymbol{\eta})$ and $\mathbf{e}_{st}(\boldsymbol{\vartheta},\boldsymbol{\eta})$ are given in (5) and (6), respectively.

It is then easy to see that the unconditional covariance between $\mathbf{s}_{\vartheta_1 t}(\boldsymbol{\vartheta},\boldsymbol{\eta})$ and $s_{\vartheta_2 t}(\boldsymbol{\vartheta},\boldsymbol{\eta})$ is

$$\begin{aligned}
& E\left\{ \left[\begin{array}{cc} \mathbf{Z}_{\boldsymbol{\vartheta}_1 lt}(\boldsymbol{\vartheta}) & \mathbf{Z}_{\boldsymbol{\vartheta}_1 st}(\boldsymbol{\vartheta}) \end{array} \right] \left[\begin{array}{cc} \mathcal{M}_{ll}(\boldsymbol{\eta}) & \mathbf{0} \\ \mathbf{0} & \mathcal{M}_{ss}(\boldsymbol{\eta}) \end{array} \right] \left[\begin{array}{c} 0 \\ \mathbf{Z}_{\vartheta_2 st}'(\boldsymbol{\vartheta}) \end{array} \right] \middle| \boldsymbol{\vartheta},\boldsymbol{\eta} \right\} \\
= & \frac{\{2\mathrm{M}_{ss}(\boldsymbol{\eta}) + N[\mathrm{M}_{ss}(\boldsymbol{\eta}) - 1]\}}{2\vartheta_2} E\left\{ \frac{1}{2}\frac{\partial vec'[\boldsymbol{\Sigma}_t^{\circ}(\boldsymbol{\vartheta}_1)]}{\partial\boldsymbol{\vartheta}_1}[\boldsymbol{\Sigma}_t^{\circ-1/2\prime}(\boldsymbol{\vartheta}_1) \otimes \boldsymbol{\Sigma}_t^{\circ-1/2\prime}(\boldsymbol{\vartheta}_1)] \middle| \boldsymbol{\vartheta},\boldsymbol{\eta} \right\} vec(\mathbf{I}_N) \\
= & \frac{\{2\mathrm{M}_{ss}(\boldsymbol{\eta}) + N[\mathrm{M}_{ss}(\boldsymbol{\eta}) - 1]\}}{2\vartheta_2} \mathbf{Z}_{\boldsymbol{\vartheta}_1 s}(\boldsymbol{\vartheta},\boldsymbol{\eta}) vec(\mathbf{I}_N),
\end{aligned}$$

with $\mathbf{Z}_{\boldsymbol{\vartheta}_1 s}(\boldsymbol{\vartheta},\boldsymbol{\eta}) = E[\mathbf{Z}_{\boldsymbol{\vartheta}_1 st}(\boldsymbol{\vartheta})|\boldsymbol{\vartheta},\boldsymbol{\eta}]$, where we have exploited the serial independence of $\boldsymbol{\varepsilon}_t^*$, as well as the law of iterated expectations, together with the results in Proposition 1.

We can use the same arguments to show that the unconditional variance of $s_{\vartheta_2 t}(\boldsymbol{\vartheta},\boldsymbol{\eta})$ will be given by

$$\begin{aligned}
& E\left\{ \left[\begin{array}{cc} 0 & \mathbf{Z}_{\vartheta_2 st}(\boldsymbol{\vartheta}) \end{array} \right] \left[\begin{array}{cc} \mathcal{M}_{ll}(\boldsymbol{\eta}) & \mathbf{0} \\ \mathbf{0} & \mathcal{M}_{ss}(\boldsymbol{\eta}) \end{array} \right] \left[\begin{array}{c} 0 \\ \mathbf{Z}_{\vartheta_2 st}'(\boldsymbol{\vartheta}) \end{array} \right] \middle| \boldsymbol{\vartheta},\boldsymbol{\eta} \right\} \\
= & \frac{1}{4\vartheta_2^2} vec'(\mathbf{I}_N)[\mathrm{M}_{ss}(\boldsymbol{\eta})\,(\mathbf{I}_{N^2} + \mathbf{K}_{NN}) + [\mathrm{M}_{ss}(\boldsymbol{\eta}) - 1])vec(\mathbf{I}_N)vec'(\mathbf{I}_N)]vec(\mathbf{I}_N) \\
= & \frac{\{2\mathrm{M}_{ss}(\boldsymbol{\eta}) + N[\mathrm{M}_{ss}(\boldsymbol{\eta}) - 1]\}N}{4\vartheta_2^2}.
\end{aligned}$$

Hence, the residuals from the unconditional regression of $\mathbf{s}_{\boldsymbol{\vartheta}_1 t}(\boldsymbol{\vartheta},\boldsymbol{\eta})$ on $s_{\vartheta_2 t}(\boldsymbol{\vartheta},\boldsymbol{\eta})$ will be:

$$\begin{aligned}
\mathbf{s}_{\vartheta_1|\vartheta_2 t}(\boldsymbol{\vartheta},\boldsymbol{\eta}) &= \mathbf{Z}_{\boldsymbol{\vartheta}_1 lt}(\boldsymbol{\vartheta})\mathbf{e}_{lt}(\boldsymbol{\vartheta},\boldsymbol{\eta}) + \mathbf{Z}_{\boldsymbol{\vartheta}_1 st}(\boldsymbol{\vartheta})\mathbf{e}_{st}(\boldsymbol{\vartheta},\boldsymbol{\eta}) \\
-\frac{4\vartheta_2^2}{\{2\mathrm{M}_{ss}(\boldsymbol{\eta})+N[\mathrm{M}_{ss}(\boldsymbol{\eta})\text{-}1]\}N} & \frac{\{2\mathrm{M}_{ss}(\boldsymbol{\eta})+N[\mathrm{M}_{ss}(\boldsymbol{\eta})\text{-}1]\}}{2\vartheta_2}\mathbf{Z}_{\boldsymbol{\vartheta}_1 s}(\boldsymbol{\vartheta})vec(\mathbf{I}_N)\frac{1}{2\vartheta_2}vec'(\mathbf{I}_N)\mathbf{e}_{st}(\boldsymbol{\vartheta},\boldsymbol{\eta}) \\
&= \mathbf{Z}_{\boldsymbol{\vartheta}_1 lt}(\boldsymbol{\vartheta})\mathbf{e}_{lt}(\boldsymbol{\vartheta},\boldsymbol{\eta}) + [\mathbf{Z}_{\boldsymbol{\vartheta}_1 st}(\boldsymbol{\vartheta}) - \mathbf{Z}_{\boldsymbol{\vartheta}_1 s}(\boldsymbol{\vartheta},\boldsymbol{\eta})]\mathbf{e}_{st}(\boldsymbol{\vartheta},\boldsymbol{\eta}).
\end{aligned}$$

The first term of $\mathbf{s}_{\vartheta_1|\vartheta_2 t}(\boldsymbol{\vartheta}_0,\boldsymbol{\eta}_0)$ is clearly conditionally orthogonal to any function of $\varsigma_t(\boldsymbol{\vartheta}_0)$. In contrast, the second term is not conditionally orthogonal to functions of $\varsigma_t(\boldsymbol{\vartheta}_0)$, but since the conditional covariance between any such function and $\mathbf{e}_{st}(\boldsymbol{\vartheta}_0,\boldsymbol{\eta}_0)$ will be time-invariant, it will be unconditionally orthogonal by the law of iterated expectations. As a result, $\mathbf{s}_{\boldsymbol{\vartheta}_1|\vartheta_2 t}(\boldsymbol{\vartheta}_0,\boldsymbol{\eta}_0)$ will be unconditionally orthogonal to the elliptically symmetric tangent set, which in turn implies that the elliptically symmetric semiparametric estimator of $\boldsymbol{\vartheta}_1$ will be ϑ_2-adaptive.

To prove Part 1b, note that Proposition 7 and (A8) imply that the elliptically symmetric semiparametric efficient score corresponding to ϑ_2 will be given by

$$\begin{aligned}\mathring{s}_{\vartheta_2 t}(\boldsymbol{\vartheta}) &= -\frac{1}{2\vartheta_2} vec'(\mathbf{I}_N) vec\left\{\delta[\varsigma_t(\boldsymbol{\vartheta}),\boldsymbol{\eta}]\boldsymbol{\varepsilon}_t^*(\boldsymbol{\vartheta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\vartheta}) - \mathbf{I}_N\right\} \\ &- \frac{N}{2\vartheta_2}\left\{\left[\delta[\varsigma_t(\boldsymbol{\vartheta}),\boldsymbol{\eta}]\frac{\varsigma_t(\boldsymbol{\vartheta})}{N} - 1\right] - \frac{2}{(N+2)\kappa+2}\left[\frac{\varsigma_t(\boldsymbol{\vartheta})}{N} - 1\right]\right\} \\ = \frac{1}{2\vartheta_2}\{\delta[\varsigma_t(\boldsymbol{\vartheta}),\boldsymbol{\eta}]\varsigma_t(\boldsymbol{\vartheta}) - N\} &- \frac{N}{2\vartheta_2}\left\{\left[\delta[\varsigma_t(\boldsymbol{\vartheta}),\boldsymbol{\eta}]\frac{\varsigma_t(\boldsymbol{\vartheta})}{N} - 1\right] - \frac{2}{(N+2)\kappa+2}\left[\frac{\varsigma_t(\boldsymbol{\vartheta})}{N} - 1\right]\right\} \\ &= \frac{N}{\vartheta_2[(N+2)\kappa+2]}\left[\frac{\varsigma_t(\boldsymbol{\vartheta})}{N} - 1\right].\end{aligned}$$

But since the iterated elliptically symmetric semiparametric estimator of $\boldsymbol{\vartheta}$ must set to 0 the sample average of this modified score, it must be the case that $\sum_{t=1}^T \varsigma_t(\mathring{\boldsymbol{\vartheta}}_T) = \sum_{t=1}^T \varsigma_t^\circ(\mathring{\boldsymbol{\vartheta}}_{1T})/\mathring{\vartheta}_{2T} = NT$, which is equivalent to (30).

To prove Part 1c note that

$$\mathbf{s}_{\vartheta_2 t}(\boldsymbol{\vartheta},\mathbf{0}) = \frac{1}{2\vartheta_2}\left[\varsigma_t(\boldsymbol{\vartheta}) - N\right] \tag{A10}$$

is proportional to elliptically symmetric semiparametric efficient score $\mathring{s}_{\vartheta_2 t}(\boldsymbol{\vartheta})$, which means that the residual covariance matrix in the theoretical regression of this efficient score on the Gaussian score will have rank $p-1$ at most. But this residual covariance matrix coincides with $\mathring{\mathcal{S}}(\boldsymbol{\phi}) - \mathcal{A}(\boldsymbol{\phi})\mathcal{B}^{-1}(\boldsymbol{\phi})\mathcal{A}(\boldsymbol{\phi})$ since

$$E[\mathring{\mathbf{s}}_{\boldsymbol{\theta}t}(\boldsymbol{\phi})\mathbf{s}'_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}] = E[\mathbf{Z}_{dt}(\boldsymbol{\theta})\mathbf{e}_{dt}(\boldsymbol{\phi})\mathbf{e}'_{dt}(\boldsymbol{\theta},\mathbf{0})\mathbf{Z}'_{dt}(\boldsymbol{\theta})|\boldsymbol{\phi}] = \mathcal{A}(\boldsymbol{\theta}) \tag{A11}$$

because the regression residual

$$\left[\delta(\varsigma_t,\boldsymbol{\eta})\frac{\varsigma_t}{N} - 1\right] - \frac{2}{(N+2)\kappa_0+2}\left(\frac{\varsigma_t}{N} - 1\right)$$

is conditionally orthogonal to $\mathbf{e}_{dt}(\boldsymbol{\theta}_0,\mathbf{0})$ by the law of iterated expectations, as shown in the proof of proposition 7.

Tedious algebraic manipulations that exploit the block-triangularity of (A8) and the constancy of $\mathbf{Z}_{\vartheta_2 st}(\boldsymbol{\vartheta})$ show that the different information matrices will be block diagonal when $\mathbf{W}_{\boldsymbol{\vartheta}_1 s}(\boldsymbol{\phi}_0)$ is 0. Then, part 2a follows from the fact that $\mathbf{W}_{\boldsymbol{\vartheta}_1 s}(\boldsymbol{\phi}_0) = -E\left\{\partial d_t(\boldsymbol{\vartheta}_0)/\partial\boldsymbol{\vartheta}_1 |\, \boldsymbol{\phi}_0\right\}$ will trivially be 0 if (29) holds.

Finally, to prove Part 2b note that (A10) implies that the Gaussian PMLE will also satisfy (30). But since the asymptotic covariance matrices in both cases will be block-diagonal between $\boldsymbol{\vartheta}_1$ and ϑ_2 when (29) holds, the effect of estimating $\boldsymbol{\vartheta}_1$ becomes irrelevant. □

Proposition 13

We can directly work in terms of the $\boldsymbol{\psi}$ parameters thanks to our assumptions on the mapping $\mathbf{r}_g(.)$. Given the specification for the conditional mean and variance in (32), and the fact that $\boldsymbol{\varepsilon}_t^*$ is assumed to be *i.i.d.* conditional on $\mathbf{z}_t$ and I_{t-1}, it is tedious but otherwise straightforward to show that the score vector will be

$$\begin{bmatrix} \mathbf{s}_{\boldsymbol{\psi}_1 t}(\boldsymbol{\psi},\boldsymbol{\varrho}) \\ \mathbf{s}_{\boldsymbol{\psi}_2 t}(\boldsymbol{\psi},\boldsymbol{\varrho}) \\ \mathbf{s}_{\boldsymbol{\psi}_3 t}(\boldsymbol{\psi},\boldsymbol{\varrho}) \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{\boldsymbol{\psi}_1 lt}(\boldsymbol{\vartheta})\mathbf{e}_{lt}(\boldsymbol{\psi},\boldsymbol{\varrho}) + \mathbf{Z}_{\boldsymbol{\psi}_1 st}(\boldsymbol{\vartheta})\mathbf{e}_{st}(\boldsymbol{\psi},\boldsymbol{\varrho}) \\ \mathbf{Z}_{\boldsymbol{\psi}_2 st}(\boldsymbol{\vartheta})\mathbf{e}_{st}(\boldsymbol{\psi},\boldsymbol{\varrho}) \\ \mathbf{Z}_{\boldsymbol{\psi}_3 lt}(\boldsymbol{\vartheta})\mathbf{e}_{lt}(\boldsymbol{\psi},\boldsymbol{\varrho}) \end{bmatrix}, \tag{A12}$$

where

$$\left.\begin{aligned} &\mathbf{Z}_{\boldsymbol{\psi}_1 lt}(\boldsymbol{\psi}) = \left\{\partial\boldsymbol{\mu}_t^{\diamond\prime}(\boldsymbol{\psi}_1)/\partial\boldsymbol{\psi}_1 + \partial vec'[\boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_1)]/\partial\boldsymbol{\psi}_1\cdot(\boldsymbol{\psi}_3\otimes\mathbf{I}_N)\right\}\boldsymbol{\Sigma}_t^{\diamond -1/2\prime}(\boldsymbol{\psi}_1)\boldsymbol{\Psi}_2^{-1/2\prime}, \\ &\mathbf{Z}_{\boldsymbol{\psi}_1 st}(\boldsymbol{\psi}) = \partial vec'[\boldsymbol{\Sigma}_t^{\diamond 1/2}(\boldsymbol{\psi}_1)]/\partial\boldsymbol{\psi}_1\cdot[\boldsymbol{\Psi}_2^{1/2}\otimes\boldsymbol{\Sigma}_t^{\diamond -1/2\prime}(\boldsymbol{\psi}_1)\boldsymbol{\Psi}_2^{-1/2\prime}], \\ &\mathbf{Z}_{\boldsymbol{\psi}_2 st}(\boldsymbol{\psi}) = \mathbf{D}_N'(\mathbf{I}_N\otimes\boldsymbol{\Psi}_2^{-1/2\prime}) = \mathbf{Z}_{\boldsymbol{\psi}_2 s}(\boldsymbol{\psi}), \\ &\mathbf{Z}_{\boldsymbol{\psi}_3 lt}(\boldsymbol{\psi}) = \boldsymbol{\Psi}_2^{-1/2\prime} = \mathbf{Z}_{\boldsymbol{\psi}_3 l}(\boldsymbol{\psi}), \end{aligned}\right\} \tag{A13}$$

$\mathbf{D}_N$ is the duplication matrix of order N (see Magnus and Neudecker (1988)),

$$\begin{aligned} \mathbf{e}_{lt}(\boldsymbol{\psi},\boldsymbol{\varrho}) &= -\frac{\partial\ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi});\boldsymbol{\varrho}]}{\partial\boldsymbol{\varepsilon}^*}, \\ \mathbf{e}_{st}(\boldsymbol{\psi},\boldsymbol{\varrho}) &= -vec\left\{\mathbf{I}_N + \frac{\partial\ln f[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi});\boldsymbol{\varrho}]}{\partial\boldsymbol{\varepsilon}^*}\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\psi})\right\}, \\ \boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}) &= \boldsymbol{\Psi}_2^{-1/2}\boldsymbol{\Sigma}_t^{\diamond -1/2}(\boldsymbol{\psi}_1)[\mathbf{y}_t - \boldsymbol{\mu}_t^{\diamond}(\boldsymbol{\psi}_1) - \boldsymbol{\Sigma}_t^{\diamond 1/2}\boldsymbol{\psi}_3], \end{aligned} \tag{A14}$$

and $f(\boldsymbol{\varepsilon}^*;\boldsymbol{\varrho})$ is the conditional density of $\boldsymbol{\varepsilon}_t^*$ given $\mathbf{z}_t$, I_{t-1} and the shape parameters $\boldsymbol{\varrho}$.

It is then easy to see that the unconditional covariance between $\mathbf{s}_{\boldsymbol{\psi}_1 t}(\boldsymbol{\psi},\boldsymbol{\varrho})$ and the remaining elements of the score will be given by

$$\begin{bmatrix} \mathbf{Z}_{\boldsymbol{\psi}_1 l}(\boldsymbol{\psi},\boldsymbol{\varrho}) & \mathbf{Z}_{\boldsymbol{\psi}_1 s}(\boldsymbol{\psi},\boldsymbol{\varrho}) \end{bmatrix} \begin{bmatrix} \mathcal{M}_{ll}(\boldsymbol{\varrho}) & \mathcal{M}_{ls}(\boldsymbol{\varrho}) \\ \mathcal{M}_{ls}'(\boldsymbol{\varrho}) & \mathcal{M}_{ss}(\boldsymbol{\varrho}) \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{Z}_{\boldsymbol{\psi}_3 l}'(\boldsymbol{\psi}) \\ \mathbf{Z}_{\boldsymbol{\psi}_2 s}'(\boldsymbol{\psi}) & \mathbf{0} \end{bmatrix}$$

with $\mathbf{Z}_{\boldsymbol{\psi}_1 l}(\boldsymbol{\psi},\boldsymbol{\varrho}) = E[\mathbf{Z}_{\boldsymbol{\psi}_1 lt}(\boldsymbol{\psi})|\boldsymbol{\psi},\boldsymbol{\varrho}]$ and $\mathbf{Z}_{\boldsymbol{\psi}_1 s}(\boldsymbol{\psi},\boldsymbol{\varrho}) = E[\mathbf{Z}_{\boldsymbol{\psi}_1 st}(\boldsymbol{\psi})|\boldsymbol{\psi},\boldsymbol{\varrho}]$, where we have exploited the serial independence of $\boldsymbol{\varepsilon}_t^*$ and the constancy of $\mathbf{Z}_{\boldsymbol{\psi}_2 st}(\boldsymbol{\psi})$ and $\mathbf{Z}_{\boldsymbol{\psi}_3 lt}(\boldsymbol{\psi})$, together with the law of iterated expectations and the definition

$$\begin{bmatrix} \mathcal{M}_{ll}(\boldsymbol{\varrho}) & \mathcal{M}_{ls}(\boldsymbol{\varrho}) \\ \mathcal{M}_{ls}'(\boldsymbol{\varrho}) & \mathcal{M}_{ss}(\boldsymbol{\varrho}) \end{bmatrix} = V\left[\begin{array}{c} \mathbf{e}_{lt}(\boldsymbol{\psi},\boldsymbol{\varrho}) \\ \mathbf{e}_{st}(\boldsymbol{\psi},\boldsymbol{\varrho}) \end{array}\middle|\boldsymbol{\psi},\boldsymbol{\varrho}\right].$$

Similarly, the unconditional covariance matrix of $\mathbf{s}_{\boldsymbol{\psi}_2 t}(\boldsymbol{\psi},\boldsymbol{\varrho})$ and $\mathbf{s}_{\boldsymbol{\psi}_3 t}(\boldsymbol{\psi},\boldsymbol{\varrho})$ will be

$$\begin{bmatrix} \mathbf{0} & \mathbf{Z}_{\boldsymbol{\psi}_2 s}(\boldsymbol{\psi}) \\ \mathbf{Z}_{\boldsymbol{\psi}_3 l}(\boldsymbol{\psi}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathcal{M}_{ll}(\boldsymbol{\varrho}) & \mathcal{M}_{ls}(\boldsymbol{\varrho}) \\ \mathcal{M}_{ls}'(\boldsymbol{\varrho}) & \mathcal{M}_{ss}(\boldsymbol{\varrho}) \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{Z}_{\boldsymbol{\psi}_3 l}'(\boldsymbol{\psi}) \\ \mathbf{Z}_{\boldsymbol{\psi}_2 s}'(\boldsymbol{\psi}) & \mathbf{0} \end{bmatrix}.$$

Hence, the residuals from the unconditional least squares projection of $\mathbf{s}_{\boldsymbol{\psi}_1 t}(\boldsymbol{\psi},\boldsymbol{\varrho})$ on $\mathbf{s}_{\boldsymbol{\psi}_2 t}(\boldsymbol{\psi},\boldsymbol{\varrho})$ and $\mathbf{s}_{\boldsymbol{\psi}_3 t}(\boldsymbol{\psi},\boldsymbol{\varrho})$ will be:

$$\begin{aligned} \mathbf{s}_{\boldsymbol{\psi}_1|\boldsymbol{\psi}_2,\boldsymbol{\psi}_3 t}(\boldsymbol{\psi},\boldsymbol{\varrho}) &= \mathbf{Z}_{\boldsymbol{\psi}_1 lt}(\boldsymbol{\psi})\mathbf{e}_{lt}(\boldsymbol{\psi},\boldsymbol{\varrho}) + \mathbf{Z}_{\boldsymbol{\psi}_1 st}(\boldsymbol{\vartheta})\mathbf{e}_{st}(\boldsymbol{\psi},\boldsymbol{\varrho}) \\ &\quad - \begin{bmatrix} \mathbf{Z}_{\boldsymbol{\psi}_1 l}(\boldsymbol{\psi},\boldsymbol{\varrho}) & \mathbf{Z}_{\boldsymbol{\psi}_1 s}(\boldsymbol{\psi},\boldsymbol{\varrho}) \end{bmatrix} \begin{bmatrix} \mathbf{e}_{lt}(\boldsymbol{\psi},\boldsymbol{\varrho}) \\ \mathbf{e}_{st}(\boldsymbol{\psi},\boldsymbol{\varrho}) \end{bmatrix} \\ &= [\mathbf{Z}_{\boldsymbol{\psi}_1 lt}(\boldsymbol{\psi}) - \mathbf{Z}_{\boldsymbol{\psi}_1 l}(\boldsymbol{\psi},\boldsymbol{\varrho})]\mathbf{e}_{lt}(\boldsymbol{\psi},\boldsymbol{\varrho}) + [\mathbf{Z}_{\boldsymbol{\psi}_1 st}(\boldsymbol{\psi}) - \mathbf{Z}_{\boldsymbol{\psi}_1 s}(\boldsymbol{\psi},\boldsymbol{\varrho})]\mathbf{e}_{st}(\boldsymbol{\psi},\boldsymbol{\varrho}), \end{aligned}$$

because both $\mathbf{Z}_{\psi_2 s}(\psi)$ and $\mathbf{Z}_{\psi_3 l}(\psi)$ have full row rank when $\mathbf{\Psi}_2$ has full rank.

Although neither $\mathbf{e}_{lt}(\psi, \varrho)$ nor $\mathbf{e}_{st}(\psi, \varrho)$ will be conditionally orthogonal to arbitrary functions of ε_t^*, their conditional covariance with any such function will be time-invariant. Hence, $\mathbf{s}_{\psi_1|\psi_2,\psi_3 t}(\psi, \varrho)$ will be unconditionally orthogonal to $\partial \ln f[\varepsilon_t^*(\psi); \varrho]/\partial \varrho_i$ by virtue of the law of iterated expectations, which in turn implies that the unrestricted semiparametric estimator of ψ_1 will be (ψ_2, ψ_3)-adaptive.

To prove Part 1b note that the semiparametric efficient scores corresponding to ψ_2 and ψ_3 will be given by

$$\begin{bmatrix} \mathbf{0} & \mathbf{Z}_{\psi_2 s}(\psi) \\ \mathbf{Z}_{\psi_3 l}(\psi) & \mathbf{0} \end{bmatrix} \mathcal{K}(0)\mathcal{K}^+(\varrho_0) \left\{ \begin{array}{c} \varepsilon_t^*(\psi) \\ vec[\varepsilon_t^*(\psi)\varepsilon_t^{*\prime}(\psi) - \mathbf{I}_N] \end{array} \right\}$$

because $\mathbf{Z}_{\psi_2 st}(\vartheta) = \mathbf{Z}_{\psi_2 s}(\vartheta)$ and $\mathbf{Z}_{\psi_3 lt}(\vartheta) = \mathbf{Z}_{\psi_3 l}(\vartheta)$ $\forall t$. But if (35) and (36) hold, then the sample averages of $\mathbf{e}_{lt}[\psi_1, \psi_{2T}(\psi_1), \psi_{3T}(\psi_1); \mathbf{0}]$ and $\mathbf{e}_{st}[\psi_1, \psi_{2T}(\psi_1), \psi_{3T}(\psi_1); \mathbf{0}]$ will be 0, and the same is true of the semiparametric efficient score.

To prove Part 1c note that

$$\begin{bmatrix} \mathbf{s}_{\psi_2 t}(\psi, \mathbf{0}) \\ \mathbf{s}_{\psi_3 t}(\psi, \mathbf{0}) \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{Z}_{\psi_2 s}(\psi) \\ \mathbf{Z}_{\psi_3 l}(\psi) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \varepsilon_t^*(\psi) \\ vec[\varepsilon_t^*(\psi)\varepsilon_t^{*\prime}(\psi) - \mathbf{I}_N] \end{bmatrix}, \tag{A15}$$

which implies that the residual covariance matrix in the theoretical regression of the semiparametric efficient score on the Gaussian score will have rank $p - N(N+3)/2$ at most because both $\mathbf{Z}_{\psi_2 s}(\psi)$ and $\mathbf{Z}_{\psi_3 l}(\psi)$ have full row rank when $\mathbf{\Psi}_2$ has full rank. But as we saw in the proof of Proposition 8, that residual covariance matrix coincides with $\mathcal{S}(\phi_0) - \mathcal{A}(\theta)\mathcal{B}^{-1}(\phi)\mathcal{A}(\theta)$.

Tedious algebraic manipulations that exploit the block structure of (A13) and the constancy of $\mathbf{Z}_{\psi_2 st}(\psi)$ and $\mathbf{Z}_{\psi_3 lt}(\psi)$ show that the different information matrices will be block diagonal when $\mathbf{Z}_{\psi_1 l}(\psi, \varrho)$ and $\mathbf{Z}_{\psi_1 s}(\psi, \varrho)$ are both 0. But those are precisely the necessary and sufficient conditions for $\mathbf{s}_{\psi_1 t}(\psi, \varrho)$ to be equal to $\mathbf{s}_{\psi_1|\psi_2,\psi_3 t}(\psi, \varrho)$, which is also guaranteed by (34).

Finally, to prove Part 2b simply note that (A15) implies that the Gaussian PMLE will also satisfy (35) and (36). But since the asymptotic covariance matrices in both cases will be block-diagonal between ψ_1 and (ψ_2, ψ_3) when (34) holds, the effect of estimating ψ_1 becomes irrelevant. □

Proposition 14

The proof of the first part trivially follows from Proposition 8 and the fact that the partitioned inverse formula implies that

$$\mathcal{I}^{\eta\eta}(\phi_0) = \mathcal{I}_{\eta\eta}^{-1}(\phi_0) + \mathcal{I}_{\eta\eta}^{-1}(\phi_0)\mathcal{I}'_{\theta\eta}(\phi_0)\mathcal{I}^{\theta\theta}(\phi_0)\mathcal{I}_{\theta\eta}(\phi_0)\mathcal{I}_{\eta\eta}^{-1}(\phi_0).$$

To prove that $\mathcal{F}(\phi_0) \leq \mathcal{J}(\phi_0)$ it is convenient to note that both these matrices can also be decomposed into a component that reflects the asymptotic variance of these estimators if θ_0

were known, plus a second component that reflects the sample variability in the PML estimator $\tilde{\boldsymbol{\theta}}_T$. With respect to the first component, it is clear that $\mathcal{I}_{\eta\eta}^{-1}(\boldsymbol{\phi}_0) \leq \mathcal{L}(\boldsymbol{\phi}_0)/\mathcal{N}^2(\boldsymbol{\phi}_0)$. As for the second component, we must compare

$$\mathcal{I}'_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_0)\mathcal{C}(\boldsymbol{\phi}_0)\mathcal{I}_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_0)/\mathcal{I}_{\eta\eta}^2(\boldsymbol{\phi}_0) = \left[\frac{2\,(N+2)\,\nu^2}{(\nu-2)\,(N+\nu)\,(N+\nu+2)\,\mathcal{I}_{\eta\eta}(\boldsymbol{\phi}_0)}\right]^2 \mathbf{W}'_s(\boldsymbol{\phi}_0)\mathcal{C}(\boldsymbol{\phi}_0)\mathbf{W}_s(\boldsymbol{\phi}_0)$$

with

$$\mathcal{Q}'(\boldsymbol{\phi}_0)\mathcal{C}(\boldsymbol{\phi}_0)\mathcal{Q}(\boldsymbol{\phi}_0)/\mathcal{N}^2(\boldsymbol{\phi}_0) = \left[\frac{4(\nu-2)\,(\nu-4)}{N\nu^2\,(\nu-6)}\right]^2 \mathbf{W}'_s(\boldsymbol{\phi}_0)\mathcal{C}(\boldsymbol{\phi}_0)\mathbf{W}_s(\boldsymbol{\phi}_0).$$

The second expression will be larger than the first one if and only if

$$\mathcal{I}_{\eta\eta}(\boldsymbol{\phi}_0) - \frac{(N+2)\,N\nu^4\,(\nu-6)}{2\,(\nu-2)^2\,(\nu-4)\,(N+\nu)\,(N+\nu+2)} \geq 0.$$

We can then show that this inequality will be true for $N+2$ if it is true for N by using the recursion $\psi'(\nu/2) - \psi'(1+\nu/2) = -4\nu^2$ (see Abramowitz and Stegun (1964)), which reduces the problem to proving the inequality for $N=1$ and $N=2$. The proof for $N=2$ immediately follows from the same recursion. The proof for $N=1$ is more tedious, as it involves the asymptotic expressions for $\psi'(.)$ in Abramowitz and Stegun (1964).

To prove the last statement, it is also convenient to decompose the asymptotic variance of $\breve{\eta}_T$ into two components, namely:

$$\mathcal{G}(\boldsymbol{\phi}_0) = [\mathcal{E}(\boldsymbol{\phi}_0) - \mathcal{D}'(\boldsymbol{\phi}_0)\mathcal{B}^{-1}(\boldsymbol{\phi}_0)\mathcal{D}(\boldsymbol{\phi}_0)]/\mathcal{N}^2(\boldsymbol{\phi}_0)$$
$$+\{[\mathcal{R}(\boldsymbol{\phi}_0) - \mathcal{D}'(\boldsymbol{\phi}_0)\mathcal{B}^{-1}(\boldsymbol{\phi}_0)\mathcal{A}(\boldsymbol{\phi}_0)]'\mathcal{C}(\boldsymbol{\phi}_0)[\mathcal{R}(\boldsymbol{\phi}_0) - \mathcal{D}'(\boldsymbol{\phi}_0)\mathcal{B}^{-1}(\boldsymbol{\phi}_0)\mathcal{A}(\boldsymbol{\phi}_0)]\}/\mathcal{N}^2(\boldsymbol{\phi}_0)$$

In this set up, it is straightforward to prove that

$$[\mathcal{R}(\boldsymbol{\phi}_0) - \mathcal{D}'(\boldsymbol{\phi}_0)\mathcal{B}^{-1}(\boldsymbol{\phi}_0)\mathcal{A}(\boldsymbol{\phi}_0)] = \mathcal{Q}(\boldsymbol{\phi}_0)$$

if condition (37) holds. As for the first component, since $\mathcal{L}(\boldsymbol{\phi}_0)$ is the residual variance in the regression of $m_{\eta t}(\boldsymbol{\theta}_0, \eta_0)$ on $\varsigma_t/N-1$, while $\mathcal{E}(\boldsymbol{\phi}_0) - \mathcal{D}'(\boldsymbol{\phi}_0)\mathcal{B}^{-1}(\boldsymbol{\phi}_0)\mathcal{D}(\boldsymbol{\phi}_0)$ is the residual variance in the regression of $m_{\eta t}(\boldsymbol{\theta}_0, \eta_0)$ on $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0, 0)$, and the Gaussian pseudo-score can be written as $\mathbf{W}_s(\boldsymbol{\phi}_0)[\varsigma_t/N-1]$ plus an extra term that is orthogonal to ς_t, it is clear that

$$\mathcal{L}(\boldsymbol{\phi}_0) \leq \mathcal{E}(\boldsymbol{\phi}_0) - \mathcal{D}'(\boldsymbol{\phi}_0)\mathcal{B}^{-1}(\boldsymbol{\phi}_0)\mathcal{D}(\boldsymbol{\phi}_0),$$

with equality if and only if $[\varsigma_t/N-1]$ can be written as an exact linear combination of $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0, 0)$, as in (38). □

Proposition 15

The consistency of the Gaussian PML derives from the fact that $E[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0,0)|\mathbf{z}_t,I_{t-1};\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0]=\mathbf{0}$. Thus, if the pseudo-true value of η, η_∞ say, is 0, then the student-t based pseudo-true values of the conditional mean and variance parameters, $\boldsymbol{\theta}_\infty$ say, will coincide with their true values $\boldsymbol{\theta}_0$ by the law of iterated expectations. But since η is estimated subject to the inequality constraint $\eta\geq 0$, the population KT conditions that define η_∞ will be

$$E[s_{\eta t}(\boldsymbol{\theta}_\infty,\eta_\infty)|\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0]+\lambda_{\eta\infty}=0;\quad \eta_\infty\geq 0;\quad \lambda_{\eta\infty}\geq 0;\quad \eta_\infty\cdot\lambda_{\eta\infty}=0,$$

where $\lambda_{\eta\infty}$ is the pseudo-true value of the KT multiplier, and the expectation is taken with respect to the true unconditional distribution of the observations (see Calzolari, Fiorentini and Sentana (2004)). Hence, $\eta_\infty=0$ if and only if $E[s_{\eta t}(\boldsymbol{\theta}_0,0)|\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0]\leq 0$.

Given that $\varsigma_t(\boldsymbol{\theta}_0)=\boldsymbol{\varepsilon}_t^{*\prime}\boldsymbol{\varepsilon}_t^*$, we can write

$$\begin{aligned} s_{\eta t}(\boldsymbol{\theta}_0,0) &= \frac{N(N+2)}{4}-\frac{N+2}{2}\varsigma_t(\boldsymbol{\theta}_0)+\frac{1}{4}\varsigma_t^2(\boldsymbol{\theta}_0)\\ &= \frac{N(N+2)}{4}\left[\frac{(\boldsymbol{\varepsilon}_{\mathbf{t}}^{*\prime}\boldsymbol{\varepsilon}_t^*)^2}{N(N+2)}-1\right]+\frac{N+2}{2}[(\boldsymbol{\varepsilon}_{\mathbf{t}}^{*\prime}\boldsymbol{\varepsilon}_t^*)-N]. \end{aligned}$$

But since we have normalised the innovations so that $E(\boldsymbol{\varepsilon}_{\mathbf{t}}^*\boldsymbol{\varepsilon}_t^{*\prime}|\mathbf{z}_t,I_{t-1};\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0)=\mathbf{I}_N$, then

$$N=tr(\mathbf{I}_N)=tr[E(\boldsymbol{\varepsilon}_{\mathbf{t}}^*\boldsymbol{\varepsilon}_t^{*\prime}|\mathbf{z}_t,I_{t-1};\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0)]=E[tr(\boldsymbol{\varepsilon}_{\mathbf{t}}^*\boldsymbol{\varepsilon}_t^{*\prime})|\mathbf{z}_t,I_{t-1};\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0]=E(\boldsymbol{\varepsilon}_{\mathbf{t}}^{*\prime}\boldsymbol{\varepsilon}_t^*|\mathbf{z}_t,I_{t-1};\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0)$$

by the linearity of the expectation and trace operators. Therefore, it immediately follows that

$$\lambda_{\eta\infty}=\min\{0,-E[s_{\eta t}(\boldsymbol{\theta}_0,0)|\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0]\}=\min\left\{0,-\frac{N(N+2)}{4}\kappa_0\right\}$$

in view of the definition of κ_0. Therefore, $\eta_\infty=0$ if and only if $\kappa_0\leq 0$.

To prove the second and third parts, we can use Propositions 1 and 2 in Calzolari, Fiorentini and Sentana (2004) if we regard the student t based estimator $\hat{\boldsymbol{\phi}}_T$ as the "inequality restricted" PML estimator of $\boldsymbol{\phi}$, and the Gaussian-based estimator $\tilde{\boldsymbol{\phi}}_T=(\tilde{\boldsymbol{\theta}}_T,0)$ as its "equality restricted" counterpart, both of which share not only the pseudo-true values $(\boldsymbol{\theta}_0,0,\lambda_{\eta\infty})$ when $\kappa_0\leq 0$, but also the modified pseudo-score $\mathbf{m}_t(\boldsymbol{\theta}_0,0,\lambda_{\eta\infty})=s_{\phi t}(\boldsymbol{\theta}_0,0)+\mathbf{e}_{p+1}\cdot\lambda_{\eta\infty}$, where $\mathbf{e}_{p+1}$ is the $(p+1)^{th}$ column of $\mathbf{I}_{p+1}$, as well as the expected value of the average Hessian $\mathcal{H}(\boldsymbol{\phi}_\infty;\boldsymbol{\varphi}_0)=E[\bar{\mathbf{h}}_T(\boldsymbol{\phi}_0)|\boldsymbol{\theta}_0,\boldsymbol{\varrho}_0]$.

Specifically, Proposition 1 in Calzolari, Fiorentini and Sentana (2004) implies here that

$$\lambda_{\eta\infty}\cdot\sqrt{T}\hat{\eta}_T=o_p(1),$$

while their Proposition 2 implies that

$$\left[\begin{array}{cc}\mathcal{H}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0) & \mathcal{H}_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0)\\ \mathcal{H}'_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0) & \mathcal{H}_{\eta\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0)\end{array}\right]\sqrt{T}\left(\begin{array}{c}\hat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}_0\\ \hat{\eta}_T\end{array}\right)+\mathbf{e}_{p+1}\sqrt{T}(\hat{\lambda}_{\eta T}-\lambda_{\eta\infty})$$
$$-\sqrt{T}\bar{\mathbf{m}}_T(\boldsymbol{\theta}_0,0,\lambda_{\eta\infty})=o_p(1),$$
$$\left[\begin{array}{cc}\mathcal{H}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0) & \mathcal{H}_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0)\\ \mathcal{H}'_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0) & \mathcal{H}_{\eta\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0)\end{array}\right]\sqrt{T}\left(\begin{array}{c}\tilde{\boldsymbol{\theta}}_T-\boldsymbol{\theta}_0\\ 0\end{array}\right)+\mathbf{e}_{p+1}\sqrt{T}(\tilde{\lambda}_{\eta T}-\lambda_{\eta\infty})$$
$$-\sqrt{T}\bar{\mathbf{m}}_T(\boldsymbol{\theta}_0,0,\lambda_{\eta\infty})=o_p(1),$$

where $\hat{\lambda}_{\eta T}$ and $\tilde{\lambda}_{\eta T}$ are the sample versions of the KT and Lagrange multipliers associated to the constraint $\eta = 0$. As a consequence,

$$\left[\begin{array}{cc}\mathcal{H}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0) & \mathcal{H}_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0)\\ \mathcal{H}'_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0) & \mathcal{H}_{\eta\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0)\end{array}\right]\sqrt{T}\left(\begin{array}{c}\hat{\boldsymbol{\theta}}_T-\tilde{\boldsymbol{\theta}}_T\\ \hat{\eta}_T\end{array}\right)+\mathbf{e}_{p+1}\sqrt{T}(\hat{\lambda}_{\eta T}-\tilde{\lambda}_{\eta T})=o_p(1).$$

Part 2 immediately follows from the fact that $\lambda_{\eta\infty} > 0$ when $\kappa_0 < 0$. Similarly, the first statement of Part 3 follows from the fact that $\lambda_{\eta\infty} = 0$ when $\kappa_0 = 0$. As for the condition (39), which derives directly from the expression for $\mathbf{h}_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi})$ in FSC evaluated at $(\boldsymbol{\theta}_0, 0)$, its role is to guarantee that $\mathcal{H}_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_{\infty};\boldsymbol{\varphi}_0) = \mathbf{0}$. In this sense, it is worth mentioning that condition (39) will be satisfied for instance if $\boldsymbol{\varepsilon}^*_t|\mathbf{z}_t, I_{t-1};\boldsymbol{\phi}_0$ is *i.i.d.* $s(\mathbf{0},\mathbf{I}_N,\boldsymbol{\eta}_0)$ with $\kappa_0 = 0$ irrespective of whether or not it is Gaussian because in that case

$$E\{[N+2-\varsigma_t(\boldsymbol{\theta}_0)]\boldsymbol{\varepsilon}^*_t(\boldsymbol{\theta}_0)|\mathbf{z}_t, I_{t-1};\boldsymbol{\theta}_0,\boldsymbol{\eta}_0] = E[(N+2-\varsigma_t)\sqrt{\varsigma_t}\mathbf{u}_t|\boldsymbol{\eta}_0] = \mathbf{0}$$

by the serial and mutual independence of ς_t and $\mathbf{u}_t$, and the fact that $E(\mathbf{u}_t) = \mathbf{0}$, while

$$E\{[N+2-\varsigma_t(\boldsymbol{\theta}_0)]\boldsymbol{\varepsilon}^*_t(\boldsymbol{\theta}_0)\boldsymbol{\varepsilon}^{*\prime}_t(\boldsymbol{\theta}_0)|\mathbf{z}_t, I_{t-1},\boldsymbol{\phi}_0\} = E[(N+2-\varsigma_t)\varsigma_t\mathbf{u}_t\mathbf{u}'_t|\boldsymbol{\eta}_0]$$
$$= N^{-1}E[(N+2-\varsigma_t)\varsigma_t|\boldsymbol{\eta}_0]\mathbf{I}_N = \mathbf{0}$$

by the definition of κ_0 and the fact that $E(\mathbf{u}_t\mathbf{u}'_t) = N^{-1}\mathbf{I}_N$. □

Proposition 16

As in the proof of Proposition 12, we can directly work in terms of the $\boldsymbol{\vartheta}$ parameters thanks to our assumptions on the mapping $\mathbf{r}_s(.)$. Let us initially keep η fixed to some positive value. The student t score vector for the remaining parameters will then be given by (A9). But since

$$\boldsymbol{\varepsilon}^*_t(\boldsymbol{\vartheta}_{10},\vartheta_2) = \sqrt{1/\vartheta_2}\boldsymbol{\Sigma}_t^{\circ-1/2}(\boldsymbol{\vartheta}_{10})[\mathbf{y}_t-\boldsymbol{\mu}_t(\boldsymbol{\vartheta}_{10})] = \sqrt{\vartheta_{20}/\vartheta_2}\boldsymbol{\varepsilon}^*_t,$$

so that

$$\varsigma_t(\boldsymbol{\vartheta}_{10},\vartheta_{2\infty}) = (\vartheta_{20}/\vartheta_2)\varsigma_t,$$

we will have that

$$\mathbf{e}_{lt}(\boldsymbol{\vartheta}_{10},\vartheta_2,\eta)=\delta[(\vartheta_{20}/\vartheta_2)\varsigma_t,\eta]\sqrt{\vartheta_{20}/\vartheta_2}\boldsymbol{\varepsilon}^*_t=\delta[(\vartheta_{20}/\vartheta_2)\varsigma_t,\eta]\sqrt{\vartheta_{20}/\vartheta_2}\sqrt{\varsigma_t}\mathbf{u}_t,$$
$$\mathbf{e}_{st}(\boldsymbol{\vartheta}_{10},\vartheta_2,\eta)=vec\left[\delta[(\vartheta_{20}/\vartheta_2)\varsigma_t,\eta](\vartheta_{20}/\vartheta_2)\boldsymbol{\varepsilon}^*_t\boldsymbol{\varepsilon}^{*\prime}_t-\mathbf{I}_N\right]=vec\left[\delta[(\vartheta_{20}/\vartheta_2)\varsigma_t,\eta](\vartheta_{20}/\vartheta_2)\varsigma_t\mathbf{u}_t\mathbf{u}'_t-\mathbf{I}_N\right].$$

Then, it follows that $E[\mathbf{e}_{lt}(\boldsymbol{\vartheta}_{10}, \vartheta_2, \eta)|\mathbf{z}_t, I_{t-1}; \boldsymbol{\varphi}_0] = \mathbf{0}$ regardless of ϑ_2 and η because of the serial and mutual independence of ς_t and $\mathbf{u}_t$, and the fact that $E(\mathbf{u}_t) = \mathbf{0}$. On the other hand,

$$E[\mathbf{e}_{st}(\boldsymbol{\vartheta}_{10}, \vartheta_2, \eta)|\mathbf{z}_t, I_{t-1}; \boldsymbol{\varphi}_0] = E\left\{\delta[(\vartheta_{20}/\vartheta_2)\varsigma_t, \eta](\vartheta_{20}/\vartheta_2)(\varsigma_t/N) - 1|\,\boldsymbol{\varphi}_0\right\} vec(\mathbf{I}_N)$$

because of the serial and mutual independence of ς_t and $\mathbf{u}_t$, and the fact that $E(\mathbf{u}_t\mathbf{u}_t') = N^{-1}\mathbf{I}_N$.

If we define $\vartheta_{2\infty}(\eta)$ as the value that solves the implicit equation

$$E\left[\delta\{[\vartheta_{20}/\vartheta_2(\eta)]\varsigma_t, \eta\}[\vartheta_{20}/\vartheta_2(\eta)](\varsigma_t/N) - 1|\,\boldsymbol{\varphi}_0\right] = 0, \tag{A16}$$

then it is straightforward to show that

$$E\{\mathbf{s}_{\boldsymbol{\vartheta}t}[\boldsymbol{\vartheta}_{10}, \vartheta_{2\infty}(\eta), \eta]|\mathbf{z}_t, I_{t-1}; \boldsymbol{\varphi}_0\} = \mathbf{0}. \tag{A17}$$

Finally, if we choose η_∞ as the solution to the implicit equation

$$E\{s_{\eta t}[\boldsymbol{\vartheta}_{10}, \vartheta_{2\infty}(\eta), \eta]|\boldsymbol{\varphi}_0\} = 0, \tag{A18}$$

then it is clear that $\boldsymbol{\vartheta}_{10}, \vartheta_{2\infty}(\eta_\infty)$ and η_∞ will be the pseudo-true values of the parameters.

To obtain the variance of the t-score under misspecification, we can follow exactly the same steps as in the proof of Proposition 1 by exploiting the fact that (A16), (A17) and (A18) hold at the pseudo-true parameter values $\boldsymbol{\phi}_\infty$.

These three conditions also allow us to obtain the expected value of the Hessian along the lines of Proposition 1.

As we mentioned in the proof of Proposition (12), we can tediously show that the condition for block-diagonality of the expected value of the Hessian and the covariance matrix of the score is $E[\mathbf{W}_{\vartheta_1 st}(\boldsymbol{\vartheta}_{10}, \vartheta_{2\infty})|\boldsymbol{\varphi}_0] = 0$. But this condition will be satisfied if (29) holds because $\mathbf{W}_{\vartheta_1 st}(\boldsymbol{\vartheta}_{10}, \boldsymbol{\vartheta}_{2\infty})$ coincides with $\mathbf{W}_{\vartheta_1 st}(\boldsymbol{\vartheta}_{10}, \vartheta_{20})$ in view of (A8). □

Proposition 17

As in the proof of Proposition 13, we can directly work in terms of the $\boldsymbol{\psi}$ parameters thanks to our assumptions on the mapping $\mathbf{r}_g(.)$. Let us initially keep η fixed to some positive value. The student t score vector for the remaining parameters will then be given by (A12), where the elements of $\mathbf{Z}_t$ are defined in (A13), $\mathbf{e}_{lt}(\boldsymbol{\psi}, \eta)$ and $\mathbf{e}_{st}(\boldsymbol{\psi}, \eta)$ are analogous to (5) and (6), respectively, and $\boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi})$ is defined in (A14).

We can immediately see from (A14) that

$$\boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3) = \boldsymbol{\Psi}_2^{-1}(\boldsymbol{\psi}_{30} - \boldsymbol{\psi}_3) + \boldsymbol{\Psi}_2^{-1}\boldsymbol{\Psi}_{20}\boldsymbol{\varepsilon}_t^*,$$

so that both this variable and $\varsigma_t(\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3) = \boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3)\boldsymbol{\varepsilon}_t^*(\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3)$ will be *i.i.d.* conditional on $\mathbf{z}_t$ and I_{t-1}. Let $\boldsymbol{\psi}_{2\infty}(\eta)$ and $\boldsymbol{\psi}_{3\infty}(\eta)$ solve the implicit system of $N+N(N+1)/2$ equations

$$
\begin{aligned}
E\left[\left.\frac{N\eta+1}{1-2\eta+\eta\varsigma_t[\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_{2\infty}(\eta), \boldsymbol{\psi}_{3\infty}(\eta)]}\boldsymbol{\varepsilon}_t^*[\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_{2\infty}(\eta), \boldsymbol{\psi}_{3\infty}(\eta)]\right|\boldsymbol{\varphi}_0\right] &= \mathbf{0}, \\
vech\left\{E\left[\left.\frac{N\eta+1}{1-2\eta+\eta\varsigma_t[\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_{2\infty}(\eta), \boldsymbol{\psi}_{3\infty}(\eta)]}\boldsymbol{\varepsilon}_t^*[\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_{2\infty}(\eta), \boldsymbol{\psi}_{3\infty}(\eta)]\right|\boldsymbol{\varphi}_0\right]\right\} &= \mathbf{0}.
\end{aligned}
$$

Then, it is follows that

$$E\{s_{\boldsymbol{\psi}t}[\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_{2\infty}(\eta), \boldsymbol{\psi}_{3\infty}(\eta)]|\mathbf{z}_t, I_{t-1}; \boldsymbol{\varphi}_0\} = \mathbf{0},$$

where we have exploited the symmetry of the matrix

$$\frac{N\eta+1}{1-2\eta+\eta\varsigma_t[\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_{2\infty}(\eta), \boldsymbol{\psi}_{3\infty}(\eta)]}\boldsymbol{\varepsilon}_t^*[\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_{2\infty}(\eta), \boldsymbol{\psi}_{3\infty}(\eta) \cdot \boldsymbol{\varepsilon}_t^{*\prime}[\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_{2\infty}(\eta), \boldsymbol{\psi}_{3\infty}(\eta)]].$$

Finally, if we choose η_∞ as the solution to the implicit equation

$$E\{s_{\eta t}[\boldsymbol{\psi}_{10}, \boldsymbol{\psi}_{2\infty}(\eta), \boldsymbol{\psi}_{3\infty}(\eta)]|\boldsymbol{\varphi}_0\} = 0,$$

then it is clear that $\boldsymbol{\psi}_{10}$, $\boldsymbol{\psi}_{2\infty}(\eta_\infty)$, $\boldsymbol{\psi}_{3\infty}(\eta_\infty)$ and η_∞ will be the pseudo-true values of the parameters. □

Proposition 18

Let $\boldsymbol{\phi}_\infty$ denote the pseudo-true values of $\boldsymbol{\phi}$ corresponding to the student t-based log-likelihood function, which can be implicitly characterised by the moment conditions

$$
\begin{aligned}
E[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_\infty, \eta_\infty)|\boldsymbol{\varphi}_0] = \mathbf{0}, \\
E[s_{\boldsymbol{\eta}t}(\boldsymbol{\theta}_\infty, \eta_\infty)|\boldsymbol{\varphi}_0] = \mathbf{0}.
\end{aligned}
\tag{A19}
$$

The score version of the Hausman test can be regarded as an unconditional moment test of

$$E[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_\infty, 0)|\boldsymbol{\varphi}_0] = \mathbf{0}, \tag{A20}$$

which will hold if the conditional distribution of $\boldsymbol{\varepsilon}_t^*$ is *i.i.d.* $t(\mathbf{0}, \mathbf{I}, \eta_0)$ because $\boldsymbol{\theta}_\infty = \boldsymbol{\theta}_0$ in that case. If we knew $\boldsymbol{\theta}_\infty$, it would be straightforward to test whether (A20) holds. But since we do not know $\boldsymbol{\theta}_\infty$, we replace it by its consistent estimator $\hat{\boldsymbol{\theta}}_T$, where $\hat{\boldsymbol{\theta}}_T$ and $\hat{\eta}_T$ satisfy the sample analogues of (A19). In order to account for the sampling variability that this introduces, we can compute the limiting unconditional least squares regression of $\sqrt{T}\bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\boldsymbol{\theta}_\infty, 0)$ on $\sqrt{T}\bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\boldsymbol{\theta}_\infty, \eta_\infty)$ and $\sqrt{T}\bar{s}_{\boldsymbol{\eta}T}(\boldsymbol{\theta}_\infty, \eta_\infty)$, and retain the residuals. But since $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0, 0)$, $\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}_0, \eta_0)$ and $s_{\boldsymbol{\eta}t}(\boldsymbol{\theta}_0, \eta_0)$ are martingale difference sequences under the null, we can simply regress the first on the last two. To do so, we need their joint asymptotic distribution, which in view of Propositions 1, 2 and 3 will be given by

$$\sqrt{T}\left[\begin{array}{c}\bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\boldsymbol{\theta}_0,0)\\ \bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\boldsymbol{\theta}_0,\eta_0)\\ \bar{s}_{\eta T}(\boldsymbol{\theta}_0,\eta_0)\end{array}\right]\stackrel{d}{\to}N\left\{\left(\begin{array}{c}\mathbf{0}\\ \mathbf{0}\\ 0\end{array}\right),\left[\begin{array}{ccc}\mathcal{B}(\boldsymbol{\phi}_0) & \mathcal{A}(\boldsymbol{\phi}_0) & \mathbf{0}\\ \mathcal{A}(\boldsymbol{\phi}_0) & \mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) & \mathcal{I}_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_0)\\ \mathbf{0}' & \mathcal{I}'_{\boldsymbol{\theta}\eta}(\boldsymbol{\phi}_0) & \mathcal{I}_{\eta\eta}(\boldsymbol{\phi}_0)\end{array}\right]\right\}.$$

Hence, we can use standard arguments to show that

$$\sqrt{T}\bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\hat{\boldsymbol{\theta}}_T,0)\stackrel{d}{\to}N[\mathbf{0},\mathcal{B}(\boldsymbol{\phi}_0)-\mathcal{A}(\boldsymbol{\phi}_0)\mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\mathcal{A}(\boldsymbol{\phi}_0)]$$

and

$$\sqrt{T}\left[\begin{array}{c}\tilde{\boldsymbol{\theta}}_T-\boldsymbol{\theta}_0\\ \hat{\boldsymbol{\theta}}_T-\boldsymbol{\theta}_0\end{array}\right]\stackrel{d}{\to}N\left\{\left(\begin{array}{c}\mathbf{0}\\ \mathbf{0}\end{array}\right),\left[\begin{array}{cc}\mathcal{C}(\boldsymbol{\phi}_0) & -\mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\\ -\mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0) & \mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\end{array}\right]\right\},$$

whence we can easily prove that

$$\sqrt{T}\bar{\mathbf{s}}_{\boldsymbol{\theta}T}(\hat{\boldsymbol{\theta}}_T,0)-\mathcal{A}(\boldsymbol{\phi}_0)\sqrt{T}(\tilde{\boldsymbol{\theta}}_T-\hat{\boldsymbol{\theta}}_T)=o_p(1),$$
$$\sqrt{T}(\tilde{\boldsymbol{\theta}}_T-\hat{\boldsymbol{\theta}}_T)\to N\left[\mathbf{0},\mathcal{C}(\boldsymbol{\phi}_0)-\mathcal{I}^{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\right],$$

as well as the asymptotic chi-square distribution of $H^W_{\boldsymbol{\theta}T}$. □

Proposition 19

The proof proceeds along the same lines of the previous one once we show that

$$E[\mathring{\mathbf{s}}_{\boldsymbol{\theta}t}(\boldsymbol{\phi})\mathbf{s}'_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}]=-\partial E[\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta},\mathbf{0})|\boldsymbol{\phi}]/\partial\boldsymbol{\theta} \tag{A21}$$

and

$$E[\mathring{\mathbf{s}}_{\boldsymbol{\theta}t}(\boldsymbol{\phi})\mathring{\mathbf{s}}'_{\boldsymbol{\theta}t}(\boldsymbol{\phi})|\boldsymbol{\phi}]=-\partial E[\mathring{\mathbf{s}}_{\boldsymbol{\theta}t}(\boldsymbol{\phi})|\boldsymbol{\phi}]/\partial\boldsymbol{\theta}. \tag{A22}$$

Condition (A21) follows immediately from (A11) and the generalised information matrix equality. As for (A22), we can use the same equality together with some of the arguments in the proof of Proposition 7 to show that

$$-\frac{\partial E[\mathring{\mathbf{s}}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}_0)|\boldsymbol{\phi}_0]}{\partial\boldsymbol{\theta}}=E[\mathring{\mathbf{s}}_{\boldsymbol{\theta}t}(\boldsymbol{\phi}_0)\mathbf{s}'_{\boldsymbol{\theta}t}(\boldsymbol{\phi}_0)|\boldsymbol{\phi}]=E[\mathbf{Z}_{dt}(\boldsymbol{\theta}_0)\mathbf{e}_{dt}(\boldsymbol{\phi}_0)\mathbf{e}'_{dt}(\boldsymbol{\phi}_0)\mathbf{Z}'_{dt}(\boldsymbol{\theta}_0)|\boldsymbol{\phi}_0]$$
$$-E\left\{\mathbf{W}_s(\boldsymbol{\phi}_0)\left[\left[\delta(\varsigma_t,\boldsymbol{\eta}_0)\frac{\varsigma_t}{N}-1\right]-\frac{2}{(N+2)\kappa_0+2}\left(\frac{\varsigma_t}{N}-1\right)\right]\mathbf{e}'_{dt}(\boldsymbol{\phi}_0)\mathbf{Z}'_{dt}(\boldsymbol{\theta}_0)\right|\boldsymbol{\phi}_0\left.\vphantom{\frac{1}{1}}\right\}$$
$$=\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)-\mathbf{W}_s(\boldsymbol{\phi}_0)E\left\{\left[\left\{\delta(\varsigma_t,\boldsymbol{\eta}_0)\frac{\varsigma_t}{N}-1\right\}-\frac{2}{(N+2)\kappa_0+2}\left(\frac{\varsigma_t}{N}-1\right)\right]\mathbf{e}'_{dt}(\boldsymbol{\phi}_0)\right|\boldsymbol{\phi}_0\left.\vphantom{\frac{1}{1}}\right\}\mathbf{Z}_d(\boldsymbol{\theta}_0)$$
$$=\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)\text{-}\mathbf{W}_s(\boldsymbol{\phi}_0)E\left[\left\{\left[\delta(\varsigma_t,\boldsymbol{\eta}_0)\frac{\varsigma_t}{N}\text{-}1\right]\text{-}\frac{2}{(N+2)\kappa_0+2}\left(\frac{\varsigma_t}{N}\text{-}1\right)\right\}\left[\delta(\varsigma_t,\boldsymbol{\eta}_0)\frac{\varsigma_t}{N}\text{-}1\right]\right|\boldsymbol{\phi}_0\left.\vphantom{\frac{1}{1}}\right]\mathbf{W}'_s(\boldsymbol{\phi}_0)$$
$$=\mathcal{I}_{\boldsymbol{\theta\theta}}(\boldsymbol{\phi}_0)-\mathbf{W}_s(\boldsymbol{\phi}_0)\mathbf{W}'_s(\boldsymbol{\phi}_0)\cdot\left\{\left[\frac{N+2}{N}\mathrm{M}_{ss}(\boldsymbol{\eta}_0)-1\right]-\frac{4}{N\left[(N+2)\kappa_0+2\right]}\right\}=\mathring{\mathcal{S}}(\boldsymbol{\phi}_0).$$

□

B Computational issues

B.1 Elliptically symmetric efficient score and semiparametric efficiency bound

If we combine model (27) with the conditional variance specification in (40), then $\boldsymbol{\theta} = (\boldsymbol{\pi}', \boldsymbol{\rho}', \mathbf{c}', \boldsymbol{\gamma}', \alpha, \beta)'$ after normalising the unconditional variance parameter λ to 1.

The Jacobian matrices of $\boldsymbol{\mu}_t(\boldsymbol{\theta})$ and $\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$ are:

$$\frac{\partial \boldsymbol{\mu}_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'} = [\mathbf{I}_N - diag(\boldsymbol{\rho})]\frac{\partial \boldsymbol{\pi}}{\partial \boldsymbol{\theta}'} + diag(\mathbf{y}_{t-1} - \boldsymbol{\pi})\frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{\theta}'}$$

and

$$\frac{\partial vec\,[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})]}{\partial \boldsymbol{\theta}'} = (\mathbf{I}_{N^2} + \mathbf{K}_{NN})[\lambda_t(\boldsymbol{\theta})\mathbf{c} \otimes \mathbf{I}_N]\frac{\partial \mathbf{c}}{\partial \boldsymbol{\theta}'} + \mathbf{E}_N \frac{\partial \boldsymbol{\gamma}}{\partial \boldsymbol{\theta}'} + (\mathbf{c} \otimes \mathbf{c})\frac{\partial \lambda_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}'},$$

respectively, where $\mathbf{E}_N' = (\mathbf{e}_1\mathbf{e}_1'|\ldots|\mathbf{e}_N\mathbf{e}_N')$, with $(\mathbf{e}_1|\ldots|\mathbf{e}_N) = \mathbf{I}_N$, is the unique $N^2 \times N$ "diagonalisation" matrix that transforms $vec(\mathbf{A})$ into $vecd(\mathbf{A})$ as $vecd(\mathbf{A}) = \mathbf{E}_N' vec(\mathbf{A})$ (see Magnus (1988)).

After some straightforward algebraic manipulations, (2)-(6) lead to:

$$\begin{aligned}
\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}) &= \begin{pmatrix} [\mathbf{I}_N - diag(\boldsymbol{\rho})]\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] \\ diag(\mathbf{y}_{t-1} - \boldsymbol{\pi})\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] \\ \boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t'(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\lambda_t(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] - \boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\lambda_t(\boldsymbol{\theta}) \\ \frac{1}{2}vecd\left[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t'(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] - \boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\right] \\ 0 \\ 0 \end{pmatrix} \\
&\quad + \frac{1}{2}\frac{\partial \boldsymbol{\lambda}_t'(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\left[\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t'(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}] - \mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\right], \\
\frac{\partial \lambda_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} &= \alpha\left[2f_{kt-1}(\boldsymbol{\theta})\frac{\partial f_{kt-1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \frac{\partial \omega_{t-1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right] + \beta\frac{\partial \lambda_{t-1}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \\
&\quad + [f_{kt-1}^2(\boldsymbol{\theta}) + \omega_{t-1}(\boldsymbol{\theta}) - 1]\frac{\partial \alpha}{\partial \boldsymbol{\theta}} + [\lambda_{t-1}(\boldsymbol{\theta}) - 1]\frac{\partial \beta}{\partial \boldsymbol{\theta}}.
\end{aligned}$$

Finally, if we take as initial conditions $\boldsymbol{\mu}_1(\boldsymbol{\theta}) = \boldsymbol{\pi}$ and $\lambda_1(\boldsymbol{\theta}) = 1$, then $\partial\boldsymbol{\mu}_1(\boldsymbol{\theta})/\partial\boldsymbol{\theta}' = \partial\boldsymbol{\pi}/\partial\boldsymbol{\theta}'$ and $\partial\lambda_1(\boldsymbol{\theta})/\partial\boldsymbol{\theta}' = \mathbf{0}$.

If $\boldsymbol{\gamma} > \mathbf{0}$, we can use the Woodbury formula to prove that

$$\begin{aligned}
f_{kt}(\boldsymbol{\theta}) &= \omega_t(\boldsymbol{\theta})\mathbf{c}'\boldsymbol{\Gamma}^{-1}\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}), \\
\omega_t(\boldsymbol{\theta}) &= [\lambda_t^{-1}(\boldsymbol{\theta}) + \mathbf{c}'\boldsymbol{\Gamma}^{-1}\mathbf{c}]^{-1}, \\
\varsigma_t(\boldsymbol{\theta}) &= \boldsymbol{\varepsilon}_t'(\boldsymbol{\theta})\boldsymbol{\Gamma}^{-1}\boldsymbol{\varepsilon}_t(\boldsymbol{\theta}) - f_{kt}^2(\boldsymbol{\theta})/\omega_t(\boldsymbol{\theta}), \\
\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta}) &= \boldsymbol{\Gamma}^{-1} - \omega_t(\boldsymbol{\theta})\boldsymbol{\Gamma}^{-1}\mathbf{c}\mathbf{c}'\boldsymbol{\Gamma}^{-1},
\end{aligned}$$

$$\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c} = \boldsymbol{\Gamma}^{-1}\mathbf{c}\omega_t(\boldsymbol{\theta})/\lambda_t(\boldsymbol{\theta}),$$

$$\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c} = \mathbf{c}\boldsymbol{\Gamma}^{-1}\mathbf{c}\omega_t(\boldsymbol{\theta})/\lambda_t(\boldsymbol{\theta})$$

$$\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t'(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\lambda_t(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] - \boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\lambda_t(\boldsymbol{\theta}) = \boldsymbol{\Gamma}^{-1}[\mathbf{v}_t(\boldsymbol{\theta})f_{kt}(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] - \mathbf{c}\omega_t(\boldsymbol{\theta})],$$

$$\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t'(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] - \boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta}) = \boldsymbol{\Gamma}^{-1}[\mathbf{v}_t(\boldsymbol{\theta})\mathbf{v}_t'(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] + \omega_{kt}(\boldsymbol{\theta})\mathbf{c}\mathbf{c}' - \boldsymbol{\Gamma}]\boldsymbol{\Gamma}^{-1}$$

and

$$\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t'(\boldsymbol{\theta})\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] - \mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c} = \frac{f_{kt}^2(\boldsymbol{\theta})}{\lambda_t^2(\boldsymbol{\theta})}\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] - \frac{\omega_t(\boldsymbol{\theta})}{\lambda_t(\boldsymbol{\theta})}\mathbf{c}'\boldsymbol{\Gamma}^{-1}\mathbf{c},$$

where $\mathbf{v}_t(\boldsymbol{\theta}) = \boldsymbol{\varepsilon}_t(\boldsymbol{\theta}) - \mathbf{c}f_{kt}(\boldsymbol{\theta})$, which greatly simplifies the computations (see Sentana (2000)). Specifically,

$$\mathbf{s}_{\boldsymbol{\theta}t}(\boldsymbol{\theta}) = \begin{pmatrix} [\mathbf{I}_N - diag(\boldsymbol{\rho})]\boldsymbol{\Gamma}^{-1}\mathbf{v}_t(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] \\ diag(\mathbf{y}_{t-1} - \boldsymbol{\pi})\boldsymbol{\Gamma}^{-1}\mathbf{v}_t(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] \\ \boldsymbol{\Gamma}^{-1}[\mathbf{v}_t(\boldsymbol{\theta})f_{kt}(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] - \mathbf{c}\omega_t(\boldsymbol{\theta})] \\ \frac{1}{2}vecd\left\{\boldsymbol{\Gamma}^{-1}[\mathbf{v}_t(\boldsymbol{\theta})\mathbf{v}_t'(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}] + \omega_t(\boldsymbol{\theta})\mathbf{c}\mathbf{c}' - \boldsymbol{\Gamma}]\boldsymbol{\Gamma}^{-1}\right\} \\ 0 \\ 0 \end{pmatrix}$$
$$+\frac{1}{2}\frac{\partial\lambda_t'(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\left[\frac{f_{kt}^2(\boldsymbol{\theta})\delta[\varsigma_t(\boldsymbol{\theta}),\boldsymbol{\eta}]}{\lambda_t^2(\boldsymbol{\theta})} - \frac{\omega_t(\boldsymbol{\theta})\mathbf{c}'\boldsymbol{\Gamma}^{-1}\mathbf{c}}{\lambda_t(\boldsymbol{\theta})}\right].$$

The last two items that we require for the score are

$$\begin{aligned}\frac{\partial f_{kt}(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} &= \mathbf{c}'\boldsymbol{\Gamma}^{-1}\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\frac{\partial\omega_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} + \frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}\boldsymbol{\Gamma}^{-1}\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\omega_t(\boldsymbol{\theta}) \\ &\quad -\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}\mathbf{E}_N'[\boldsymbol{\Gamma}^{-1}\boldsymbol{\varepsilon}_t(\boldsymbol{\theta})\otimes\omega_t(\boldsymbol{\theta})\boldsymbol{\Gamma}^{-1}\mathbf{c}] - \frac{\partial\boldsymbol{\mu}_t'(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\mathbf{c}'\boldsymbol{\Gamma}^{-1}\omega_t(\boldsymbol{\theta})\end{aligned}$$

and

$$\frac{\partial\omega_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}} = -2\omega_t^2(\boldsymbol{\theta})\frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}\boldsymbol{\Gamma}^{-1}\mathbf{c} + \omega_t(\boldsymbol{\theta})\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}\mathbf{E}_N'(\boldsymbol{\Gamma}^{-1}\mathbf{c}\otimes\boldsymbol{\Gamma}^{-1}\mathbf{c}) + \frac{\omega_t^2(\boldsymbol{\theta})}{\boldsymbol{\lambda}_t^2(\boldsymbol{\theta})}\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}.$$

To compute the elliptically symmetric semiparametric bound we need expressions for

$$\frac{\partial\boldsymbol{\mu}_t'(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\frac{\partial\boldsymbol{\mu}_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'},$$
$$\frac{\partial vec'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]}{\partial\boldsymbol{\theta}}[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\frac{\partial vec\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]}{\partial\boldsymbol{\theta}'},$$

and

$$\frac{\partial vec'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]}{\partial\boldsymbol{\theta}}vec[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]vec'[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\frac{\partial vec\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]}{\partial\boldsymbol{\theta}'}.$$

The first term will be given by

$$\begin{aligned}\frac{\partial\boldsymbol{\mu}_t'(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\frac{\partial\boldsymbol{\mu}_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'} &= \frac{\partial\boldsymbol{\pi}'}{\partial\boldsymbol{\theta}}[\mathbf{I}_N - diag(\boldsymbol{\rho})]\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})[\mathbf{I}_N - diag(\boldsymbol{\rho})]\frac{\partial\boldsymbol{\pi}}{\partial\boldsymbol{\theta}'} \\ &\quad +\frac{\partial\boldsymbol{\rho}'}{\partial\boldsymbol{\theta}}diag(\mathbf{y}_{t-1} - \boldsymbol{\pi})\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})diag(\mathbf{y}_{t-1} - \boldsymbol{\pi})\frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{\theta}'}\end{aligned}$$
$$+\frac{\partial\boldsymbol{\rho}'}{\partial\boldsymbol{\theta}}diag(\mathbf{y}_{t-1} - \boldsymbol{\pi})\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})[\mathbf{I}_N - diag(\boldsymbol{\rho})]\frac{\partial\boldsymbol{\pi}}{\partial\boldsymbol{\theta}'} + \frac{\partial\boldsymbol{\pi}'}{\partial\boldsymbol{\theta}}[\mathbf{I}_N - diag(\boldsymbol{\rho})]\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})diag(\mathbf{y}_{t-1} - \boldsymbol{\pi})\frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{\theta}'},$$

which effectively has four non-zero blocks only, two of which are equal by symmetry.

The second term is also straightforward. Specifically:

$$\frac{\partial vec'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]}{\partial\boldsymbol{\theta}}[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\frac{\partial vec\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]}{\partial\boldsymbol{\theta}'}$$

$$= \frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}[\lambda_t(\boldsymbol{\theta})\mathbf{c}'\otimes\mathbf{I}_N](\mathbf{I}_{N^2}+\mathbf{K}_{NN})[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})](\mathbf{I}_{N^2}+\mathbf{K}_{NN})[\lambda_t(\boldsymbol{\theta})\mathbf{c}\otimes\mathbf{I}_N]\frac{\partial\mathbf{c}}{\partial\boldsymbol{\theta}'}$$

$$+\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}\mathbf{E}_N'[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\mathbf{E}_N\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\theta}'}+\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}(\mathbf{c}'\otimes\mathbf{c}')[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})](\mathbf{c}\otimes\mathbf{c})\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}$$

$$+\frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}[\lambda_t(\boldsymbol{\theta})\mathbf{c}'\otimes\mathbf{I}_N](\mathbf{I}_{N^2}+\mathbf{K}_{NN})[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\mathbf{E}_N\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\theta}'}$$

$$+\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}\mathbf{E}_N'[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})](\mathbf{I}_{N^2}+\mathbf{K}_{NN})[\lambda_t(\boldsymbol{\theta})\mathbf{c}\otimes\mathbf{I}_N]\frac{\partial\mathbf{c}}{\partial\boldsymbol{\theta}'}$$

$$+\frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}[\lambda_t(\boldsymbol{\theta})\mathbf{c}'\otimes\mathbf{I}_N](\mathbf{I}_{N^2}+\mathbf{K}_{NN})[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})](\mathbf{c}\otimes\mathbf{c})\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}$$

$$+\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}(\mathbf{c}'\otimes\mathbf{c}')[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})](\mathbf{I}_{N^2}+\mathbf{K}_{NN})[\lambda_t(\boldsymbol{\theta})\mathbf{c}\otimes\mathbf{I}_N]\frac{\partial\mathbf{c}}{\partial\boldsymbol{\theta}'}$$

$$+\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}\mathbf{E}_N'[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})](\mathbf{c}\otimes\mathbf{c})\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}+\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}(\mathbf{c}'\otimes\mathbf{c}')[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\mathbf{E}_N\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\theta}'}$$

$$= 2\boldsymbol{\lambda}_t^2(\boldsymbol{\theta})\frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}\{[\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\cdot\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})+\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\frac{\partial\mathbf{c}}{\partial\boldsymbol{\theta}'}$$

$$+\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\odot\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\theta}'}+[\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}]^2\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}$$

$$+2\lambda_t(\boldsymbol{\theta})\frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}[\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\mathbf{E}_N\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\theta}'}+2\lambda_t(\boldsymbol{\theta})\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}\mathbf{E}_N'[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\frac{\partial\mathbf{c}}{\partial\boldsymbol{\theta}'}$$

$$+2\lambda_t(\boldsymbol{\theta})[\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}]\frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}+2\lambda_t(\boldsymbol{\theta})[\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}]\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\mathbf{c}'\Sigma_t^{-1}(\boldsymbol{\theta})\frac{\partial\mathbf{c}}{\partial\boldsymbol{\theta}'}$$

$$+\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}\odot\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}]\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}'}+\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}[\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\odot\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\frac{\partial\boldsymbol{\gamma}}{\partial\boldsymbol{\theta}'},$$

where $\odot$ denotes Hadamard products.

But if we assume that $\boldsymbol{\gamma}>\mathbf{0}$, we can use again the Woodbury formula to considerably simplify the previous expressions. The only slightly complex term left is

$$[\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\otimes\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]\mathbf{E}_N$$

But if we exploit the explicit shape of $\mathbf{E}_N$, then we can show that the (i,j)th element of this matrix takes the following form

$$\frac{\omega_t(\boldsymbol{\theta})}{\lambda_t(\boldsymbol{\theta})}\frac{b_j}{\gamma_j}\left[\frac{I(i=j)}{\gamma_i}-\frac{b_ib_j}{\gamma_i\gamma_j}\omega_t(\boldsymbol{\theta})\right],$$

where $I(.)$ is the usual indicator function.

Finally,

$$\mathbf{W}_{st}(\boldsymbol{\theta})=\frac{1}{2}\frac{\partial vec'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]}{\partial\boldsymbol{\theta}}vec[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]=\frac{1}{2}\frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}[\lambda_t(\boldsymbol{\theta})\mathbf{c}'\otimes\mathbf{I}_N](\mathbf{I}_{N^2}+\mathbf{K}_{NN})vec[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]$$

$$+\frac{1}{2}\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}\mathbf{E}_N'vec[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]+\frac{1}{2}\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}(\mathbf{c}'\otimes\mathbf{c}')vec[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]$$

$$= 2\lambda_t(\boldsymbol{\theta})\frac{\partial\mathbf{c}'}{\partial\boldsymbol{\theta}}vec[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c}]+\frac{1}{2}\frac{\partial\boldsymbol{\gamma}'}{\partial\boldsymbol{\theta}}ved[\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})]+\frac{1}{2}\frac{\partial\lambda_t(\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}\mathbf{c}'\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{\theta})\mathbf{c},$$

whose computation can also be greatly simplified by using the Woodbury formula.

To estimate $\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]$ non-parametrically, we can exploit expression (A4) to write

$$-\frac{2\partial g[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]}{\partial \varsigma} = -\frac{2\partial \ln h[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]}{\partial \varsigma} + \frac{N-2}{2}\frac{1}{\varsigma_t(\boldsymbol{\theta})}.$$

Then, we can compute $h[\varsigma_t(\boldsymbol{\theta}); \boldsymbol{\eta}]$ either directly by using a kernel for positive random variables (see Chen (2000)), or indirectly by using a faster standard Gaussian kernel after exploiting the Box-Cox-type transformation $v = \varsigma^k$ (see Hodgson, Linton and Vorkink (2002)). In the second case, the usual change of variable formula yields

$$p(v; \boldsymbol{\eta}) = \frac{\pi^{N/2}}{k\Gamma(N/2)} v^{-1+N/2k} \exp[c(\boldsymbol{\eta}) + g(v^{1/k}; \boldsymbol{\eta})],$$

whence

$$g(v^{1/k}; \boldsymbol{\eta}) = \ln p(v; \boldsymbol{\eta}) + \left(1 - \frac{N}{2k}\right) \ln v - \frac{N}{2} \ln 2\pi + \ln k - \ln \Gamma(N/2) - c(\boldsymbol{\eta})$$

and

$$\frac{\partial g(v^{1/k}; \boldsymbol{\eta})}{\partial v^{1/k}} = k\frac{\partial \ln f(v; \boldsymbol{\eta})}{\partial v} v^{1-1/k} + \frac{k - N/2}{v^{1/k}}.$$

We use the second procedure in our Monte Carlo simulations because the distribution of $\varsigma_t(\boldsymbol{\theta})$ becomes more normal-like as N increases, which reduces the advantages of using kernels for positive variables. Still, we use a cubic root transformation to improve the approximation, with a common bandwidth parameter for both the density and its first derivative.

The last thing we need is to estimate $\mathrm{M}_{ll}(\boldsymbol{\eta})$ and $\mathrm{M}_{ss}(\boldsymbol{\eta})$. In our experience, the sample analogue of the OOS expression for $\mathrm{M}_{ll}(\boldsymbol{\eta})$ in Proposition 16 based on the nonparametric estimators of $\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]$ tends to overestimate $\mathrm{M}_{ll}(\boldsymbol{\eta})$ even in fairly large samples because $\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]$ is imprecisely estimated when ς_t is either very small or very large. For that reason, we have considered an alternative estimator based on the following equivalent expression:

$$\mathrm{M}_{ll}(\boldsymbol{\eta}) = cov\left\{ \delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}], \delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]\frac{\varsigma_t}{N} \middle| \boldsymbol{\eta} \right\} + (N-2)E[\varsigma^{-1}(\boldsymbol{\theta})|\boldsymbol{\eta}],$$

where we have exploited (A5), as well as Lemma 1 applied to $m(1) = 1$, which yields

$$E[\delta(\varsigma_t, \boldsymbol{\eta})] = -(N-2)E[\varsigma^{-1}|\boldsymbol{\eta}], \qquad \text{(B23)}$$

as long as $E[\varsigma^{-1}|\boldsymbol{\eta}]$ is bounded, which in the Gaussian case, for instance, requires $N \geq 3$. Importantly, note that (B23) does not depend at all on the semiparametric estimator. Still, its sample analogue typically underestimates $\mathrm{M}_{ll}(\boldsymbol{\eta})$, for which reason in the end we average the two estimators.

As for $\mathrm{M}_{ss}(\boldsymbol{\eta})$, our experience is that the sample analogue of the OOS expression for $\mathrm{M}_{ss}(\boldsymbol{\eta})$ in Proposition 16 tends to underestimate it. For that reason, we divide it by the square of the sample mean of $\delta[\varsigma_t(\boldsymbol{\theta}), \boldsymbol{\eta}]\varsigma_t/N$, which converges in probability to 1 asymptotically in view of (A5).

In order to make sure that $\mathring{\mathcal{S}}(\boldsymbol{\phi}_0) - \mathcal{S}(\boldsymbol{\phi}_0)$ is positive semidefinite, we also impose the theoretical restrictions $\mathrm{M}_{ll}(\boldsymbol{\eta}_0) \geq 1$ and

$$V\left[\left\{\frac{\delta(\varsigma_t, \boldsymbol{\eta})\varsigma_t}{N} - 1\right\} - \frac{2}{(N+2)\kappa_0 + 2}\left(\frac{\varsigma_t}{N} - 1\right)\right] = \left[\frac{N+2}{N}\mathrm{M}_{ss}(\boldsymbol{\eta}_0) - 1\right] - \frac{4}{N\left[(N+2)\kappa_0 + 2\right]} \geq 0,$$

after replacing κ_0 by its sample analogue. These restrictions also guarantee that our estimates of $\mathcal{C}(\boldsymbol{\phi}_0) - \mathring{\mathcal{S}}^{-1}(\boldsymbol{\phi}_0)$ will be positive semidefinite too as long as we evaluate these matrices at the same parameter values using the analytical expressions in Propositions 2 and 7. Finally, we deal with the fact that $rank[\mathcal{C}(\boldsymbol{\phi}_0) - \mathring{\mathcal{S}}^{-1}(\boldsymbol{\phi}_0)] \leq p - 1$ in view of Proposition 12.1.c by setting to 0 those eigenvalues that are smaller than $10^{-7}/T$ in computing the Moore-Penrose inverse of the difference between those matrices.

B.2 The semiparametric efficient score

As pointed out by Mencía and Sentana (2005), the first thing to note regarding a non-elliptical distribution function for the innovations is that the choice of $\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})$ affects the value of the log-likelihood function and its score. In what follows, we shall use the standard (i.e. lower triangular) Cholesky decomposition of $\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$ because it is much faster to compute than its symmetric square root matrix, which requires the spectral decomposition of $\boldsymbol{\Sigma}_t(\boldsymbol{\theta})$ for each t. As a result, we will have that

$$dvec(\boldsymbol{\Sigma}_t) = [(\boldsymbol{\Sigma}_t^{1/2} \otimes \mathbf{I}_N) + (\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{1/2})\mathbf{K}_{NN}]dvec(\boldsymbol{\Sigma}_t^{1/2}).$$

Unfortunately, this transformation is singular, which means that we must find an analogous transformation between the corresponding $dvech's$. In this sense, we can write the previous expression as

$$dvech(\boldsymbol{\Sigma}_t) = [\mathbf{L}_N(\boldsymbol{\Sigma}_t^{1/2} \otimes \mathbf{I}_N)\mathbf{L}_N' + \mathbf{L}_N(\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{1/2})\mathbf{K}_{NN}\mathbf{L}_N']dvech(\boldsymbol{\Sigma}_t^{1/2}),$$

where $\mathbf{L}_N$ is the elimination matrix (see Magnus, 1988). We can then use the results in chapter 5 of Magnus (1988) to show that the above mapping will be lower triangular of full rank as long as $\boldsymbol{\Sigma}_t^{1/2}$ has full rank, which means that we can readily obtain the Jacobian matrix of $vech(\boldsymbol{\Sigma}_t^{1/2})$ from the Jacobian matrix of $vech(\boldsymbol{\Sigma}_t)$.

In the case of the symmetric square root matrix, the analogous transformation would be

$$dvech(\boldsymbol{\Sigma}_t) = [\mathbf{D}_N^+(\boldsymbol{\Sigma}_t^{1/2} \otimes \mathbf{I}_N)\mathbf{D}_N + \mathbf{D}_N^+(\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{1/2})\mathbf{D}_N]dvech(\boldsymbol{\Sigma}_t^{1/2}),$$

where $\mathbf{D}_N$ is the duplication matrix and $\mathbf{D}_N^+ = (\mathbf{D}_N'\mathbf{D}_N)^{-1}\mathbf{D}_N'$ its Moore-Penrose inverse (see Magnus and Neudecker, 1988).

From a numerical point of view, the calculation of both $\mathbf{L}_N(\boldsymbol{\Sigma}_t^{1/2} \otimes \mathbf{I}_N)\mathbf{L}'_N$ and $\mathbf{L}_N(\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{1/2})\mathbf{K}_{NN}\mathbf{L}'_N$ is straightforward. Specifically, given that $\mathbf{L}_N vec(\mathbf{A}) = vech(\mathbf{A})$ for any square matrix $\mathbf{A}$, the effect of premultiplying by the $\frac{1}{2}N(N+1) \times N^2$ matrix $\mathbf{L}_N$ is to eliminate rows N+1, 2N+1 and 2N+2, 3N+1, 3N+2 and 3N+3, etc. Similarly, given that $\mathbf{L}_N\mathbf{K}_{NN}vec(\mathbf{A}) = vech(\mathbf{A}')$, the effect of postmultiplying by $\mathbf{K}_{NN}\mathbf{L}'_N$ is to delete all columns but those in positions 1, N+1, 2N+1,...,N+2, 2N+2,..., N+3, 2N+3,..., N^2.

Let $\mathbf{F}_t$ denote the transpose of the inverse of $\mathbf{L}_N(\boldsymbol{\Sigma}_t^{1/2} \otimes \mathbf{I}_N)\mathbf{L}'_N + \mathbf{L}_N(\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{1/2})\mathbf{K}_{NN}\mathbf{L}'_N$, which will be upper triangular. The fastest way to compute

$$\frac{\partial vec'[\boldsymbol{\Sigma}_t^{1/2}(\boldsymbol{\theta})]}{\partial \boldsymbol{\theta}}[\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{-1/2}(\boldsymbol{\theta})] = \frac{1}{2}\frac{\partial vech'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]}{\partial \boldsymbol{\theta}}\mathbf{F}_t\mathbf{L}_N(\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{-1/2})$$

is as follows:

1. From the expression for $\partial vec'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]/\partial\boldsymbol{\theta}$ we can readily obtain $\partial vech'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]/\partial\boldsymbol{\theta}$ by simply avoiding the computation of the duplicated columns

2. Then we postmultiply the resulting matrix by $\mathbf{F}_t$

3. Next, we construct the matrix

$$\mathbf{L}_N(\mathbf{I}_N \otimes \boldsymbol{\Sigma}_t^{1/2}) = \mathbf{L}_N \begin{pmatrix} \boldsymbol{\Sigma}_t^{-1/2} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_t^{-1/2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \boldsymbol{\Sigma}_t^{-1/2} \end{pmatrix}$$

   by eliminating the first row from the second block, the first two rows from the third block, ..., and all the rows but the last one from the last block

4. Finally, we premultiply the resulting matrix by $\partial vech'\left[\boldsymbol{\Sigma}_t(\boldsymbol{\theta})\right]/\partial\boldsymbol{\theta} \cdot \mathbf{F}_t$.

The last task that we must perform is the computation of $\mathcal{K}(0)\mathcal{K}^+(\boldsymbol{\varrho})\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0})$. The two main problems here are the singular nature of $\mathcal{K}(\boldsymbol{\varrho})$, and its positive semidefiniteness. The first problem is easy to solve because

$$\mathcal{K}(0)\mathcal{K}^+(\boldsymbol{\varrho})\mathbf{e}_{dt}(\boldsymbol{\theta},\mathbf{0}) = \underset{\cdot}{\mathrm{K}}(0)\underset{\cdot}{\mathrm{K}}^{-1}(\boldsymbol{\varrho})\underset{\cdot}{\mathrm{e}}_{dt}(\boldsymbol{\theta},\mathbf{0}),$$

where

$$\underset{\cdot}{\mathrm{K}}(0) = \begin{pmatrix} \mathbf{I}_N & \mathbf{0} \\ \mathbf{0} & 2\mathbf{D}_N^{+\prime} \end{pmatrix}, \ \underset{\cdot}{\mathrm{K}}(\boldsymbol{\rho}) = \begin{pmatrix} \mathbf{I}_N & \boldsymbol{\Phi}' \\ \boldsymbol{\Phi} & \boldsymbol{\Upsilon} \end{pmatrix}, \ \underset{\cdot}{\mathrm{e}}_{dt}(\boldsymbol{\theta},\mathbf{0}) = \left\{ \begin{matrix} \boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta}) \\ vech\left[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}) - \mathbf{I}_N\right] \end{matrix} \right\},$$

$$\boldsymbol{\Phi} = E\{vech[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}) - \mathbf{I}_N] \cdot \boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta})|\boldsymbol{\theta},\boldsymbol{\varrho}\}$$

and

$$\boldsymbol{\Upsilon} = E\{vech[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}) - \mathbf{I}_N] \cdot vech'[\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})\boldsymbol{\varepsilon}_t^{*\prime}(\boldsymbol{\theta}) - \mathbf{I}_N]|\boldsymbol{\theta},\boldsymbol{\varrho}\}.$$

As for the second problem, there are two alternative solutions:

1. Re-centre and orthogonalise $\varepsilon_t^*(\boldsymbol{\theta})$ as $\varepsilon_t^{**}(\boldsymbol{\theta}) = \bar{\mathbf{P}}_T^{-1/2}[\varepsilon_t^*(\boldsymbol{\theta}) - \bar{\mathbf{p}}_T]$, where $\bar{\mathbf{p}}_T$ is the sample mean of $\varepsilon_t^*(\boldsymbol{\theta})$ and $\bar{\mathbf{P}}_T$ its sample covariance. In this way, the sample covariance matrix of the vector $\{\varepsilon_t^{**\prime}(\boldsymbol{\theta}), vech'[\varepsilon_t^{**}(\boldsymbol{\theta})\varepsilon_t^{**\prime}(\boldsymbol{\theta})]\}$ will have exactly the same structure as $\underset{\cdot}{\mathrm{K}}(\boldsymbol{\varrho})$.

2. Replace $\underset{\cdot}{\mathrm{K}}(\boldsymbol{\varrho})$ by either the sample covariance matrix or the second moment matrix of the vector $\underset{\cdot}{\mathrm{e}}_{dt}(\boldsymbol{\theta}, \mathbf{0})$.

The advantage of the first procedure is that we can exploit the fact that the sample covariance matrix of $\varepsilon_t^{**}(\boldsymbol{\theta})$ will be the identity matrix in using the partitioned inverse formula for $\underset{\cdot}{\mathrm{K}}(\boldsymbol{\varrho})$. On the other hand, the advantage of the second procedure is that there is no need to standardise again the standardised innovations $\varepsilon_t^*(\boldsymbol{\theta})$, which in our experience makes it more attractive.

It is also worth mentioning that the most convenient way to compute $\underset{\cdot}{\mathrm{K}}(0)\underset{\cdot}{\mathrm{K}}^{-1}(\boldsymbol{\varrho})\underset{\cdot}{\mathrm{e}}_{dt}(\boldsymbol{\theta}, \mathbf{0})$ is by first computing $\underset{\cdot}{\mathrm{K}}^{-1}(\boldsymbol{\varrho})\underset{\cdot}{\mathrm{e}}_{dt}(\boldsymbol{\theta}, \mathbf{0})$, and then exploiting the shape of $\underset{\cdot}{\mathrm{K}}(0)$ as follows: (a) copy the first N elements of $\underset{\cdot}{\mathrm{K}}^{-1}(\boldsymbol{\varrho})\underset{\cdot}{\mathrm{e}}_{dt}(\boldsymbol{\theta}, \mathbf{0})$; and (b) duplicate the remaining $\frac{1}{2}$N(N+1) elements, but doubling the ones in the following positions: N+1, 2N+1, 3N, 4N-1, 5N-2,...,N+N^2. Intuitively, in doing so we are simply using the fact that $2\mathbf{D}_N^{+\prime} vech(\mathbf{A}_L) = vec(\mathbf{A}_L + \mathbf{A}_L')$ for any lower triangular matrix $\mathbf{A}_L$.

Finally, we use a multivariate spherical Gaussian kernel to compute the density of $\boldsymbol{\varepsilon}_t^*(\boldsymbol{\theta})$ and its derivatives with a common bandwidth parameter.

References

Abramowitz, M. and Stegun, I.A. (1964): *Handbook of mathematic functions*, AMS 55, National Bureau of Standards.

Amengual, D. and Sentana (2007): "A comparison of mean-variance efficiency tests", mimeo, Princeton.

Andrews, D.W.K. (1999): "Estimation when a parameter in on the boundary", *Econometrica* 67, 1341-1383.

Balestra, P., and Holly, A. (1990), "A general Kronecker formula for the moments of the multivariate normal distribution", DEEP Cahier 9002, University of Lausanne.

Berkane, M. and Bentler, P.M. (1986): "Moments of elliptically distributed random variates", *Statistics and Probability Letters* 4, 333-335.

Bickel, P.J. (1982): "On adaptive estimation", *Annals of Statistics* 10, 647-671.

Bollerslev, T. (1990): "Modelling the coherence in short-run nominal exchange rates: a multivariate Generalized Arch model", *Review of Economics and Statistics* 72, 498-505.

Bollerslev, T., and Wooldridge, J. M. (1992): "Quasi maximum likelihood estimation and inference in dynamic models with time-varying covariances", *Econometric Reviews* 11, 143-172.

Bontemps, C. and Meddahi, N. (2005): "Testing distributional assumptions: a GMM approach", mimeo, U. Montreal.

Calzolari, G., Fiorentini, G. and Sentana, E. (2004): "Constrained indirect estimation", *Review of Economic Studies* 71, 945-973.

Chen, S.X. (2000): "Probability density function estimation using gamma kernels", *Annals of the Institute of Statistical Mathematics* 52, 471-480.

Crowder, M.J. (1976): "Maximum likelihood estimation for dependent observations", *Journal of the Royal Statistical Society* B, 38, 45-53.

Davidson, R. and MacKinnon, J.G. (1998): "Graphical methods for investigating the size and power of tests statistics", *The Manchester School* 66, 1-26.

Demos, A. and Sentana, E. (1998): "Testing for Garch effects: a one-sided approach", *Journal of Econometrics*, 86, 97-127.

Drost, F.C. and Klaassen, C.A.J. (1997): "Efficient estimation in semiparametric Garch models", *Journal of Econometrics* 80, 193-221.

Drost, F.C., Klaassen, C.A.J. and Werker, B.J.M. (1997): "Adaptive estimation in time series models", *Annals of Statistics* 25, 786-817.

Engle, R.F. (1982): "Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation", *Econometrica* 50, 987-1007.

Engle, R.F. and Gonzalez-Rivera, G. (1991): "Semiparametric ARCH models", *Journal of Business and Economic Statistics* 9, 345-360.

Fang, K.-T., Kotz, S., and Ng, K.-W. (1990), *Symmetric multivariate and related distributions*, Chapman and Hall.

Fiorentini, G., Sentana, E. and Calzolari, G. (2003): "Maximum likelihood estimation and inference in multivariate conditionally heteroskedastic dynamic regression models with student t innovations", *Journal of Business and Economic Statistics* 21, 532-546.

Fiorentini, G., Sentana, E., and Calzolari, G. (2004): "On the validity of the Jarque-Bera normality test in conditionally heteroskedastic dynamic regression models", *Economics Letters* 83, 307-312.

Gonzalez-Rivera, G. (1997): "A note on adaptation in GARCH models", *Econometric Reviews* 16, 55-68.

Gonzalez-Rivera, G. and Drost, F.C. (1999): "Efficiency comparisons of maximum-likelihood-based estimators in GARCH models", *Journal of Econometrics* 93, 93-111.

Hafner, C.M. and Rombouts, J.V.K. (2007): "Semiparametric multivariate volatility models", *Econometric Theory* 23, 251-280.

Hall, P. and Yao, Q. (2003): "Inference in ARCH and GARCH models with heavy-tailed errors", *Econometrica* 71, 285-317.

Hodgson, D.J. (2000): "Unconditional pseudo-maximum likelihood and adaptive estimation in the presence of conditional heterogeneity of unknown form", *Econometric Reviews* 19, 175-206.

Hodgson, D.J., Linton, O. and Vorkink, K.P. (2002): "Testing the Capital Asset Pricing Model efficiently under elliptical symmetry: a semiparametric approach", *Journal of Applied Econometrics* 17, 617-639.

Hodgson, D.J. and Vorkink, K.P. (2003): "Efficient estimation of conditional asset pricing models", *Journal of Business and Economic Statistics* 21, 269-283.

Kotz, S. (1975): "Multivariate distributions at a cross-road", in G. P. Patil, S. Kotz and J.K. Ord (eds.) *Statistical distributions in scientific work*, vol. I, 247-270, Reidel.

Lange, K.L., Little, R.J.A., and Taylor, J.M.G. (1989): "Robust statistical modeling using the t distribution", *Journal of the American Statistical Association* 84, 881-896.

Ling, S. and McAleer, M. (2003): "Asymptotic theory for a vector ARMA-GARCH model", *Econometric Theory* 19, 280-310.

Linton, O. (1993): "Adaptive estimation in ARCH models", *Econometric Theory* 9, 539-569.

Linton, O. and Steigerwald, D. (2000): "Adaptive testing in ARCH models", *Econometric*

Reviews 19, 145-174.

Magnus, J.R. (1988): *Linear structures*, Oxford University Press, New York.

Magnus, J.R. and Neudecker, H. (1988): *Matrix differential calculus with applications in Statistics and Econometrics*, Wiley, Chichester.

Mardia, K.V. (1970): "Measures of multivariate skewness and kurtosis with applications", *Biometrika* 57, 519-530.

Maruyama, Y. and Seo, T. (2003): "Estimation of moment parameter in elliptical distributions", *Journal of the Japan Statistical Society* 33, 215-229.

Mencía, J. and Sentana, E. (2005): "Estimation and testing of dynamic models with generalised hyperbolic innovations", mimeo, CEMFI.

NAG (2001): *NAG Fortran 77 Library Mark 19 Reference Manual.*

Newey, W.K. and McFadden, D.L. (1994): "Large sample estimation and hypothesis testing", in R.F. Engle and D.L. McFadden (eds.) *Handbook of Econometrics* vol. IV, 2111-2245, Elsevier.

Newey, W.K. and Powell, J.L. (1998): "Two-step estimation, optimal moment conditions, and sample selection models", mimeo, MIT.

Newey, W.K. and Steigerwald, D.G. (1997): "Asymptotic bias for quasi-maximum-likelihood estimators in conditional heteroskedasticity models", *Econometrica* 65, 587-99.

Sentana, E. (2000): "The likelihood function of conditionally heteroskedastic factor models", *Annales d'Economie et de Statistique* 58, 1-19.

Sentana, E. (2004): "Factor representing portfolios in large asset markets", *Journal of Econometrics* 119, 257-289.

Sentana, E. and Fiorentini, G. (2001): "Identification, estimation and testing of conditionally heteroskedastic factor models", *Journal of Econometrics* 102, 143-164.

Silverman B.W. (1986): *Density estimation*, Chapman and Hall.

Sun, Y. and Stengos, T. (2006): "Semiparametric efficient adaptive estimation of asymmetric GARCH models", *Journal of Econometrics* 127, 373-386.

Table 1

Size properties of Hausman tests in finite samples

Parametric

Nominal size (%)	student t_8			
	$\bar{\pi}$		$\bar{\gamma}$	
	Wald	LM	Wald	LM
1	1.68	1.77	2.35	1.33
5	6.28	6.67	6.69	5.23
10	11.2	11.7	11.1	10.2

Semiparametric

Nominal size (%)	student t_8			
	$\bar{\pi}$		$\bar{\gamma}$	
	Wald	LM	Wald	LM
1	2.68	4.75	36.1	23.1
5	8.95	11.4	52.5	36.9
10	15.2	17.5	61.9	45.7

Nominal size (%)	normal-gamma			
	$\bar{\pi}$		$\bar{\gamma}$	
	Wald	LM	Wald	LM
1	1.13	2.53	66.0	48.4
5	5.40	7.03	80.9	66.1
10	10.5	12.2	87.0	74.5

Table 2

Size-adjusted power properties of Hausman tests in finite samples

Parametric

Actual size (%)	normal-gamma $\bar{\pi}$ Wald	$\bar{\pi}$ LM	$\bar{\gamma}$ Wald	$\bar{\gamma}$ LM
1	3.40	3.04	99.9	99.9
5	11.1	10.1	100.	100.
10	18.5	16.8	100.	100.

Actual size (%)	asymmetric t $\bar{\pi}$ Wald	$\bar{\pi}$ LM	$\bar{\gamma}$ Wald	$\bar{\gamma}$ LM
1	100.	100.	52.5	55.0
5	100.	100.	78.7	76.5
10	100.	100.	87.9	84.6

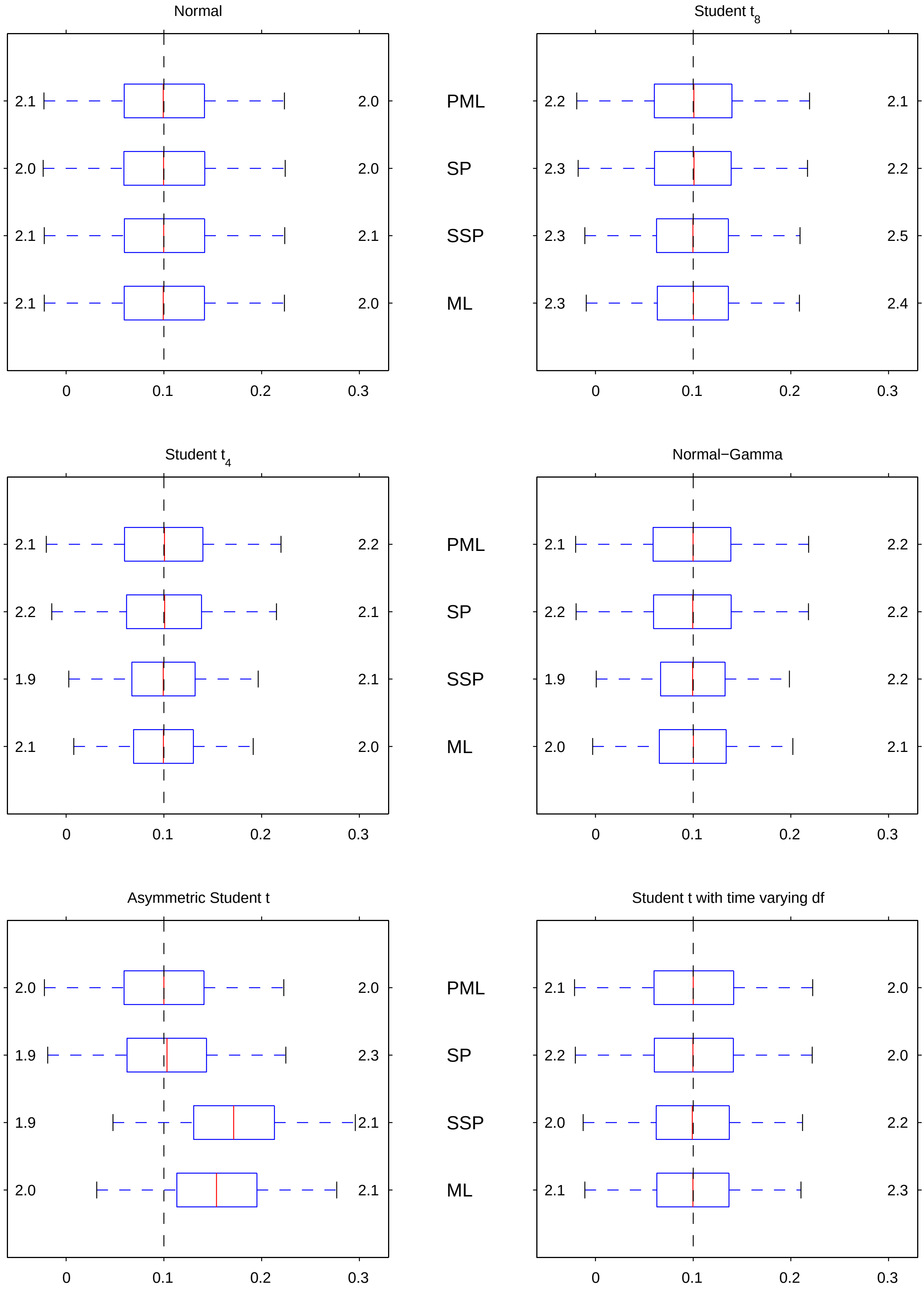
Actual size (%)	t with time-varying df $\bar{\pi}$ Wald	$\bar{\pi}$ LM	$\bar{\gamma}$ Wald	$\bar{\gamma}$ LM
1	1.03	1.09	0.59	0.65
5	4.90	5.08	4.10	4.25
10	10.3	10.3	9.55	9.83

Semiparametric

Actual size (%)	asymmetric t $\bar{\pi}$ Wald	$\bar{\pi}$ LM	$\bar{\gamma}$ Wald	$\bar{\gamma}$ LM
1	100.	50.8	99.9	0.37
5	100.	100.	100.	99.8
10	100.	100.	100.	99.9

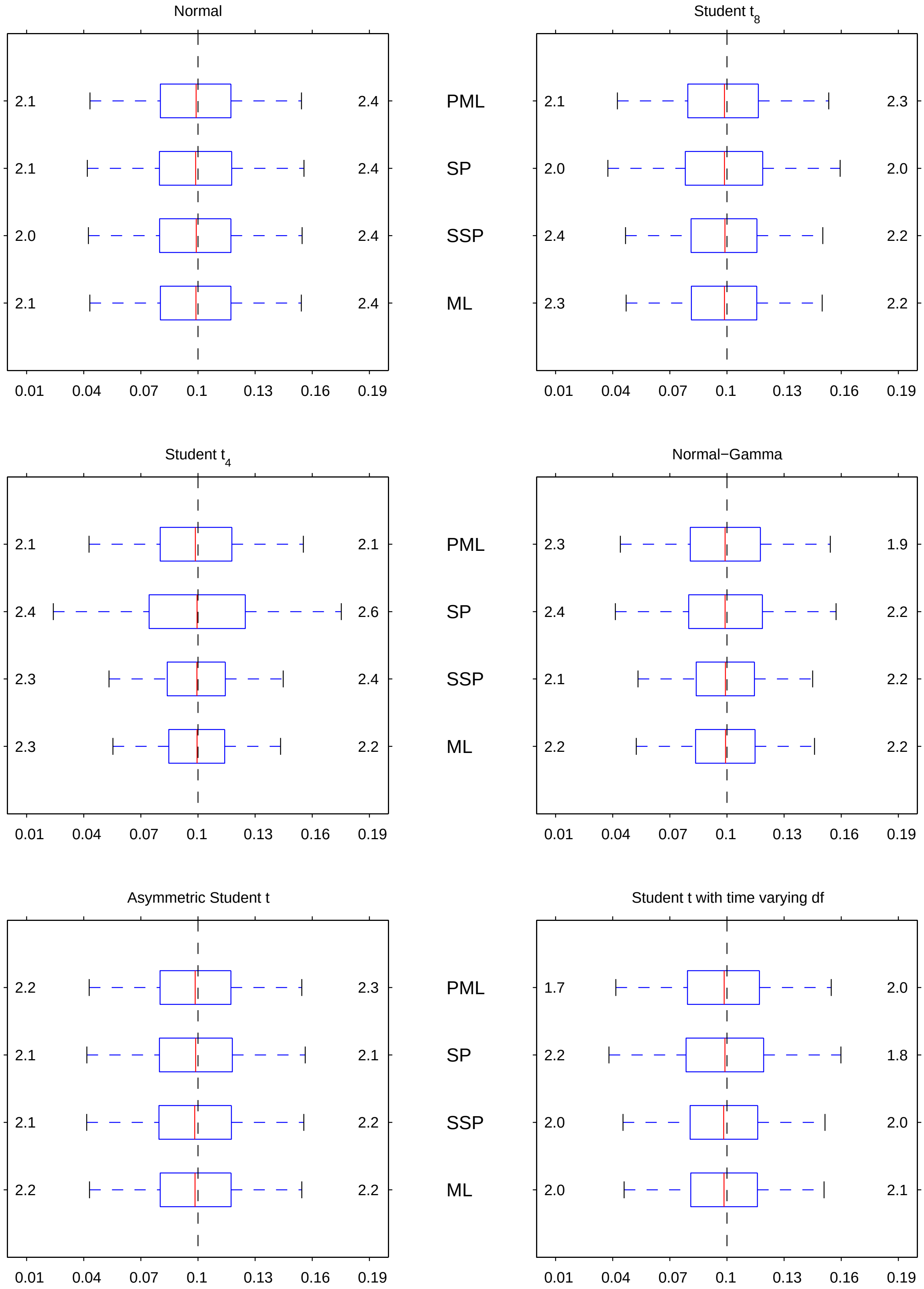
Actual size (%)	t with time-varying df $\bar{\pi}$ Wald	$\bar{\pi}$ LM	$\bar{\gamma}$ Wald	$\bar{\gamma}$ LM
1	0.94	0.85	0.98	0.63
5	5.06	5.10	5.07	4.56
10	10.2	9.71	9.37	9.19

Figure 1A: Monte Carlo distributions of estimators of unconditional mean



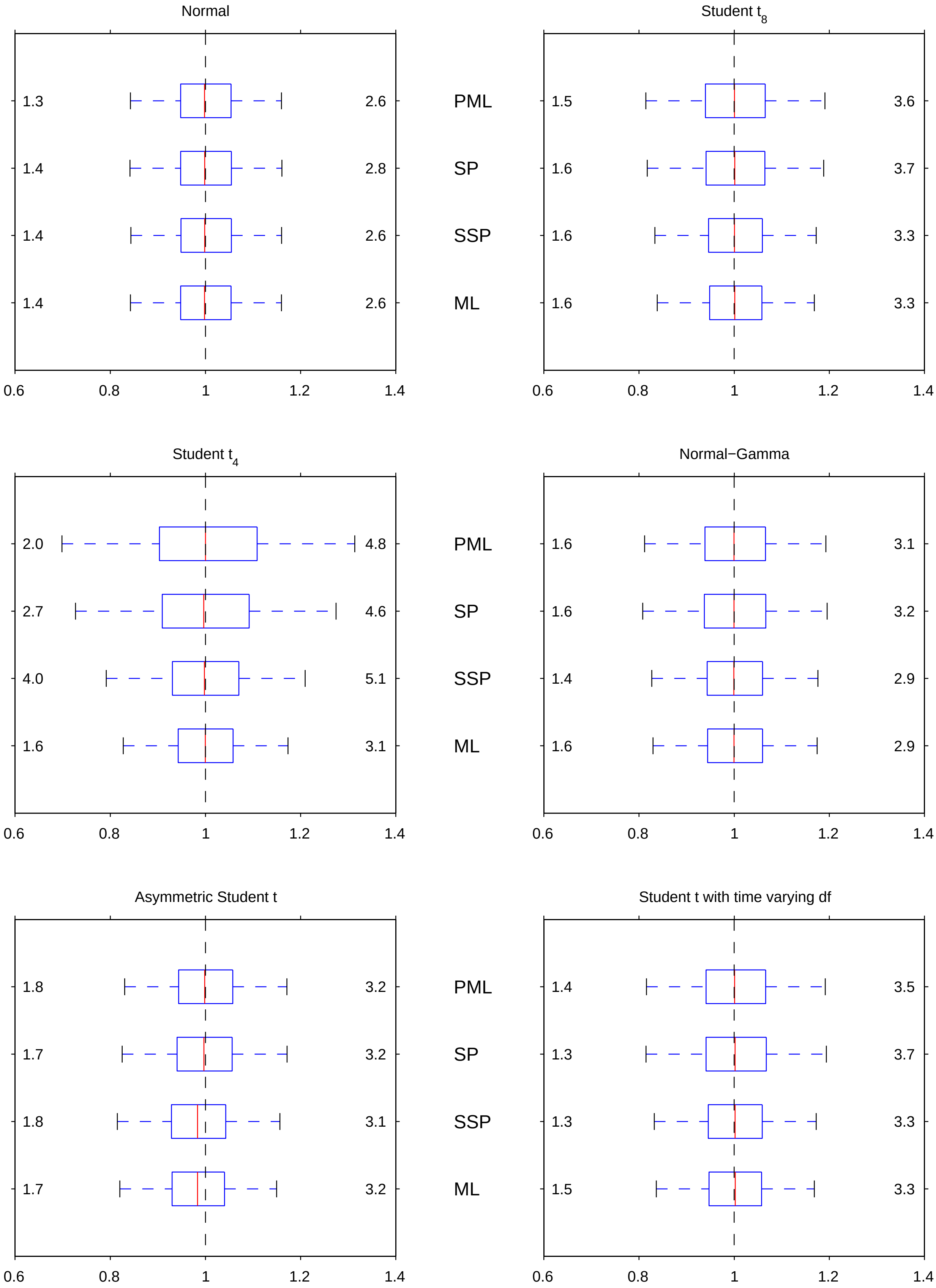
The central boxes describe the 1st and 3rd quartiles of the sampling distributions, and their median. The maximum length of the whiskers is one interquartile range. We also report the fraction of replications outside those whiskers. PML means Gaussian−based maximum likelihood estimator, ML Student t−based maximum likelihood estimator, SSP elliptically symmetric semiparametric estimator and SP unrestricted semiparametric estimator.

Figure 1B: Monte Carlo distributions of estimators of autoregressive coefficient



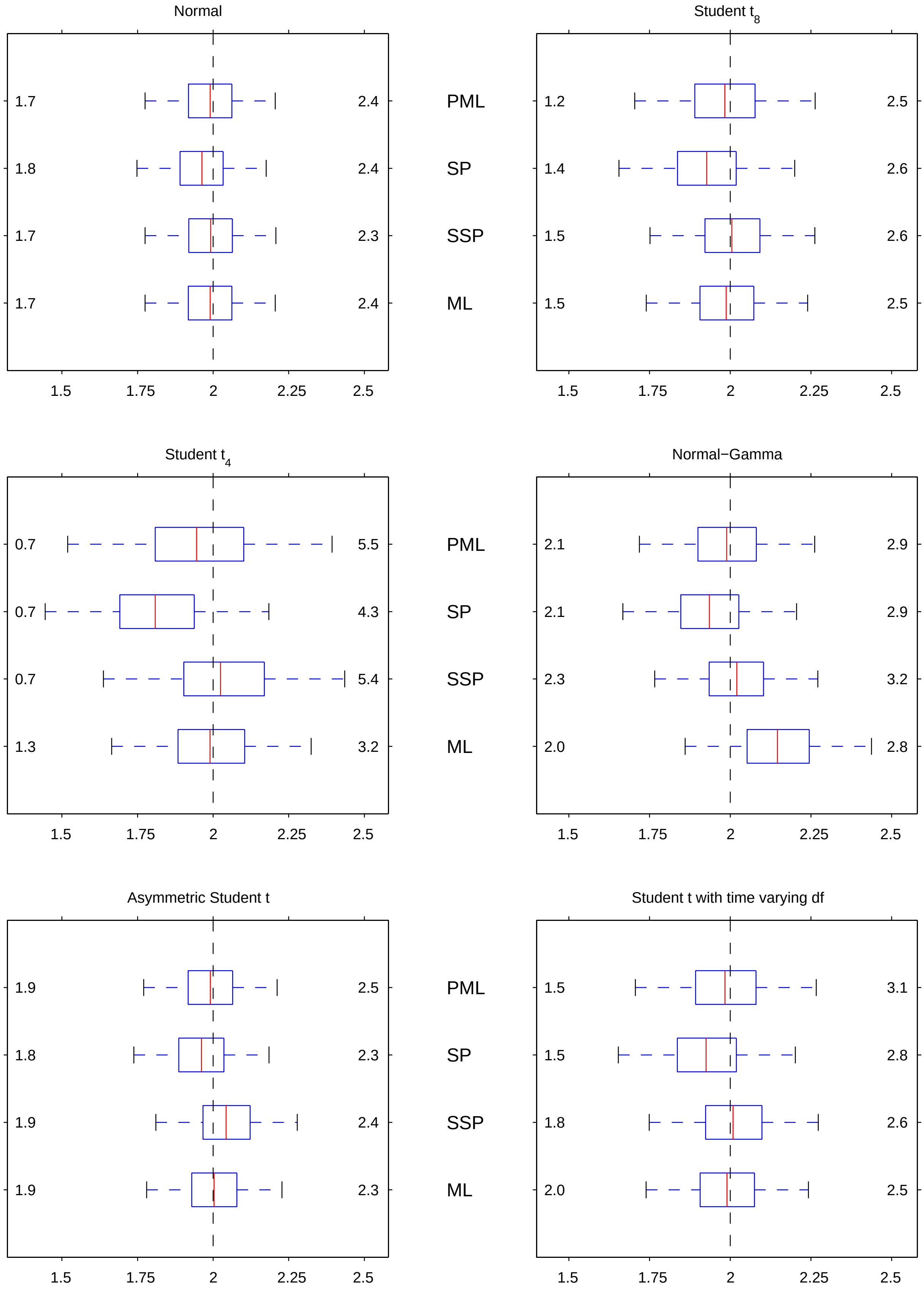
The central boxes describe the 1st and 3rd quartiles of the sampling distributions, and their median. The maximum length of the whiskers is one interquartile range. We also report the fraction of replications outside those whiskers. PML means Gaussian−based maximum likelihood estimator, ML Student t−based maximum likelihood estimator, SSP elliptically symmetric semiparametric estimator and SP unrestricted semiparametric estimator.

Figure 1C: Monte Carlo distributions of estimators of normalised factor loadings



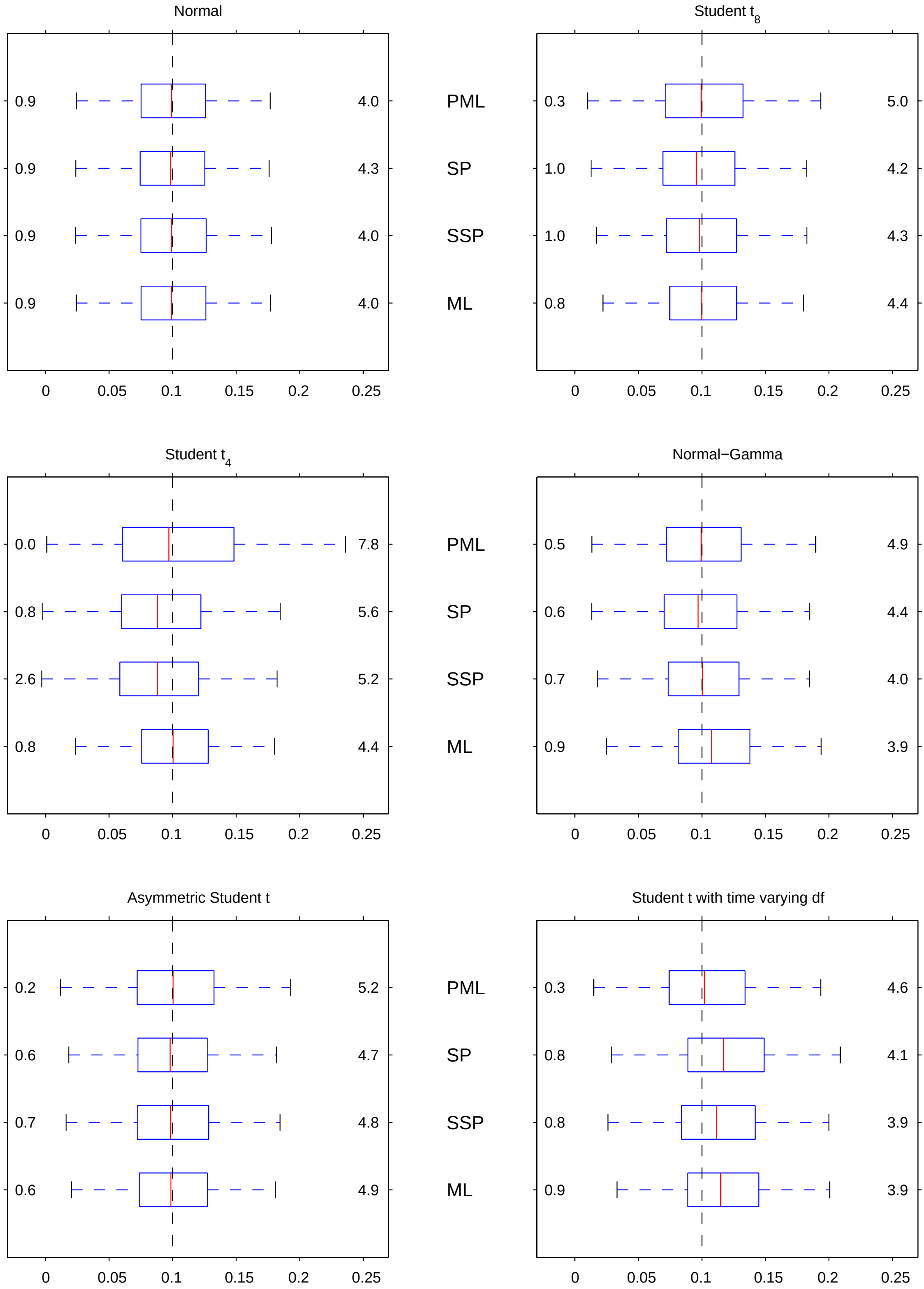
The central boxes describe the 1st and 3rd quartiles of the sampling distributions, and their median. The maximum length of the whiskers is one interquartile range. We also report the fraction of replications outside those whiskers. PML means Gaussian−based maximum likelihood estimator, ML Student t−based maximum likelihood estimator, SSP elliptically symmetric semiparametric estimator and SP unrestricted semiparametric estimator.

Figure 1D: Monte Carlo distributions of estimators of idyosincratic variances



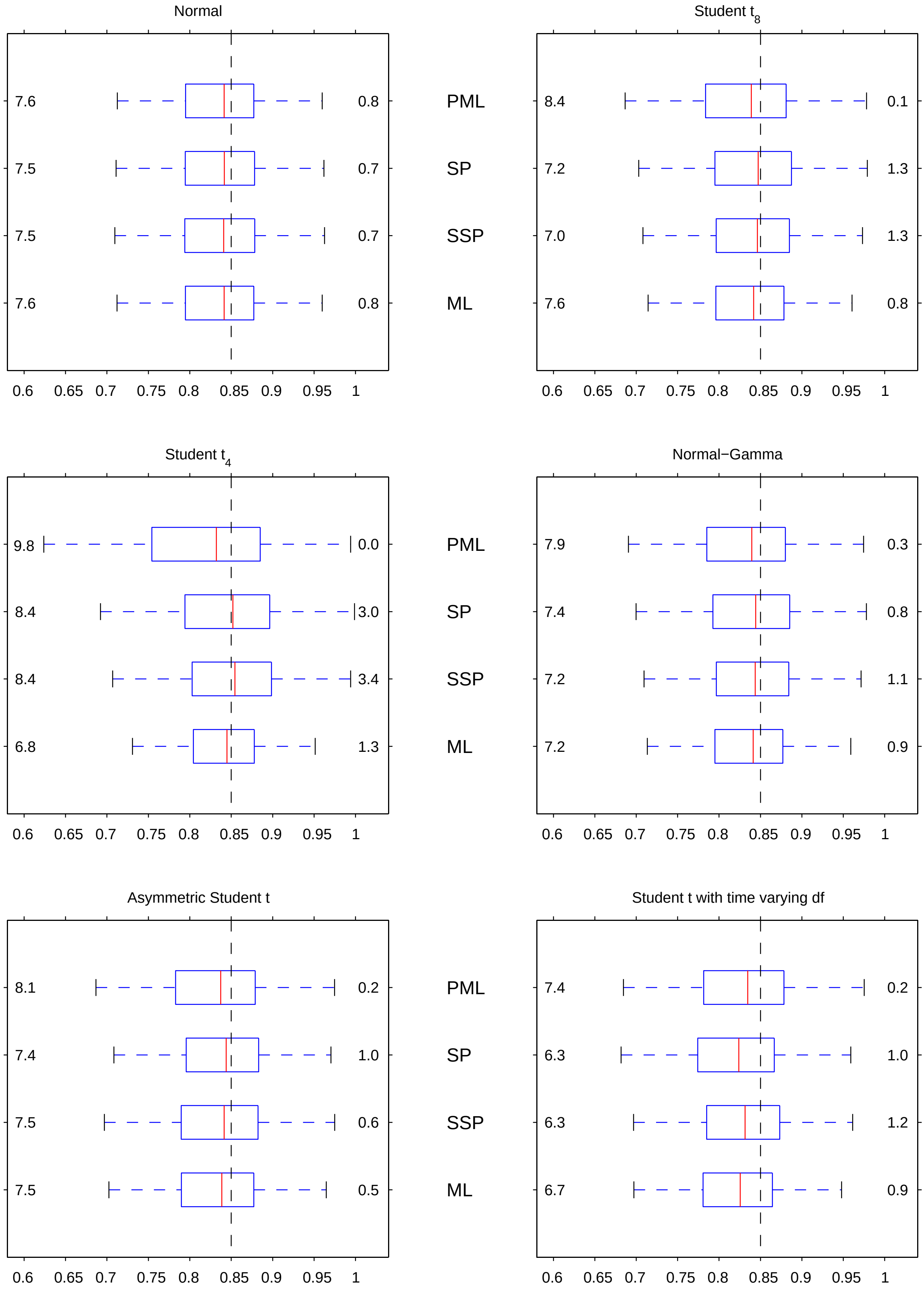
The central boxes describe the 1st and 3rd quartiles of the sampling distributions, and their median. The maximum length of the whiskers is one interquartile range. We also report the fraction of replications outside those whiskers. PML means Gaussian−based maximum likelihood estimator, ML Student t−based maximum likelihood estimator, SSP elliptically symmetric semiparametric estimator and SP unrestricted semiparametric estimator.

Figure 1E: Monte Carlo distributions of estimators of ARCH coefficent



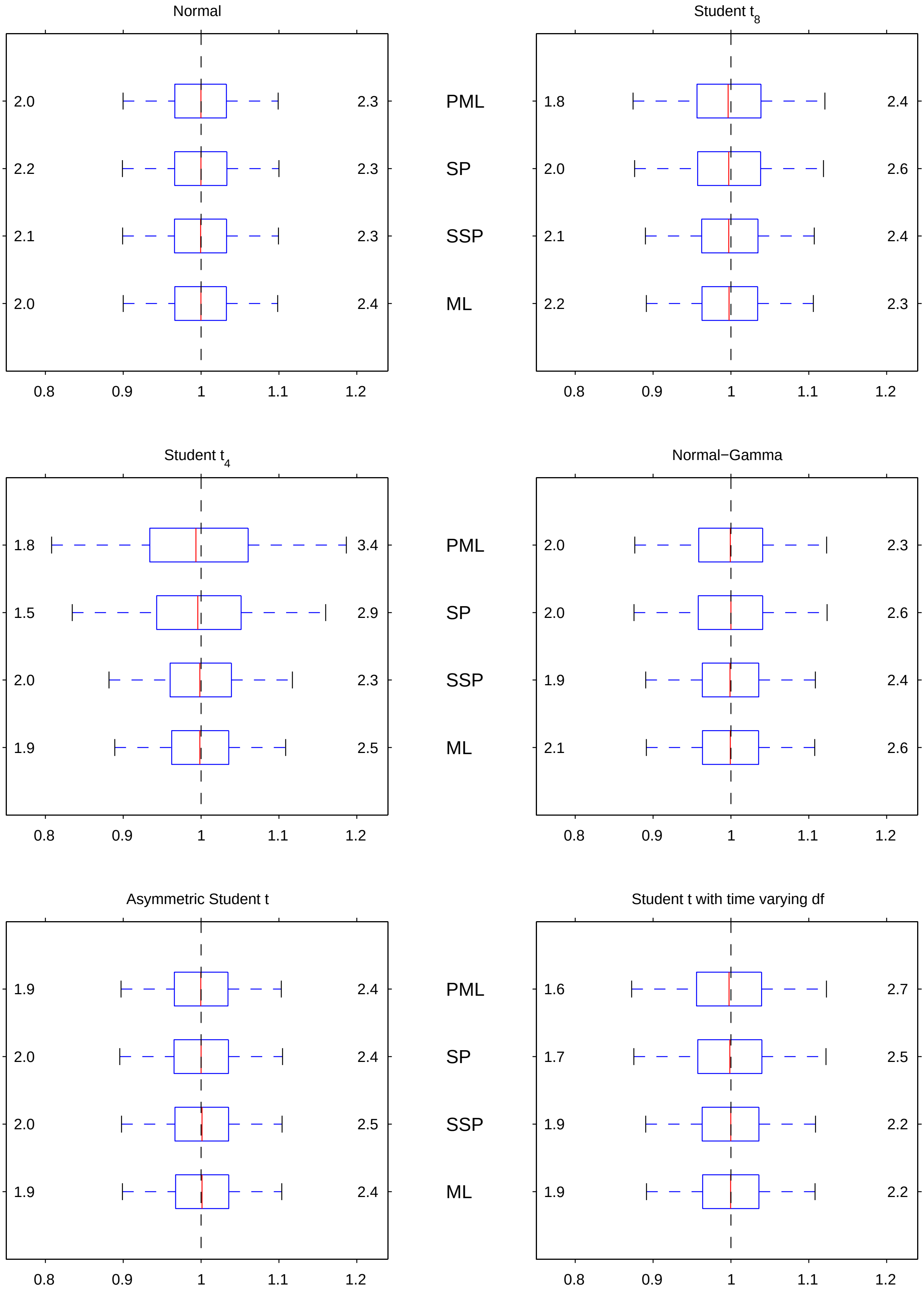
The central boxes describe the 1st and 3rd quartiles of the sampling distributions, and their median. The maximum length of the whiskers is one interquartile range. We also report the fraction of replications outside those whiskers. PML means Gaussian−based maximum likelihood estimator, ML Student t−based maximum likelihood estimator, SSP elliptically symmetric semiparametric estimator and SP unrestricted semiparametric estimator.

Figure 1F: Monte Carlo distributions of estimators of GARCH coefficent



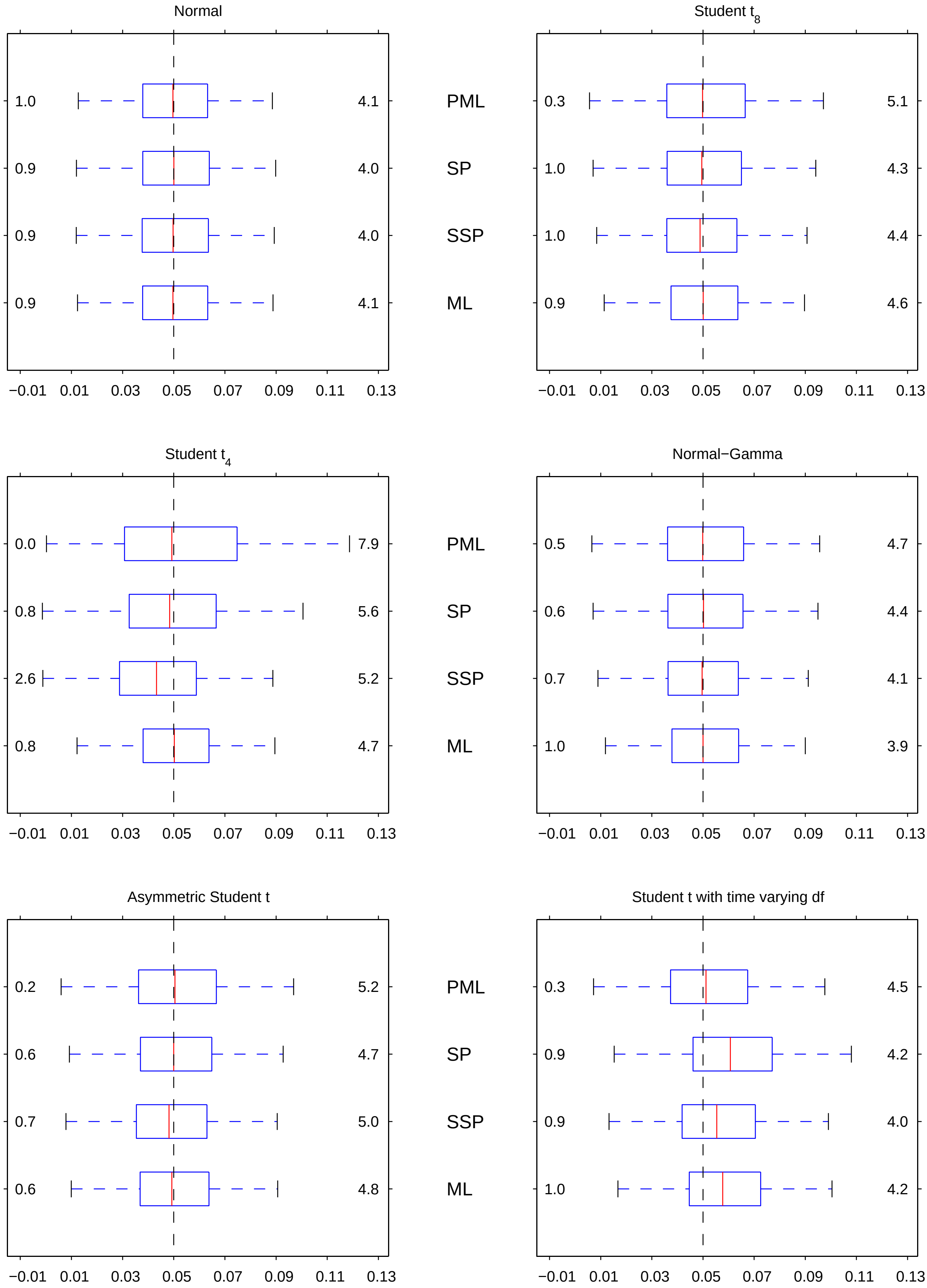
The central boxes describe the 1st and 3rd quartiles of the sampling distributions, and their median. The maximum length of the whiskers is one interquartile range. We also report the fraction of replications outside those whiskers. PML means Gaussian−based maximum likelihood estimator, ML Student t−based maximum likelihood estimator, SSP elliptically symmetric semiparametric estimator and SP unrestricted semiparametric estimator.

Figure 1G: Monte Carlo distributions of estimators of re−scaled idyosincratic variances



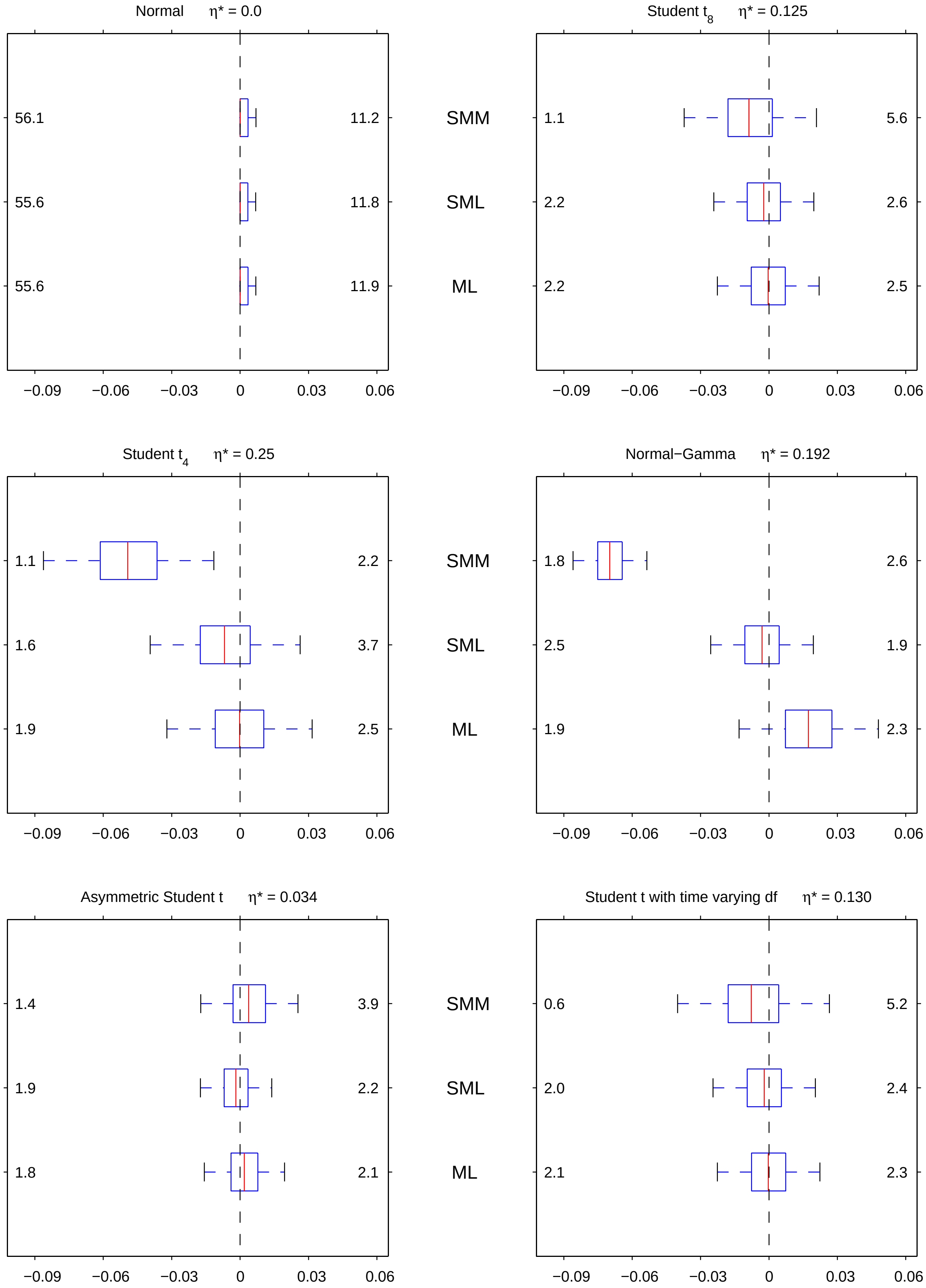
The central boxes describe the 1st and 3rd quartiles of the sampling distributions, and their median. The maximum length of the whiskers is one interquartile range. We also report the fraction of replications outside those whiskers. PML means Gaussian−based maximum likelihood estimator, ML Student t−based maximum likelihood estimator, SSP elliptically symmetric semiparametric estimator and SP unrestricted semiparametric estimator.

Figure 1H: Monte Carlo distributions of estimators of re−scaled ARCH coefficent



The central boxes describe the 1st and 3rd quartiles of the sampling distributions, and their median. The maximum length of the whiskers is one interquartile range. We also report the fraction of replications outside those whiskers. PML means Gaussian−based maximum likelihood estimator, ML Student t−based maximum likelihood estimator, SSP elliptically symmetric semiparametric estimator and SP unrestricted semiparametric estimator.

Figure 2: Monte Carlo distributions of estimators of shape parameter



The central boxes describe the 1st and 3rd quartiles of the sampling distributions, and their median. The maximum length of the whiskers is one interquartile range. We also report the fraction of replications outside those whiskers. In the Normal case the numbers on the left are the fraction of replications in which η is estimated as 0. Estimators are centred around their (SML pseudo−) true value η*. SMM means sequential method of moments estimator, SML sequential ML Student t−based maximum likelihood estimator, ML Student t−based maximum likelihood estimator.