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MAXIMIZING HUMAN DEVELOPMENT

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Abstract

The Human Development Index (HDI) is widely used as an aggregate measure of overall human well-being. We examine the allocations implied by the maximization of this index using a standard growth model. Maximization of the HDI leads to consumption (excluding education and health expenditures) being pushed to minimal levels. It also leads to the overaccumulation of education and/or health capital and possibly physical capital, relative to the standard golden rule. We propose an alternative specification of the HDI, where permanent consumption replaces income as the proxy for a decent standard of living. Maximization of this alternative index yields a "human development golden rule" which balances consumption, education and health expenditure in promoting human development. We also advocate the method of optimization subject to constraints for revealing the consequences of taking a policy measure seriously and testing its congruence with its underlying philosophy.

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"The Human Development Index is a very crude measure, but it is a better crude measure than Gross National Product or Gross Domestic Product."

Amartya Sen[1], 1998 Nobel Laureate in Economics

"The Human Development Index really helped to generate political competition. And, just as competition is very good in markets to make them efficient, political competition is also very good."

Inge Kaul[1], Director, Human Development Reports 1990-1994

1 Introduction

The progress of nations, and their relative standing, has most often been assessed using per capita measures of gross income and output as crude measures of wealth. This "income/wealth approach" has been criticized as emphasizing means over ends and for being too narrow. Building on the work of Amartya Sen and co-authors, a number of academics and policy analysts have championed the "human development approach".[2] This alternative approach sees development as the expansion of peoples' capabilities "to live better and richer lives, through more freedom and opportunity" (Anand and Sen (2000b) p84). The progress of the growth of such capabilities has been measured by outcomes ("functionings") as documented in the Human Development Reports. These Reports regularly publish country rankings for various indexes designed to evaluate aggregate outcomes. Foremost amongst these indexes is the Human Development Index (HDI), which evaluates overall human development.[3]

[1] Quoted from United Nations Development Programme (2005) video: "People First: The Human Development Reports".

[2] Both approaches have roots in longstanding traditions. Anand and Sen (2000a) trace the human development approach to the philosophies of Aristotle and Kant and describe how it "relates to the more conventional analyses to be found in the standard economics literature – from Adam Smith onwards". They relate the income/wealth approach to the 'old opulence-oriented approach'. See Pritchett and Summers (1996) for recent evidence supporting this approach, and see Anand and Ravallion (1993) and Stiglitz, Sen and Fitoussi (2009) for evidence in favor of the human development approach.

[3] The other indexes are the Inequality-adjusted Human Development Index (IHPI), Gender Inequality Index (GII), and the Multidiminensional Poverty Index (MPI). See Technical notes in the Human Development Report 2011.

The HDI annual ranking of nations is widely used and is claimed to have the effect of generating competition between nations (see the quote by Inge Kaul). The HDI, arguably, has come to rival measures of output, like per capita Gross National Income (GNI) and per capita Gross Domestic Product (GDP), as the leading measure for evaluating a country's achievement in fostering human well-being. It attempts to capture overall human development in terms of three primary dimensions: a long and healthy life, knowledge, and a decent standard of living. In formalizing the index, these three dimensions are represented, respectively, by life expectancy, education, and per capita GNI. Thus, the HDI is a more general measure of well-being than per capita income alone. Crucially for the human development approach, health and knowledge are viewed as valued ends in themselves rather than simply means to increase income.

This paper takes the HDI seriously as a basis for planning. We ask the following questions. Is maximization of the HDI a sensible basis for improving human development? How do development plans change when we alter the human development criteria, and how do they compare with income and traditional growth approaches? Consistent with the lead quote from Amartya Sen, we find that the HDI is, indeed, a crude measure on which to base planning – though a better crude measure than income. Maximization of the HDI leads to minimal consumption (excluding education and health expenditures). It also leads to the overaccumulation of education and/or health capital and possibly physical capital, relative to the standard golden rule. We therefore propose a simple change to the HDI which, we argue, more sensibly balances the primary dimensions of human development.

Our method is to analyze the long-run implications of using a generalized HDI as the key criterion for development planning in macroeconomies, using the conceptual lens of conventional economic growth theory. To do so, we construct a simple growth model which includes health, education, and income, as endogenous variables. This model is based on the extended Solow model in Mankiw, Romer, and Weil (1992), but further extends the model to include health. We identify the allocations that maximize the generalized HDI in this model, and compare them with those that satisfy the more traditional growth criterion of dynamic efficiency.

Some results are immediate and startling. Maximization of the generalized HDI, if taken literally, leads to the result that consumption (excluding education and health expenditures) should be set to zero – or, more generally, to its minimal sustainable level. This result is obtained because consumption does not enter the objective function and simply represents a cost to the planning program. To avoid this corner result, we modify the model to give consumption an instrumental role in increasing production. This modification captures the fact that labour needs to consume, at least a little, to produce. Specifically, we consider a model where output is increasing in consumption levels, up to a critical threshold consumption level. Even with this modification, the HDI-maximizing plan exhibits minimal consumption. Intuitively, the resources freed from consumption are allocated among the three types of capital: physical, educational, and health capital. Both educational and health capital are valued ends and physical capital is valuable indirectly because it increases income – another valued end. Relative to the traditional welfare criterion in the Solow model – the golden rule, which maximizes steady state consumption[4] – maximization of the HDI generically implies overaccumulation of human capital and possibly physical capital.

Interestingly, the traditional "income approach", where per capita income is the sole criterion for planning, also leads to minimal consumption. in our framework it leads to the overaccumulation of both human and physical capital. The emphasis on capital has come to be known as "capital fundamentalism" in some circles and our analysis of capital overaccumulation shows that it extends to human capital.[5] In comparison, the positive weight on education and health in the HDI leads to favoring these particular types of capital, though there may also be physical capital overaccumulation.

The usage of per capita income in the HDI is explained by Anand and Sen (p86, 2000b):

> "The use of 'command over resources' in the HDI is strictly as a residual catch-all, to reflect something of the basic capabilities not already incorporated in the measures of longevity and education."

As such,we propose a friendly amendment to the HDI, modifying it by replacing the indicator variable per capita income with per capita consumption in the dimension

[4] Phelps (1966, p.5) called this the golden rule as "... each generation saves (for future generations) that fraction of income which it would have had past generations save for it." Phelps' analysis is in the context of the Solow model in which generations are not explicitly identified.

[5] King and Levine (1994) describe capital fundamentalism as the view that increasing capital is the key to increasing per capita output. The traditional capital fundamentalism is implicit in the income approach and the new capital fundamentalism implicit in endogenous growth theory.

index for 'decent standard of living'. This is consistent with Stiglitz, Sen and Fitoussi's (2009, p12) report, which in recommendation 1 urges that material well-being be evaluated by looking at consumption rather than production. Consistent with the above quote, consumption is defined as income net of education, health and capital expenditures.

The optimal planning conditions that correspond to using consumption in the index are characterized as a "human development golden rule" in this paper. This rule, like the HDI-maximizing rule, also exhibits education and/or health capital overaccumulation relative to the traditional golden rule. However, the accumulation of capital is not inefficient on its own terms because education and health expenditures are efficiently traded-off relative to a valued end, consumption.

An important feature of the actual HDI is that it is specified with upper and lower bound values for the indicator variables. We look at cases when the steady state involves some indicators achieving their upper bounds, and we show that this does not generically change our results. Similarly, introducing exogenous technological change into the analysis does not generically change the results provided that a steady state exists. In recent years the upper bounds have been adjusted to encompass the data. Thus, the HDI can be viewed as a way of ranking nations in a relative way.

The remainder of the paper is structured as follows. Section 2 describes the HDI and introduces a general human development objective function, which nests the income approach and the human development approach as special cases. In Section 3 we present the extended Solow model, analyze its properties, and derive the traditional golden rule for the model. Section 4 develops the HDI-maximizing rule for the basic model as well as for extensions of the model. Section 5 makes the case for replacing income with consumption in the HDI. Finally, in Section 6 we discuss future directions for research and argue that achievement measures that are used to inform policy should be judged by the outcomes they imply.

2 The HDI Objective

2.1 The Human Development Index (HDI)

The HDI was first reported in the Human Development Report 1990. Over the years the index has changed considerably. In 2011, the index is the geometric mean of the component dimension indexes:

$$HDI(l, \varepsilon, y) = \left(\frac{l - 20}{83.4 - 20}\right)^{\frac{1}{3}} \left(\frac{\varepsilon - 0}{0.978 - 0}\right)^{\frac{1}{3}} \left(\frac{\ln y - \ln 100}{\ln 107721 - \ln 100}\right)^{\frac{1}{3}}$$

where l is life expectancy at birth, ε is an index of education, and y is GNI per capita (PPP US$). The logarithm of income is used in the income dimension index, $\frac{\ln y - \ln 100}{\ln 107721 - \ln 100}$. This captures the view that the transformation from income to capabilities is concave. In each dimension, the variables are bounded by the observed maximum in the data series from 1980. The minimum bounds are conceived as subsistence values.[6]

A major advantage of the HDI is that it is straightforward and built from data that is widely available. However, the form of the index is related to a more general philosophy and methodology. The specific arguments of the HDI, are called "indicators". They are intended to proxy the three most important dimensions for well-being: the dimension "long and healthy life" is represented by the indicator of life expectancy l, the dimension of "knowledge" is represented by the indicator of education ε, and the dimension of decent standard of living is proxied by per capita GNI y. As mentioned above, and explored further below, the underlying philosophy behind the choice of these dimensions is arguably not particularly well served by these indicators. However the consistent message is that these dimensions and their indicators are intended to represent *valued ends*. It is also recognized that these same primary ends of human development are also primary means, but this is not what the index is meant to capture.

[6] The methodology of the HDI is detailed in Technical note 1 of the Human Development Report 2011. The education dimension is comprised of two indexes where $e = (\textit{Mean years of schooling index})^{1/2}(\textit{Expected years of schooling index})^{1/2}$.

When the index was first created the lower and upper bounds were from the yearly data. Thus, the index was used to simply rank countries yearly. Later the index was changed to have fixed bounds in order to trace the improvement in national achievements over time. Starting in 2010, the upper bounds are updated to encompass the data series from 1980.

There have been a number of critiques of the HDI.[7] Criticism has been encouraged and has lead to revisions of the index. Prior to 2010, the HDI equaled the sum of the dimension indexes. This specification was criticized for allowing perfect substitution across dimensions. The 2010 change to a geometric index yields imperfect substitutability across dimensions. Similarly, the education indicator is now derived from a geometric combination of two subindexes. The lower bound on life expectancy was changed to 20 years based on long-run historical evidence. The income proxy has changed over time from per capita GDP to logrithm of per capita GDP to logarithm of per capita GNI.[8] A further limitation of the HDI relevant to this paper is that it incorporates no intertemporal trade-offs. In this sense it is a static concept.

2.2 A General Human Development Objective Function

We represent a general human development objective function as follows:

$$D(h(t), e(t), y(t)) \tag{1}$$

where $h(t)$ and $e(t)$ are the current per capita stocks of health and education human capital at time t. This index is assumed to be twice differentiable and strictly concave when the arguments are between the lower bounds, denoted $(\underline{h}, \underline{e}, \underline{y})$, and the upper bounds $(\overline{h}, \overline{e}, \overline{y})$. Using subscripts to denote partial derivatives, we assume that $D_j > 0$ and $D_{jj} < 0$ for $j(t) \in [\underline{j}, \overline{j})$, $D_j = 0$ for $j(t) < \underline{j}$ and $j(t) \geq \overline{j}$, where $j = h, e, y$.

Stocks of health and education human capital replace the specific indicators of the HDI in our general function for three reasons. First, our dynamic model has stocks of human capital, so the same variables can represent both means and ends. Second, these stocks are more general indicators. For example, Engineer,

[7] The criticisms and responses are reviewed by Raworth and Stewart (2005). Also, see Hicks (1997), Noorbakhsh, (1998), Mazumdar (2003), Cahill, (2005), Osberg and Sharpe (2005).

[8] GDP is the gross output of a nation. GNI is a better measure of material well being as it is the total income accruing to the citizens of a nation. The measures differ by net international income flows. The World Bank, International Comparison Program database, defines "... GNI is the sum of value added by all resident producers plus any product taxes (less subsidies) not included in the valuation of output plus net receipts of primary income (compensation of employees and property income) from abroad ..." Previously, this definition was referred to as the Gross National Production (GNP), a term the World Bank no longer uses in this regard.

Roy and Fink (2010) criticize the life expectancy indicator for not capturing the healthy part of the dimension "long and healthy" life. Health human capital is a concept that generally captures the dimension. Finally, the implications of the life expectancy and education levels have already been examined in Engineer, King and Roy (2008) – in a static analysis.

Consider a simple explicit version of the general function:

$$D(h(t), e(t), y(t); w, W) = (h(t)^{(1-W)} e(t)^{W})^{(1-w)} y(t)^{w}$$

where w is the relative exponent weight on income and $1 - w$ the weight on the health and education component. Within the health and education component, W is the relative weight on education. Equal weights on the arguments is when $w = \frac{1}{3}$ and $W = \frac{1}{2}$. Setting $w = 1$ implies that income is the only argument. This is the special case of the "income approach".

3 The Extended Solow Model

In this section, we present an extended Solow model similar to the one given in Mankiw, Romer, and Weil (1992), with education human capital in the production function, but extended further to include both health human capital and consumption in production.[9] As discussed in the introduction, we include a limited productive role for consumption to avoid the unrealistic possibility that consumption is set equal to zero to maximize the HDI value of this economy.

3.1 Production with Productive Consumption

Aggregate output $Y(t)$ is the product of two functions:

$$Y(t) = F(K(t), H(t), E(t), L(t))\Phi\left(\frac{c(t)}{c_s(t)}\right) \tag{2}$$

where $K(t), H(t), E(t)$, and $L(t)$ are, respectively, aggregate values for physical capital, health capital, education capital, and labour. The component function

[9] Rivera and Currais (1999) provide strong evidence for including health capital in an extended Mankiw et. al (1992) model.

F has the form of a Neoclassical production function: it is increasing in each of its arguments, strictly concave, satisfies the Inada conditions and has constant returns to scale. We also assume the factors of production are weak complements and somewhat symmetric complements.[10]

The Φ function captures the effect of per capita consumption on output. Below a critical "threshold consumption" level, $c_s(t) > 0$, output is increasing in per capita consumption $c(t)$. Above $c_s(t)$, further increments in $c(t)$ have no further effects on output:

$$\Phi\left(\frac{c(t)}{c_s(t)}\right) < 1, \quad \Phi' > 0, \quad \Phi'' < 0 \quad \text{for} \quad 0 \leq \frac{c(t)}{c_s(t)} < 1$$

$$\Phi\left(\frac{c(t)}{c_s(t)}\right) = 1 \quad \text{for} \quad \frac{c(t)}{c_s(t)} \geq 1$$

$$\lim_{(c/c_s)\to 0} \Phi' \to \infty, \qquad \lim_{(c/c_s)\to 1} \Phi' \to 0$$

Our production function can be thought to be generalization of the standard specification where the minimum amount of consumption needed to fully sustain production $c = c_s$ is quite small relative to per capita output. In the limit as the threshold goes to zero, $c_s \to 0$, the minumum amount of consumption need to fully sustain production also goes to zero. Observe that below the threshold, $c < c_s$, consumption is effectively a complement with the factors of production in the F function.

The essential rationale for a production function where consumption is productive goes back at least as far as Leibenstein (1957):

> The amount of work that the representative laborer can be expected to perform depends on his energy level, his health, his vitality, etc., which in turn depend on his consumption level (which depends on income level) and, most directly, on the nutritive value of his food intake.

While this seems an obvious point, the literature on productive consumption is a surprising small and scattered. Steger (2002) briefly reviews this literature which

[10]Factors are weak complements when $F_{KH}, F_{KE}, F_{KL}, F_{HE}, F_{HL}, F_{EL} \geq 0$. 'Somewhat symmetric complements' is the requirement that the cross partials are similar in magnitude. For example, the proof of Proposition 2b(ii) assumes that $|F_{KE} - F_{KH}|$ is sufficiently small.

mostly appeals to nutriutional energy requirements of work. He builds a growth model where consumption has an ongoing productive role. In his model, physical and human capital are perfect substitutues and the production function is linear. With this specification he is able to characterize the endogenous growth path of the economy.

Unlike Steger (2002), the productive role for consumption in our model is limited to a level of consumption below a threshold. We believe this is more realistic. The specification $\Phi\left(\frac{c(t)}{c_s(t)}\right)$ is is the simpliest possible and it allows us to readily compare our results to the standard model. This term acts like a multifactor productivity term in production which disappears when consumption exceeds the threshold to yield the standard Neoclassical production function.

Our results are qualitatively unchanged, if instead a similar term directly impacts the effectiveness of labour (but the growth analysis becomes cumbersome). With Cobb-Douglas production, the results are unaffected if consumption directly impacts the effectiveness of any of the inputs. For example, consider the production function $Y(t) = K(t)^{\alpha}[\phi(\frac{c(t)}{c_s(t)})H(t)]^{\beta}E(t)^{\eta}L(t)^{(1-\alpha-\beta-\eta)}$, where the coefficients are positive and $\alpha+\beta+\eta < 1$, and $[\phi(\frac{c(t)}{c_s(t)})]^{\beta} = \Phi(\frac{c(t)}{c_s(t)})$. Collecting the other terms in $F = K(t)^{\alpha}H(t)^{\beta}E(t)^{\eta}L(t)^{(1-\alpha-\beta-\eta)}$ gives back the original form of our production function. We discuss other ways that consumption might affect output and the results in Section 4.5.

It is instructive to continue the example started in Section 2.2. Substituting $y(t) = Y(t)/L(t) = k(t)^{\alpha}h(t)^{\beta}e(t)^{\eta}\Phi(\frac{c(t)}{c_s(t)})$ into the explicit HDI objective function gives the following reduced form objective function:

$$\Gamma(h(t), e(t), k(t), c(t)) = h(t)^{[w\beta+(1-w)(1-W)]}e(t)^{[w\eta+(1-w)W]}k(t)^{\alpha w}\Phi\left(\frac{c(t)}{c_s(t)}\right)^{w}$$

Indirectly both physical capital and consumption are valued in the objective. When each argument in the original objective function D is equally weighted (i.e. $w = \frac{1}{3}$ and $W = \frac{1}{2}$), the exponents on $h(t), e(t)$ and $k(t)$ respectively are $\frac{1+\beta}{3}$, $\frac{\eta+1}{3}$ and $\frac{\alpha}{3}$. Observe that the exponents on health and education human capital dominate the weight on physical capital even if human capital is not productive $\beta = \eta = 0$. The dominance of human capital becomes more pronounced when logarithm of income is used in the index.

3.2 The Model

The equations of motion for each of the inputs $K(t), H(t), E(t)$, and $L(t)$ are, respectively:

$$\dot{K}(t) = I_K(t) - \delta K(t) \tag{3}$$

$$\dot{H}(t) = I_H(t) - \delta H(t) \tag{4}$$

$$\dot{E}(t) = I_E(t) - \delta E(t) \tag{5}$$

$$L(t) = N(t), \qquad \dot{N}(t) = nN(t) \tag{6}$$

where δ is the depreciation rate, which we assume to be common for all types of capital and I_J are the aggregate investments for $J = K, H, E$. Population $N(t)$ grows at exogenous rate n, and the population equals the labour force. Dots over variables denote their time derivatives. The resource constraint is:

$$Y(t) = C(t) + I_K(t) + I_E(t) + I_H(t)$$

This constraint can be expressed in terms of savings rates:

$$C(t) = (1 - s_K(t) - s_E(t) - s_H(t))Y(t) \tag{7}$$

where $s_J(t) \equiv I_J(t)/Y(t)$. We can now express the model in per capita terms. With constant returns to scale in F, we can divide this function by $L(t)$ to find per capita income in terms of the intensive production function f. Accordingly:

$$y(t) = f(k(t), h(t), e(t))\Phi(c(t)/c_s(t)) \tag{2'}$$

$$\dot{k}(t) = s_K(t)y(t) - (n + \delta)k(t) \tag{3'}$$

$$\dot{h}(t) = s_H(t)y(t) - (n + \delta)h(t) \tag{4'}$$

$$\dot{e}(t) = s_E(t)y(t) - (n + \delta)e(t) \tag{5'}$$

$$c(t) = (1 - s_K(t) - s_H(t) - s_E(t))y(t) \tag{7'}$$

where

$$y(t) \equiv \frac{Y(t)}{L(t)}, \;\; c(t) \equiv \frac{C(t)}{L(t)}, \;\; k(t) \equiv \frac{K(t)}{L(t)}, \;\; e(t) \equiv \frac{E(t)}{L(t)}, \;\; h(t) \equiv \frac{H(t)}{L(t)}$$

We concentrate on the steady state of the model. In the steady state per capita quantities settle down to constants so that aggregate quantities grow at the rate of the population. Then, $\dot{k}(t) = \dot{h}(t) = \dot{e}(t) = 0$, and (2′)-(5′) imply:

$$y = f(k, h, e)\Phi(c/c_s) \tag{2''}$$

$$k = \frac{s_K y}{n + \delta} \tag{3''}$$

$$h = \frac{s_H y}{n + \delta} \tag{4''}$$

$$e = \frac{s_E y}{n + \delta} \tag{5''}$$

Here we assume threshold consumption $c_s(t)$ is a constant c_s. Using equations (2″)-(5″), the resource constraint for consumption can be expressed solely in terms of the capital stocks in the steady state:

$$c = f(k, h, e)\Phi(c/c_s) - (k + h + e)(n + \delta) \tag{8}$$

The last term $(k+h+e)(n+\delta)$ is steady state investment per capita and $(k+h+e)$ is the total physical and human capital stock. Totally differentiating (8) reveals the trade off between capital and consumption on the resource constraint:

$$(1 - f\Phi_c)dc = [\Phi f_k - (n + \delta)]dk + [\Phi f_h - (n + \delta)]dh + [\Phi f_e - (n + \delta)]de \tag{8'}$$

3.3 The (Consumption) Golden Rule

The traditional golden rule maximizes steady state consumption subject to the resource constraint (8). The first-order conditions are:

$$\frac{dc}{dk} = \frac{\Phi f_k - (n + \delta)}{1 - f\Phi_c} = 0 \quad \Rightarrow \quad MP_K = \Phi f_k = (n + \delta)$$

$$\frac{dc}{dh} = \frac{\Phi f_h - (n + \delta)}{1 - f\Phi_c} = 0 \quad \Rightarrow \quad MP_H = \Phi f_h = (n + \delta)$$

$$\frac{dc}{de} = \frac{\Phi f_e - (n + \delta)}{1 - f\Phi_c} = 0 \quad \Rightarrow \quad MP_E = \Phi f_e = (n + \delta)$$

where $\Phi_c = \Phi'/c_s$.Optimization implies that the capital marginal products be equated to the breakeven replacement rate, $MP_K = MP_H = MP_E = n + \delta$, and the marginal product of consumption is less than one, $MP_C = f\Phi_c < 1$.[11] These conditions identify the golden rule in our model.

There are two cases for the golden rule. The first case is $c^* \geq c_s$, where the * superscript indicates the golden rule value. In this case, consumption is not productive at the margin, $\Phi_c = 0$ and $\Phi = 1$. The golden rule condition for physical capital is completely standard, $f_k = n + \delta$, and the planner should also set $f_h = f_e = n + \delta$. The second case is where the planner can not achieve the threshold, $c^* < c_s$, so that consumption is productive at the margin, $\Phi_c > 0$ and $\Phi < 1$. Here $f_k = f_h = f_e = (n+\delta)/\Phi > n+\delta$ and our assumptions on production ensure that as c_s increases optimal consumption, investment and output fall.[12]

Which case obtains relates to the benchmark $c^{**} = f(k^{**}, h^{**}, e^{**}) - (k^{**} + h^{**} + e^{**})(n + \delta)$, where the superscript ** denotes the solution to the planner's problem in the standard formulation where consumption does not enter production, $Y(t) = F(K(t), H(t), E(t), L(t))$. Suppose threshold consumption is higher than the benchmark, $c^{**} < c_s$. Then, the second case $c^* < c_s$ obtains, as $c^* < c^{**}$ when $\Phi < 1$. This case obtains only if $c^{**} < c_s$. Otherwise, $c^{**} \geq c_s$ and the threshold is feasible. Then, the first case obtains $c^* = c^{**} \geq c_s$. The following proposition summarizes.

Proposition 1 *At the golden rule, physical and human capital* (k^*, h^*, e^*) *are allocated to equate the marginal products*

$$MP_K = MP_H = MP_E = n + \delta \ .$$

Consumption depends on the threshold, c_s*, as follows:*
(i) $c^* = c^{**}$ *if and only if* $c_s \leq c^{**}$, $MP_C = 0$*; and*

[11] Conversely, if $MP_C > 1$ then 1 unit allocated to consumption generates more than one unit of production. Greater consumption could be generated by continuing to allocate resources to consumption. However, the Inada condition on Φ requires that the marginal productivity falls with increases in c and $MP_C = 0$ for $c \geq c_s$. We assume c_s is sufficiently small that there exists a feasible allocation $MP_C < 1$ and $MP_J \geq n + \delta$ for all $J = K, H, E$ and strictly greater for one J . Then $MP_C = 1$ is non-optimal.

[12] When $c^* < c_s$, increasing c_s results in c^* falling as the previous optimum consumption level is now infeasible. Thus, Φ decreases and $(n+g)/\Phi$ increases which in turn implies lower investment (under weak complements). Output is lower because both consumption and investment are less.

(ii) $c^* < c^{**}$ *if and only* $c_s > c^{**}$, $0 < MP_C < 1$ *and* c^* *is decreasing in* c_s.

The (consumption) golden rule provides a well-know basis for comparison with the human development rules we examine later. In the standard Solow model, capital overaccumulation, $k > k^*$, results whenever there is a reduction in the marginal product of capital, $MP_K < n + \delta$. Capital overaccummulation is of concern because it implies dynamic inefficiency, which means that it is possible to increase the path of consumption by reducing investment (see De la Croix and Michel (2002) and King and Ferguson (1993)). In our model the relationships between marginal products and capital overaccumulation is complicated with three types of capital as well as productive consumption. Capital overaccumulation occurs when the capital marginal products are lower than the breakeven rate, $n + \delta$, provided that $MP_C < 1$ so that consumption isn't reduced too much.

Proposition 2 *Capital overaccumulation occurs whenever* $MP_C < 1$ *and* $MP_J \leq n + \delta$ *for all* $J(= H, E, K)$ *and* $MP_J < n + \delta$ *for at least one* J.
(a) If $c \geq c_s$ *or* $c \to c^* \leq c_s$, *then* $j > j^*$ *for all* j *when capital factors are complements, and when factors are not complements (or substitutes),* $j > j^*$ *for* $j(= h, e, k)$ *corresponding to* J *such that* $MP_J < n + \delta$ *and* $j = j^*$ *for* j *corresponding to* $MP_J = n + \delta$.
(b) If $c < c_s$, *then: (i)* $j > j^*$ *for all* j *when* $MP_H = MP_E = MP_K < n + \delta$;
(ii) $e > e^*$, $h > h^*$ *and* $k \gtreqless k^*$ *when* $MP_E = MP_H < MP_K \leq n + \delta$;
(iii) $j > j^*$ *for the factor with the lowest marginal product.*

The proof is found in the Appendix. Intuitively, consider (8′) rewritten in terms of marginal products:

$$(1 - MP_C)dc = [MP_K - (n + \delta)]dk + [MP_H - (n + \delta)]dh + [MP_E - (n + \delta)]de$$

Consider a deviation from the golden rule that allows only physical capital and consumption to vary. An increase in physical capital drives $MP_K < n + \delta$ and results in physical capital overaccumulation. As $MP_C < 1$, consumption is less, $dc < 0$. If the different types of capital are complements, then an increase in physical capital alone would result in a rise in $MP_H, MP_E > n + \delta$. To return to $MP_H = MP_E = n + \delta$ these other factors also must so increase. Thus around the golden rule, weak complements yields physical and human capital overaccumulation.

More generally, capital overaccumulation occurs away from the golden rule as long as consumption is not marginally productive (Proposition 2(a)), as in the standard Solow model. When consumption is marginally productive (Proposition 2(b)), reducing the marginal products symmetrically results in capital overaccumulation of each type of capital. When the marginal products of education and health capital are lower than physical capital, there is human capital overaccumulation and the possibility of physical capital underaccumulation or overaccumulation. This particular result is useful in the next section.

4 Maximizing the HDI

4.1 The Planner's Problem

We now consider the problem of choosing steady state values of h, e, y, k, and c to maximize the value of the HDI, as represented in (1), subject to the production ($2''$) and feasibility constraint (8). In per-capita terms, the problem becomes:

$$\max_{\{h,e,y,k,c\}} D(h,e,y) \ \text{ st } \ \begin{aligned} c &= y-(n+\delta)(k+h+e) \\ y &= f(k,h,e)\Phi(c/c_s) \end{aligned}$$

Here we assume that the bounds on the indicator variables (h, e, y) are non-binding, an assumption that is relaxed later.

It is convenient to form the Lagrangian by substituting the production function for y in to both the objective function and the feasibility constraint:

$$L = D(h,e,f(k,h,e)\Phi(c/c_s)) - \sigma(c - f(k,h,e)\Phi(c/c_s) + (n+\delta)(k+h+e))$$

where $\sigma > 0$ is the marginal value of an exogenous increase in income. The first-order conditions with respect to c, k, h, and e are, respectively

$$D_y f\Phi_c - \sigma(1 - f\Phi_c) = 0 \qquad \Rightarrow \qquad f\Phi_c = \frac{\sigma}{D_y + \sigma}$$

$$D_y f_k \Phi - \sigma(-f_k\Phi + (n+\delta)) = 0 \quad \Rightarrow \quad f_k\Phi = \frac{\sigma}{D_y + \sigma}(n+\delta)$$

$$D_h + D_y f_h \Phi - \sigma(-f_h\Phi + (n+\delta)) = 0 \quad \Rightarrow \quad f_h\Phi = \frac{\sigma(n+\delta) - D_h}{D_y + \sigma}$$

$$D_e + D_y f_e \Phi - \sigma(-f_e\Phi + (n+\delta)) = 0 \quad \Rightarrow \quad f_e\Phi = \frac{\sigma(n+\delta) - D_e}{D_y + \sigma}$$

These four first-order conditions, together with the feasibility constraint, constitute a system of five equations in five unknowns (c, k, h, e, and σ) that describe the HDI maximizing allocations in the steady state. The Inada conditions assure positive values for inputs c, k, h and e, and non-satiation implies income is valuable $\sigma > 0$.

The above conditions can be rewritten in terms of marginal products ($MP_C = f\Phi_c$, $MP_K = f_k\Phi$, $MP_H = f_h\Phi$, and $MP_E = f_e\Phi$) as follows:

$$MP_C = \frac{\sigma}{D_y + \sigma} \quad \Rightarrow \quad 0 < MP_C < 1$$

$$MP_K = MP_C(n+\delta) \quad \Rightarrow \quad MP_K < n+\delta$$

$$MP_H = MP_K - \frac{D_h}{D_y}(1 - MP_C) \quad \Rightarrow \quad MP_H < MP_K < n+\delta$$

$$MP_E = MP_K - \frac{D_e}{D_y}(1 - MP_C) \quad \Rightarrow \quad MP_E < MP_K < n+\delta$$

The MP_C is driven below 1 and consumption increases to the extent that output is valued in the objective function, $D_y > 0$. However, $MP_C > 0$ indicates that $\Phi_c > 0$ so that $c < c_s$. Consumption is only provided when it is marginally productive. All the conditions differ from the golden rule, so $c < c^*$. Thus, $c < \min[c_s, c^*]$. As $MP_H, MP_E < MP_K < n + g$, Proposition 2$b$ requires overaccumulation of at least one type of human capital. The following proposition summarizes these results.

Proposition 3 *Maximizing the HDI implies minimal consumption,* $c < \min[c_s, c^*]$*, and overaccumulation of education and/or health human capital. Physical human capital may or may not be overaccumulated.*

Consumption not only must be smaller than the golden rule level, it must also be smaller than the threshold. When the threshold is very small so is consumption, and in the limit as $c_s \to 0$ then $c \to 0$.[13] Consumption is minimal in the sense that

[13]Suppose c_s is quite small and the same across countries. Further suppose countries differ

it is only provided for the instrumental reason that it increases the index through it effect on productivity.[14]

Maximizing the HDI gives priority to human capital over physical capital, MP_H, $MP_E < MP_K < n + \delta$. This is because both types of human capital are valued not only indirectly through production, but also directly in the objective function. Both h and e are means and ends. The relative sizes of MP_H and MP_E depend on their direct weights in the objective function:

$$\frac{D_e}{D_h} = \frac{MP_K - MP_E}{MP_K - MP_H} \quad \Rightarrow \quad D_e \gtreqless D_h \quad \Leftrightarrow \quad MP_E \lesseqgtr MP_H$$

If $D_e = D_h$ then $MP_E = MP_H$ and Proposition $2b(ii)$ requires $h > h^*$ and $e > e^*$. It follows that if h and e enter both the production and objective functions symmetrically then both factors will be overaccumulated.

4.2 The Income Approach

At this point, it is worthwhile to compare the results with the "income approach", where only income is valued in the objective function. The first-order conditions take the same form as above with $D_h = D_e = 0$. Thus, the condition for consumption is unchanged requiring $0 < c < c_s$ for the same reasons as before. The optimality conditions imply that the marginal products be equated across types of capital: $MP_J = MP_C(n + \delta) < n + \delta$ for $J = H, E, K$. Human capital is not favoured over physical capital. Proposition $2(b)(i)$ then implies the capital overaccumulation for all J.

Proposition 4 *Maximizing per capita income implies minimal consumption,* $c < \min[c_s, c^*]$, *and overaccumulation of education, health and physical capital.*

The analysis reveals that the income approach is "capital fundamentalist" in emphasis. In cases where the human development approach implies physical cap-

only in multifactor productivity. Then even much more productive countries would have only somewhat higher consumption since as $c < c_s$ for all countries.

[14] The planner's problem could be reformulated in stages. Choosing c and k to minimize cost $c + (n + \delta)k$ while achieving a given level of output $f(k, e, h)\Phi(c/c_s) \geq y$ implies a cost function $\theta(h, e, y)$. This cost function with the feasibililty constraint $y = \theta(h, e, y) + (h + e)(n + \delta)$ implies a maximum output function $y(h, e)$.The planner then chooses h and e to maximize $D(h, e, y(h, e))$.

ital overaccumulation (i.e. D_y is large), it might also be described as capital fundamentalist.

4.3 Bounds on the Indicator Variables in the HDI

As discussed in Section 2, the indicator variables only effect the HDI when they are between their lower and upper bounds (i.e. $D_j > 0$ for $j(t) \in [\underline{j}, \overline{j})$, and $D_j = 0$ for $j(t) < \underline{j}$ and $j(t) \geq \overline{j}$, where $j = h, e, y$). In recent years the lower bounds have been sufficiently low that they have been exceeded by all countries. However, under the old specification of the HDI prior to 2010 some upper bounds were achieved by a few countries. Here, we consider the possibility that the planner may choose indicator variables at or above the previous upper bounds, but assume that it is infeasible to achieve all of the upper bounds simultaneously.[15] We continue to assume that the planner can and will choose the indicator variables above their lower bounds.

First consider when it is optimal to choose education at, or above, the bound, $e \geq \overline{e}$, but other indicator variables are below their upper bounds. Now the direct marginal benefit for education is $D_e = 0$ and, at the margin, education will be valued like physical capital: $MP_H < MP_E = MP_K < n + \delta$. Similarly, when health $h \geq \overline{h}$, is the only variable chosen at or above the upper bound, $MP_E < MP_H = MP_K < n+\delta$. In either case, by Proposition 2*b*, there is overaccumulation of at least one type of human capital. When income is the only indicator variable within bounds, then $MP_E = MP_H = MP_K < n + \delta$ so there is overaccumulation of all types of capital as in Proposition 4.

Now consider when it is optimal to choose income $y \geq \overline{y}$. Then $D_y = 0$ and $MP_C = 1$, so there is even less reason to provide consumption. Also, $MP_K = n+\delta$, and $MP_H = MP_K - D_h/\sigma$ or $MP_E = MP_K - D_e/\sigma$. There is overaccumulation when $D_h > 0$ and/or $D_e > 0$. Summarizing, as long as it is infeasible to simultaneously achieve the upper bounds for all the indicator variables, the results in the above propositions obtain.

[15] If, alternatively, it was feasible to achieve all the upper bounds, then a country would simply choose the variables at or above these bounds. Such a country is sufficiently productive that $\{k, h, e\} \geq \{\overline{k}, \overline{h}, \overline{e}\}$ and $c > 0$ satisfies the resource constraint (8) for a steady state.

4.4 Exogenous Technological Change

Introducing exogenous technological change into the analysis does not change the results, provided a steady state exists. With exogenous labour-augmenting technological change the production function becomes:

$$Y(t) = F\left(K(t), H(t), E(t), A(t)L(t)\right)\Phi\left(\frac{c(t)}{c_s(t)}\right)$$

where $\dot{A} = gA(t)$ and g is the exogenous rate of technological change. In terms of efficiency units the intensive production function is:

$$\hat{y}(t) = f\left(\hat{k}(t), \hat{h}(t), \hat{e}(t)\right)\Phi\left(\frac{\hat{c}(t)}{\hat{c}_s(t)}\right)$$

where $\widehat{x}(t) = x(t)/A(t)$ for $x = y, k, h, e, c, c_s$. The feasibility constraint has the same form expect for the breakeven capital term including g:

$$\hat{c}(t) = f\left(\hat{k}(t), \hat{h}(t), \hat{e}(t)\right)\Phi\left(\frac{\hat{c}(t)}{\hat{c}_s(t)}\right) - (n+g+\delta)\left(\hat{k}(t) + \hat{h}(t) + \hat{e}(t)\right)$$

If we assume that $c_s(t)$ grows at the rate of technology g, then $c_s(t)$ is constant $\hat{c}_s$. It follows that the marginal productivity conditions that describe the (consumption) golden rule are the same as before, except that the breakeven capital accumulation term includes g: $MP_H = MP_E = MP_K = n + g + \delta$.

If we further assume that the objective function is homogenous of degree one, the exogenous term $A(t)$ can be taken out as a multiplicative factor:[16]

$$D(A(t)\hat{h}(t), A(t)\hat{e}(t), A(t)\hat{y}(t)) = A(t)D(\hat{h}(t), \hat{e}(t), \hat{y}(t))$$

As the choice of $\hat{c}$, $\hat{k}$, $\hat{h}$, and $\hat{e}$ is unaffected by $A(t)$, the planner's problem is very similar to before. It follows that solution has the same qualitative features as before. The analysis also extends to the presence of bounds on the indicator variables analogous to above.

If the bounds change at a different rate than g, then a steady state with the

[16] Anand and Sen (1994) advocate homogeneity of degree one as a desirable feature of a development index. Assuming homogeneity of degree one is perhaps not unreasonable when the objective includes health and education capital.

indicator variables interior to the bounds may not exist.[17] For example, if the bounds on education and health grow at a rate less than g then eventually the planner achieves these upper bounds. Income will be left as the only indicator within bounds and the results in Proposition 4 obtain. The current practice is that the upper bounds are adjusted to encompass the data of the leading country in each indicator.

4.5 Robustness

Our benchmark growth model makes standard assumptions but also has some unusual features. Relaxing some of the standard assumptions lead to well know complications. For example, in cases where we found all types of capital are overaccummulated (under the assumption that factors are weak complements in F), we would find that only a subset of factors would be overaccumulated when factors are substitutes. Interestingly, with factor substitutes the extent of overaccumulation of human capital could be much greater. The reason is that Planner would have a greater incentive to lean heavily towards those factors that are directly weighted in the objective function and away from physical capital. Similarly, nonconvexities would tend to give this type of result. Our finding of minimal consumption depends only on nonsatiation of the objective and is not at threat from relaxing any of the standard assumptions on production.

A more unusual feature is the inclusion of health capital in the growth model. However, our qualitative results are unchanged if we excluded both education and health capital from the production component F. In this case, a HDI-maximizing planner would provide some human capital for its intrinsic value whereas the golden rule requires zero levels of human capital.

The productive consumption feature in the model is not driving the results. Consider a simpler model without productive consumption, $Y(t) = F(K(t), H(t), E(t), L(t))$, but with a minimum consumption constraint, $c \geq \underline{c}$ where $\underline{c} < c^{**}$. With this model,

[17] This is also the case for $c_s(t)$. When $c_s(t)$ grows at a rate less than g, there is no steady state without the additional constraint $c(t)/y(t) \geq \theta$, where $\theta > 0$ is a small fraction (that might be motivated by social or political considerations). With this constraint binding, $c(t) = \theta y(t)$ grows at rate g, and $\widehat{c}/\widehat{c}_s(t) \geq 1$ which is sufficient for $\Phi = 1$. Here minimal consumption, redefined as $\widehat{c} = \theta\widehat{y}$, does not directly effect production. The steady state has $MP_H, MP_E < MP_K < n+g+\delta$. Overaccumulation of all types of capital follows from a straightforward extension of Proposition 2a.

it can be shown that a HDI-maximizing planner sets $c = \underline{c}$ and $MP_H, MP_E < MP_K < n + \delta$. This implies overaccumulation of capital as stated in Proposition 3. Consumption is minimal because the Kuhn-Tucker condition on consumption in the Planner's problem always binds as long as there is nonsatiation of the HDI objective function. Relative to this model, the inclusion of the multifactor productivity term $\Phi(\frac{c}{c_s})$ does not change the results.

The model suggests other ways in which consumption might be productive. The lack of consumption could reduce the rate of accummulation of human capital. For example, the effective savings rate in the health capital differential equation might be modelled as decreasing when consumption falls short of a threshold c_{ss}. Similarly, the rate of population growth might decrease when consumption falls short of a threshold c_{sss}. In a general model a number of productive consumption margins might be present.[18] However, the absence of a larger literature on productive consumption suggests that the consumption thresholds corresponding to these margins are at quite low levels. Were the planner constrained, for political or other reasons, to not force consumption to the absolute minimal level (e.g. $c \geq \underline{c} \geq c_{ss}, c_{sss}$), human capital overaccumulation obtains as explained above. Below we show that capital overaccumulation obtains when the planner cares about consumption.

5 The HDI Modified with Alternative Proxies for Decent Standard of Living

As we have seen, the planning criterion of HDI maximization leads to problematic outcomes, at least in the steady state. In particular, it implies that consumption would be set to minimal levels. Moreover, it involves the maximization of per

[18] The tiny literature on productive consumption gives little guidance on how to model these margins and we suggest this as an area for future research. Our results would go through unchanged in a general model if multifactor productivity needed the most consumption for production; for example, $c_s > c_{ss}, c_{sss}$ and at the optimum consumption c lies $c_s > c > c_{ss}, c_{sss}$.

If the planner chooses $c < c_{ss}$, then consumption reduces the rate at which health capital is acccumulated, and the planner substitutes towards other forms of capital. If health capital were not valued much and factors are not complements, less health capital might be accumulated. On the other hand, if $c < c_{sss}$ then consumption reduces the rate at population growth n, and steady state capital stocks increase.

capita GNI for its own sake – over other valued ends like health and education – which goes against the philosophy of the human development approach. As such, the indicator variable per capita GNI does not have the intended consequence of proxying the dimension ‘decent standard of living’.

A way forward is suggested in Stiglitz, Sen, and Fitoussi (2009), "Report of the Commission on the Measurement of Economic Performance and Social Progress". Their first recommendation (p12) is to look at household income and consumption rather than production when evaluating material well-being. We find that using per capita disposable income as the indicator variable leads to similar problems as per capita GNI. In constrast, using per capita consumption as the indicator variable yields a “human development golden rule” which efficiently balances expenditures on consumption, education and health.

5.1 Maximizing the HDI modified with Disposable Income Replacing Income in the Index

We define disposable income as income net of expenditures on education and health.[19] Expressed in terms of the intensive variables, per capita disposable income in the steady state is:

$$d \equiv c + (n + \delta)k = y - (n + \delta)(e + h)$$

The objective function, now with disposable income replacing income (as the indicator in the dimension index for decent standard) is:

$$D^d(h(t), e(t), d(t))$$

As before, the objective function is strictly concave. Below, for simplicity, the indicators are assumed to be chosen within their bounds.

[19] Anand and Sen (2000b, p86) report that the indicator for a ‘decent standard of living’ was meant to “reflect something of basic capabilities not already incorporated in measures of longevity and education”. Using disposible income avoids the double counting of education and health in the modified index. In terms of public finance and individual choice, this definition works well when expenditures on education and health are mostly in the public sector whereas saving for physical capital formatoin is a household choice.

In the steady state, the Lagrangian for the planner's problem is:

$$L^d = D^d(h, e, c + (n+\delta)k) - \sigma^d(c - f(k, h, e)\Phi(c/c_s) + (n+\delta)(k + h + e))$$

Again, we can express the first-order conditions in terms of marginal products and marginal rates of substitution:

$$MP_C = 1 - \frac{D_d^d}{\sigma^d} \qquad \Rightarrow \qquad MP_C < 1$$

$$MP_K = MP_C(n+\delta) \qquad \Rightarrow \qquad MP_K < n+\delta$$

$$MP_H = n + \delta - \frac{D_h^d}{D_d^d}(1 - MP_C) \qquad \Rightarrow \qquad MP_H < n+\delta$$

$$MP_E = n + \delta - \frac{D_e^d}{D_d^d}(1 - MP_C) \qquad \Rightarrow \qquad MP_E < n+\delta$$

Since $MP_K > 0$ it follows that $MP_C = MP_K/(n+\delta) > 0$. Thus $\Phi'(c/c_s) > 0$ and $0 < c < c_s$. As $MP_C < 1$, the marginal products are below the breakeven rate $MP_J < n + \delta$ for all $J = H, E, K$. If $\frac{D_h^d}{D_d^d} \gtreqless n + \delta$, then $MP_H \lesseqgtr MP_K$, and if $\frac{D_e^d}{D_d^d} \gtreqless n + \delta$, then $MP_E \lesseqgtr MP_K$. The type of capital overaccumulation depends on the relative sizes of the marginal products as described in Proposition 2(b). For example if $\frac{D_e^d}{D_d^d} = \frac{D_h^d}{D_d^d} = n + \delta$ then $MP_J = MP_C(n+\delta)$ for all J, and there is physical and human capital overaccumulation. If $\frac{D_e^d}{D_d^d} = \frac{D_h^d}{D_d^d} > n + \delta$ then $MP_E = MP_H < MP_K = MP_C(n+\delta)$; there is human capital overaccumulation, and there may or may not be physical capital overaccumulation. Conversely, if $\frac{D_e^d}{D_d^d} = \frac{D_h^d}{D_d^d} < n + \delta$, then $MP_K < MP_E = MP_H < n + \delta$; there is physical human capital overaccumulation, and there may or may not be human capital overaccumulation. The following proposition summarizes.

Proposition 5 *Maximizing the disposable income modified HDI gives minimal consumption* $c < \min[c_s, c^*]$, *and physical and/or human capital overaccummulation depending on the marginal valuation of health and education capital relative to disposible income in the index.*

Modifying the HDI in this way does not alter the minimal consumption conclusion. The planner increases disposable income $d = c + (n+\delta)k$ by increasing k

at the margin rather than by increasing c. However, compared to the original HDI, the priority is no longer necessarily on human capital. This is because disposable income nets out education and health capital, $d = y - (n+\delta)(e+h)$ which are otherwise effectively double counted in the objective function. The next alternative, replacing output with consumption, further avoids double counting by eliminating the accumulation of physical capital for its own sake.

5.2 Maximizing the HDI when Consumption Replaces Income in the Index

Now we replace income in the HDI with consumption. By consumption we mean output less expenditures on all capital investments. Consumption is a better proxy for a "decent standard of living" than disposable income, because it also excludes physical capital investment and hence avoids physical capital overaccumulation. Per capita consumption, in the steady state, is given by:

$$c = y - (n + \delta)(k + h + e)$$

The objective function, now with consumption, is denoted:

$$D^c(h(t), e(t), c(t))$$

As before, the objective function is strictly concave and, outside of the upper bounds, the marginal values D^c_j are zero. We initially assume, for simplicity, that the indicator variables are interior to their bounds so that $D^c_j > 0$ for $j = h, e, c$.

The corresponding Lagrangian is

$$L^c = D^c(h, e, c) - \sigma^c(c - f(k, h, e)\Phi(c/c_s) + (n + \delta)(k + h + e))$$

The first-order conditions with respect to c, k, h, and e, respectively, imply:

$$D^c_c = \sigma^c(1 - f\Phi_c)$$

$$f_k\Phi = (n + \delta)$$

$$D^c_h = \sigma^c((n + \delta) - f_h\Phi)$$

$$D_e^c = \sigma^c((n+\delta) - f\Phi_e)$$

The first-order conditions and the feasibility constraint constitute a system of five equations in five unknowns (c, k,h, e, and σ^c) that describe the optimal policy.

The first-order conditions can be rewritten in terms of marginal products in what we refer to as the *human development golden rule*:

$$MP_C = 1 - \frac{D_c^c}{\sigma^c} \quad \Rightarrow \quad 0 \leq MP_C < 1 \tag{9}$$

$$MP_K = n + \delta \tag{10}$$

$$MP_H = MP_K - \frac{D_h^c}{D_c^c}(1 - MP_C) \quad \Rightarrow \quad MP_H < MP_K \tag{11}$$

$$MP_E = MP_K - \frac{D_e^c}{D_c^c}(1 - MP_C) \quad \Rightarrow \quad MP_E < MP_K \tag{12}$$

For consumption there are two cases to consider. If $0 < c < c_s$, then $0 < MP_C < 1$ and $D_c^c < \sigma^c$. Conversely, if $c \geq c_s$ then $MP_C = 0$ and $D_c^c = \sigma^c$. This latter case prevails when c_s is relatively small relative to output (so that $c_s < c^{**}$) and consumption is valued enough in the objective function. Then at the optimum $c_s \leq f(h, e, k) - (n+\delta)(h+e+k)$. This is the standard case in economics where consumption does not increase marginal productivity but rather is only provided as a direct benefit. Hence, this might be considered the "standard case" for the steady state, except for perhaps the poorest nations.

The condition for the physical capital stock is $MP_K = n+\delta$. As with the HDI, human capital is given priority over physical capital in the sense that $MP_H, MP_E < MP_K = n + \delta$. Thus, if $c < c_s$ we get the same pattern of overaccummulation relative to the consumption golden rule as with the HDI. However, in the standard case $c \geq c_s$, Proposition $2(a)$ requires that $e > e^*$, $h > h^*$ and $k \geq k^*$. Further, if factors are complements, then $k > k^*$ so that physical and human capital are overaccumulated relative to the consumption golden rule.

The marginal rates of substitution between consumption and health and education human capital are:

$$\frac{D_h^c}{D_c^c} = \frac{MP_K - MP_H}{1 - MP_C}, \qquad \frac{D_e^c}{D_c^c} = \frac{MP_K - MP_E}{1 - MP_C}$$

where $MP_K = n+\delta$. These conditions have no analog in the existing HDI analysis. The condition describing the division of human capital takes the same form as for the HDI-maximizing rule. The following proposition summaries.

Proposition 6 *The human development golden rule, equations (9)-(12), efficiently trades off consumption, education and health variables in the human development index* $D^c(h(t), e(t), c(t))$. *Consumption may or may not exceed threshold consumption,* $c \gtreqless c_s$. *The standard case,* $c \geq c_s$, *obtains when* c_s *is small relative to output and consumption is valued enough in the index. Relative to the (consumption) golden rule, consumption is less,* $c < c^*$, *and there is education and/or health capital overaccumulation. In the standard case,* $c \geq c_s$, *there is human and physical capital overaccumulation when factors are complements.*

In interpreting these results there are several points to bear in mind. First, we could have made the comparison the other way around: i.e. the (consumption) golden rule in comparison to the human development golden rule displays excess consumption and underaccumulation of capital, particularly human capital. This fits the tenor of much of the human development literature. Second, consumption in this steady state analysis corresponds to permanent consumption. Consumption should be sustainable. A flow measure which values a temporary increased without taking into account a future decline misses the mark. Third, the consumption variable we use is unusual in excluding expenditures on education and health. Consumption should be valued in a direct trade off with education and health expenditures. Even with these qualifications, the consumption variable is still a crude indirect aggregate measure of the capabilities underlying decent standard of living (see Anand and Sen (2002b)).

The standard case in Proposition 6 has $c \geq c_s$. In contrast to the other human development maximizing rules, consumption is not productive at the margin. Nevertheless, in the standard case, there is capital overacccumulation. This demonstrates capital overaccumulation is not an artifact of productive consumption. Rather overaccumulation of all types of capital is unambiguously present because of the objective function $D^c(h(t), e(t), c(t))$. Increasing the relative weight on consumption in the objective results in an increase in consumption and a decline in capital stocks. In the extreme case, where the planner preference shifts entirely to consumption, yields $c \rightarrow c^{**}$ and the (consumption) golden rule.

5.2.1 Bounds and Technological Change

Introducing exogenous technological change into the analysis (as in Section 4.3) does not change the qualitative results. When $c_s(t)$ grows at the growth rate g then $\widehat{c}_s(t) = \widehat{c}_s$, the analysis is parallel to the one given above: there are two cases – one where $0 < \widehat{c} < \widehat{c}_s$ and the standard case where $\widehat{c} \geq \widehat{c}_s$.[20] If $c_s(t)$ grows at rate less than g, the standard case $\widehat{c} > \widehat{c}_s(t)$ obtains in the steady state. This is because consumption $\widehat{c}(t)$ is directly valued in the objective function, and the planner will eventually find it feasible and desirable to provide consumption above the threshold.

As before, in the presence of technological growth, the bounds must also grow at rate g for there to be a steady state with the indicator variables interior to the bounds. If, for example, the bounds on education and health grow at a rate less than g, then eventually the planner achieves these upper bounds and per capita consumption remains the only indicator variable within the bounds. In this case, the planner maximizes steady state per capita consumption. This implies a (consumption) golden rule with exogenous growth as described in Section 4.4.

6 Conclusion

In this paper we have taken the unusual methodological approach of evaluating a well-known overall achievement index, the human development index (HDI), by examining the optimal dynamic plans it implies. Our motivation for maximizing the HDI has been both positive and normative, as encapsulated in the opening quotations. We believe this method has been quite revealing in uncovering unintended consequences of using the index as a guide for development planning. The optimal plans for the HDI imply minimal consumption, human capital overaccumulation and possible physical capital overaccumulation. This motivated us to try to modify the index in a way that respected the philosophy behind the HDI. The modified index with per capita consumption replacing per capita GNI yields a "human development golden rule" which balances the ends of health, education and a 'decent standard of living'.

[20] If $c_s(t)$ grows at rate less than g, the standard case $\widehat{c} \geq \widehat{c}_s$ obtains. When consumption $\widehat{c}(t)$ is directly valued in the objective, the planner will eventually find it feasible and desirable to provide consumption above the threshold.

There has been a great deal of work critiquing the use of aggregate production measures for evaluating material well-being. For example, Recommendation 1 in the Stiglitz, Sen, and Fitoussi's (2009) Report is: "When evaluating material well-being, look at income and consumption rather than production."[21] In this paper, we find that per capita consumption is an appropriate measure for balancing material well-being with other dimensions of well-being. On the other hand, we find that the use of per capita income and per capita disposable income can lead to perverse outcomes – both lead to minimal consumption and capital overaccumulation.

The method that we have used here evaluates a criterion by examining its implied economic outcomes – effectively using a macroeconomic model as a laboratory to test the implications of the criterion. This method can be applied to other indexes and issues. For example, Engineer, King and Roy (2008) compare the pre-2010 versions of the HDI and Gender Development Index in a static model.[22] They find plausible assumptions under which maximizing both indexes yield the same optimal plan, despite the gender index treating the sexes asymmetrically and being sensitive to inequality. It would be interesting to pursue a number of questions: Under what circumstances would maximizing the HDI yield high achievement as measured by other dimensions or indexes (e.g. Inequality-adjusted HDI, Multidimensional Poverty Index)? Does adding other dimensions or indexes to the HDI substantially change development plans?[23] What new dimensions and principles (e.g. in Stiglitz, Sen, and Fitoussi (2009)) should be considered in improving the HDI? When are differences in country rankings by HDI and per capita GNI useful in determining which countries are more efficiently pursuing development?

The analysis of intertemporal planning with multi-dimensional objectives is

[21] Easterlin (2010) finds that Report main contribution is its compelling arguments for using a measure of overall well-being which include non-economic components. In reviewing the Report, Frank (2010) stresses that that a society that uses per capita GDP as the sole measure of progress will tilt its policies toward promoting economic growth at the expense of other things known to promote well-being.

[22] As far as we are aware, the only other paper to take this methodological approach is Bourguignon and Fields (1990). They minimize poverty indices subject to redistribution constraints. The implied policies can differ dramatically among indexes. Though it appears that many theorist understand that maximizing output yields zero consumption, this point is not explicitly mentioned in any in policy analyzes (e.g. Stiglitz et. al. (2009)) we could find.

[23] Jones and Klenow (2010), for example, propose an index based on consumption, leisure, inequality, and mortality.

inherently complex. This is particularly so when there are state variables in the objective function which feedback to production. We have explored the implications of the optimal dynamic plan in an extension of the simplest well-known dynamic model, the Solow model, to get a feel for the issues. Our analysis has exclusively concentrated on the steady state. Steady state analysis is straightforward and provides relatively simple and unambiguous conditions which can be compared with a classic benchmark, the golden rule. Arguably, steady state analysis is an appropriate counterpart for an index which has no intertemporal dimension because variables are constant over time. Also, Anand and Sen (2000a) argue that the sustainability of human development should be a primary value, rather than having it developed from welfarist criteria.[24] In this light, the indicators in the HDI should be modified to capture long run averages that can be sustained.

Nevertheless, in growth analysis there is perhaps an overreliance on interpreting steady states as analogues of the long run and ignoring transitional effects. Particularly, in modelling development issues, analysis of transition paths seems more appropriate. Indeed, it might be reasonably argued that human development should be thought of as a process of transition growth to a developed state. Our analysis is more consistent with the view that there is no end to human development – where human development is an ongoing expansion of peoples' abilities to make choices. Optimal transition analysis could include a more sophisticated objective with discounting. We would be surprised if the results of our steady state analysis – minimal consumption, and capital overaccumulation – did not obtain on the transition path. Still transition analysis would generate new results related to other dynamic issues, such as speed of adjustment. Second-best considerations and individual incentive and participation constraints are likely more threatening to the results in this paper. They suggest a new field of dynamic public finance for human development.

Another issue, which we believe would be interesting to explore, would be the

[24] As in the analysis of the golden rule in the Solow model, our analysis is not welfarist in the sense of maximizing a welfare function derived from individual agents' utility functions. Such models may not have an optimal path that is a steady state. Here we concentrate on steady states and so our analysis implicitly has sustainable plans.

Anand and Sen (2000a) show that sustainable plans are not necessarily optimal in terms of a welfarist criterion. They argue for the normative primacy of sustainable plans. Our paper can be thought of as extending their work to optimal sustainable human development plans. See Pessy (1992) for an evaluation of sustainable development concepts.

tournament aspect implied by the quotation from Inge Kaul, in the beginning of this paper. In particular, the analysis of tournaments in, for example, Lazear and Rosen (1981) and Green and Stokey (1983) could shed light on the relationship between maximizing a country's rank in the HDI and the actual value of the HDI itself.

The method advanced in this paper assesses the usefulness of an achievement measure in terms of its policy implications. The rigor of maximization subject to feasibility constraints is a check for evaluating welfarist and non-welfarist indexes. This methodology makes policy trade-offs explicit and reveals the effective goals implicit in taking an index seriously. Though this methodology has revealed a critique of the current version of the HDI, it has also suggested a simple remedy. Similarly, we believe other measures can be improved. Making explicit connections from achievement measures to desireable policy outcomes should give policy makers more confidence in seriously pursuing policies towards maximizing human development.

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7 Appendix: Proof of Proposition 2

The endogenous change in capital allocations is derived by exogenously changing the marginal products. Recall the capital marginal products can be expressed: $MP_K = \Phi f_k$, $MP_H = \Phi f_h$, and $MP_E = \Phi f_e$. Taking total derivatives and substituting for dc from (8′) yields:

$$\begin{matrix} dMP_K \\ dMP_H \\ dMP_E \end{matrix} = \begin{matrix} \Phi f_{kk} + \Phi_c f_k \gamma_k/\zeta & \Phi f_{kh} + \Phi_c f_k \gamma_h/\zeta & \Phi f_{ke} + \Phi_c f_k \gamma_e/\zeta \\ \Phi f_{hk} + \Phi_c f_h \gamma_k/\zeta & \Phi f_{hh} + \Phi_c f_h \gamma_h/\zeta & \Phi f_{he} + \Phi_c f_h \gamma_e/\zeta \\ \Phi f_{ek} + \Phi_c f_e \gamma_k/\zeta & \Phi f_{eh} + \Phi_c f_e \gamma_h/\zeta & \Phi f_{ee} + \Phi_c f_e \gamma_e/\zeta \end{matrix} \begin{matrix} dk \\ dh \\ de \end{matrix}$$

where $\zeta \equiv 1 - MP_C > 0$, $\gamma_k \equiv MP_K - (n+\delta) \leq 0$, $\gamma_h \equiv MP_H - (n+\delta) \leq 0$, $\gamma_e \equiv MP_E - (n+\delta) \leq 0$. The Hessian of this system is negative and can be expressed:

$$\begin{aligned} |\Psi| &= \Phi^3 |f| + \Phi^2 \Phi_c \zeta^{-1} [(f_k A^e + f_h B^e + f_e C^e)\gamma_e \\ &\qquad + \left(f_k A^k + f_h C^k + f_e B^k\right)\gamma_k + \left(f_k B^h + f_h A^h + f_e C^h\right)\gamma_h] \end{aligned}$$

where $A^k \equiv f_{ee} f_{hh} - f_{he} f_{eh} > 0$, $B^k \equiv f_{he} f_{kh} - f_{ke} f_{hh} \geq 0$, $C^k \equiv f_{eh} f_{ke} - f_{kh} f_{ee} \geq 0$, $A^e \equiv f_{kk} f_{hh} - f_{hk} f_{kh} > 0$, $B^e \equiv f_{hk} f_{eh} - f_{hh} f_{eh} \geq 0$, $C^e \equiv f_{ek} f_{kh} - f_{eh} f_{kk} \geq 0$, $A^h \equiv f_{kk} f_{ee} - f_{ke} f_{ek} > 0$, $B^h \equiv f_{ek} f_{he} - f_{hk} f_{ee} \geq 0$, $C^h \equiv f_{ke} f_{hk} - f_{kk} f_{he} \geq 0$. Here $|f| = f_{ee} A^e + f_{ke} B^e + f_{he} C^e < 0$ is the determinant of the Hessian of the strictly concave intensive production function f.

We now derive de, dh and dk from exogenous changes in dMP_E, dMP_H and dMP_K. First consider $dMP_E < 0$ while holding $dMP_H = dMP_K = 0$. Solving for de yields:

$$de = \frac{dMP_E}{|\Psi|} \left[\Phi^2 A^e + \Phi \Phi_c \zeta^{-1} \left((f_{kk} f_h - f_{kh} f_k)\gamma_h + (f_{hh} f_k - f_{kh} f_h)\gamma_k\right)\right]$$

The own effect of decreasing the marginal product is positive; i.e. $dMP_E < 0$

implies $de > 0$, since $A^e > 0$. The cross effects on dk and dh from $dMP_E < 0$ are:

$$\begin{aligned} dk &= \frac{dMP_E}{|\Psi|}\left[\Phi^2 B^k + \Phi\Phi_c\zeta^{-1}((f_{kh}f_k - f_{hh}f_k)\gamma_e + (f_{he}f_k - f_{ke}f_h)\gamma_h)\right] \\ dh &= \frac{dMP_E}{|\Psi|}\left[\Phi^2 C^h + \Phi\Phi_c\zeta^{-1}((f_{hk}f_k - f_{kk}f_h)\gamma_e + (f_{ke}f_h - f_{he}f_k)\gamma_k)\right] \end{aligned}$$

If capital inputs are not complements, then $dk = dh = 0$. When capital factors are complements, then $B^k > 0$ and $C^h > 0$. In the traditional case $\Phi_c = 0$, so $dk > 0$ and $dh > 0$.Evaluating around the golden rule, $\gamma_j = 0$ for all j, also implies $dk > 0$ and $dh > 0$.Analysis of changes dMP_H and dMP_K are symmetric. This establishes part (a) of the proposition.

Part (b) examines productive consumption, $\Phi_c > 0$.To prove $b(i)$, that $j > j^*$ for all j if $MP_H = MP_E = MP_K < n + \delta$, we show that $dj > 0$ for all j when the marginal products are symmetrically lowered $dMP_H = dMP_E = dMP_K < 0$ such that $\gamma_e = \gamma_h = \gamma_k < 0$. Note that $\gamma_e = \gamma_h = \gamma_k$ requires $f_e = f_h = f_k$. First consider the effect of $dMP_H = dMP_E = dMP_K < 0$ on de. The effect for dMP_E is given above. Changes induced by dMP_H and dMP_K are:

$$\begin{aligned} &\frac{dMP_H}{|\Psi|}\left[\Phi^2 C^e + \Phi\Phi_c\zeta^{-1}((f_{ek}f_k - f_{kk}f_e)\gamma_h + (f_{kh}f_e - f_{eh}f_k)\gamma_k)\right] \\ &\frac{dMP_K}{|\Psi|}\left[\Phi^2 B^k + \Phi\Phi_c\zeta^{-1}((f_{hk}f_e - f_{ek}f_h)\gamma_h + (f_{eh}f_h - f_{hh}f_e)\gamma_k)\right] \end{aligned}$$

The total effect on de reduces to $de = \frac{dMP_E}{|\Psi|}\Phi^2[A^e + B^k + C^e] > 0$. The analysis of dk and dh are similar. Thus, $j > j^*$ for all j when $\gamma_h = \gamma_e = \gamma_k < 0$.

Consider (ii): $e > e^*$ and $h > h^*$ if $MP_H = MP_E < MP_K \le n + \delta$. Thus, $\gamma_h = \gamma_e = \gamma_k \le 0$. We first show that $de > 0$ when the marginal products MP_H and MP_E are symmetrically lowered, $dMP_H = dMP_E < 0$. The total effect de is just the sum of the terms corresponding to dMP_H and dMP_E expressions given above.

$$de = \frac{dMP_E}{|\Psi|}\Phi^2[A^e + C^e + \frac{\Phi_c}{\Phi\zeta}f_k((f_{ke} - f_{kh})\gamma_h + (f_{hh} - f_{eh})\gamma_k)]$$

Here $A^e > 0$ and all the other terms imply a non-negative effect except for f_{ke}.

Similarly, the total effect on dh is:

$$dh = \frac{dMP_E}{|\Psi|}\Phi^2[A^h + C^h + \frac{\Phi_c}{\Phi\zeta} f_k((f_{kh} - f_{ke})\gamma_e + (f_{ee} - f_{he})\gamma_k]$$

where $A^h > 0$ and all the other terms imply a non-negative effect except for f_{kh}. Both de and dh are positive when f_{ke} and f_{kh} are sufficiently small or when they largely negate each other so that $|f_{ke} - f_{kh}|$ is sufficiently small. The assumption of 'somewhat symmetry weak complements' made in the text corresponds to $|f_{ke} - f_{kh}| = |F_{KE} - F_{KH}| /AL$ being sufficiently small. This assumption is satisfied by Cobb-Douglas production when $MP_H \approx MP_E$. (Without this assumption, we can prove that the sum of the human capital changes is positive, $de + dh > 0$, when $dMP_H = dMP_E < 0$.)

To prove (ii) consider starting from the situation like (i) where $\gamma_e = \gamma_h = \gamma_k \leq 0$, so that $j \geq j^*$. From this point a marginal decrease $dMP_e = dMP_h < 0$ results in $e > e^*$ and $h > h^*$. Any further symmetric decrease in the marginal products of human capital such that $\gamma_e = \gamma_h < \gamma_k$ results in a further increase in e and h. (Without the somewhat symmetric weak complement assumption, it can be shown that the sum $e + h$ increases and $e + h > e^* + h^*$.) In case (ii) $k < k^*$ is possible. Suppose $\gamma_k = 0$. Then $f_k = \frac{n+g}{\Phi} > n+g$ and if factors are not complements $k < k^*$.

To prove (iii) consider starting from the situation like (i) where $\gamma_e = \gamma_h = \gamma_k \leq 0$, so that $j \geq j^*$. The own effect of $dMP_e < 0$ is an increase in e and $e > e^*$. A further decrease in MP_e such that $\gamma_e < \gamma_h = \gamma_k$ results in a further increase in e. Now start from (ii) where $\gamma_e = \gamma_h < \gamma_k \leq 0$. Again $dMP_e < 0$ results in an increase in e and $e > e^*$. A further decrease in MP_e such that $\gamma_e < \gamma_h < \gamma_k$ results in a further increase in e and $e > e^*$. Under somewhat symmetric complements, the above argument equally applies to k and h. Thus, $j > j^*$ for the factor with the lowest marginal product. ■