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HIERARCHICAL SHRINKAGE PRIORS FOR DYNAMIC REGRESSIONS WITH MANY PREDICTORS

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Hierarchical shrinkage priors for dynamic regressions with many predictors

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Abstract

This paper builds on a simple unified representation of shrinkage Bayes estimators based on hierarchical Normal-Gamma priors. Various popular penalized least squares estimators for shrinkage and selection in regression models can be recovered using this single hierarchical Bayes formulation. Using 129 U.S. macroeconomic quarterly variables for the period 1959 – 2010 I exhaustively evaluate the forecasting properties of Bayesian shrinkage in regressions with many predictors. Results show that for particular data series hierarchical shrinkage dominates factor model forecasts, and hence it becomes a valuable addition to existing methods for handling large dimensional data.

Keywords: Forecasting; shrinkage; factor model; variable selection; Bayesian LASSO

JEL Classification: C11, C22, C52, C53, C63, E37

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I Introduction

Responding to the vastly increasing need of applied economists in business and government for accurate economic forecasts using as much information as possible, academic econometricians have recently devoted significant effort to develop and test various methods for handling large macroeconomic and financial datasets. For many years, the dynamic factor model of Geweke (1977) has been used to successfully address the problem of summarizing datasets with hundreds of variables. In that respect, Stock and Watson (2002a,b), among many others, show that estimating dynamic factors, or just extracting principal components, can improve forecasts over simple ARMA models and also more complicated nonlinear time series models. In the last few years, many new statistical methods have emerged that do not explicitly summarize all the information in large datasets. Rather, they shrink their dimension by reducing or completely removing the impact of irrelevant predictors. These methods include statistical algorithms adopted in econometric forecasting, such as bagging (Inoue and Kilian, 2008), least absolute shrinkage and selection operator (LASSO) (De Mol, Giannone, and Reichlin, 2008), boosting (Bai and Ng, 2007), Bayesian model averaging (Koop and Potter, 2004) and dynamic model averaging (Koop and Korobilis, 2009).

More recently, Stock and Watson (2011) provide a flexible shrinkage representation of dynamic regression models with many orthogonal predictors. Their results are encouraging because they show that a global representation of many shrinkage estimators is possible, including pretest methods, Bayesian model averaging, empirical Bayes, and bagging. Their contribution is twofold since the theoretical properties of various shrinkage methods presented in previous literature depend on the specific modelling assumptions made, and empirical differences in the performance of these shrinkage methods rely on the data and implementation details of each study. Similarly, De Mol, Giannone, and Reichlin (2008) compare in one integrated setting the shrinkage and model selection properties of Bayesian LASSO and ridge regression estimators as opposed to principal component shrinkage. Both sets of authors identify that there is large potential in forecasting performance by shrinking the coefficients of a large number of predictors.

From a Bayesian point of view, the idea of a unified approach to shrinkage is not new. Long ago Bayes and Empirical Bayes priors which lead to shrinkage posterior estimators have been used successfully, with probably the most notable example in economics being the Minnesota prior for vector autoregressions of Litterman (1979) and the g-prior of Zellner. Empirical Bayes estimators in particular depend on a few hyperparameters which control the amount of shrinkage of each regression coefficient based on some information in the data sample. Additionally, early research identified the connection of Empirical Bayes methods with admissible estimators which dominate unrestricted simple least squares, like the Bayesian variant of the James-Stein estimator (see the results of Efron and Morris, 1975). Nowadays, modern Markov Chain Monte Carlo (MCMC) simulation methods can be used to consider a larger set of regularization ill-posed regression problems. In particular, the stochastic form of MCMC methods can be used to provide *adaptive* shrinkage and recover many estimators which may dominate least squares in a mean-square error sense.

In this paper I provide a quite general representation of Bayesian variants of penalized regression estimators. I show that by using hierarchical Normal-Gamma priors many popular estimators can be recovered such as the LASSO (Tibshirani,1996) , fused LASSO (Tibshirani et al., 2005) and Elastic Net (Zou and Hastie, 2005). These priors are straightforward extensions of the typical Bayesian ridge regression priors used in regression models (Koop, 2003), and simple posterior expressions are available. In fact, following Kyung et al. (2010) the Bayes *posterior mode* of regression coefficients β in the typical regression problem $y = \beta x + \varepsilon$ under a hierarchical Normal-Gamma prior admits a single general form which corresponds to the solution to a "generic" penalized regression problem of the form

$$\widetilde{\beta}^{BAYES} = \arg\min_{\beta} \left\{ \|y - \beta x\|_{\ell_2} + \lambda_1 \|h_1(\beta)\|_{\ell_1} + \lambda_2 \|h_2(\beta)\|_{\ell_q} \right\},$$

where $\|\cdot\|_{\ell_p}$ denotes the ℓ_p-norm. This representation restricts the least squares estimator by adding two penalty terms $\lambda_1 \|h_1(\beta)\|_{\ell_1}$ and $\lambda_2 \|h_2(\beta)\|_{\ell_q}$. Various specifications of the Normal-Gamma prior correspond to specific choices of $h_1(\cdot)$, $h_2(\cdot)$ and q, as well as the use of one or two regularization parameters λ_1, λ_2. The benefit of a Bayesian approach using MCMC is that it is trivial to treat uncertainty about the regularization parameters λ_1, λ_2, as well as relax the assumption of using the same amount of shrinkage for each regression coefficient ("adaptive shrinkage") to obtain the oracle property (Zou, 2006).

The main goal of this paper is to empirically examine the shrinkage performance of the Normal-Gamma Bayes estimators using a data set with 129 quarterly macroeconomic and financial time series. For that reason I focus on five special cases of shrinkage estimators from the Normal-Gamma family and I set near improper (uninformative) priors on the regularization parameters λ_1, λ_2 as a default automatic choice. As Park and Casella (2008) note, scale invariance is not a compelling criterion for these parameters because they are unitless. However the purpose of this paper is to examine from a "practitioner's point of view" if such automatic uninformative prior choices make sense for macroeconomic forecasting. In that case the frequentist econometrician can view hierarchical Bayes shrinkage as a pragmatic device and a useful tool for statistical inference (see for example the popularity of Bayesian model averaging in macroeconomic forecasting; Koop and Potter, 2004). The paper concludes with an application of shrinkage on forecasting GDP using many predictors focusing only on the Bayesian LASSO estimator. In this case I also perform a sensitivity analysis and compare the uninformative priors on the regularization parameters with some informative values (selected "subjectively"), as well as a semi-automatic method to estimate the regularization parameters based on marginal maximum likelihood.

The paper is organized as follows. Section 2 describes the econometric methodology: the general dynamic regression problem with many predictors; a unified shrinkage representation of Bayes estimators; their tuning; and how they compare with traditional shrinkage. Section 3 reports the results from the out-of-sample exercise for five special shrinkage estimators applied on 129 series. This section concludes with a sensitivity analysis. Section 4 concludes and provides an assessment of the empirical value of hierarchical shrinkage priors.

II Bayes shrinkage formulations for dynamic regressions

In this paper I consider univariate forecasting models of the form

$$y_{t+h} = \alpha z_t + \beta x_t + \varepsilon_{t+h}, \tag{1}$$

where ε_{t+h} is the h-quarters ahead forecasts error distributed $\varepsilon_t \overset{iid}{\sim} N\left(0, \sigma^2\right)$, for $t = 1, ..., T$. In this type of regressions y_{t+h} is the h-quarters ahead value of the variable of interest, z_t is the $q \times 1$ vector of unrestricted predictors always included in the forecasting model, like the intercept, dummies and lags of the dependent variables, and x_t is the $p \times 1$ vector of *many* (say $p \to \infty$ or p grows at a faster rate than T) *standardized* exogenous predictors whose dimension we would like to shrink.

The unrestricted coefficients α and the variance σ^2 can be integrated out using the uninformative priors $\pi(\alpha) \propto 1$ and $\pi\left(\sigma^2\right) \propto 1/\sigma^2$ respectively. This allows closer focus on the regression coefficient vector β which has individual elements β_j, $j = 1, ..., p$.

II.1 Classical shrinkage

A noninformative prior, like the one assigned to the coefficients a, leads to a Bayes estimator centered at the unrestricted LS quantities. This choice would obviously pose a problem for estimating the "large" number of coefficients β, especially when $p > T$. Traditionally, Normal priors of the form

$$\pi(\beta) \sim N(0, V), \tag{2}$$

have been used, because they are conjugate to the likelihood and allow easy calculations of the Bayes posterior. The $p \times p$ matrix V is the prior covariance matrix of the regression coefficients which we want to elicit for this "large p" problem. For instance, a common choice is the case $V = \tau^2 I_p$ which leads to the classical *ridge regression* shrinkage. Ignoring for now the effect of the regressors z_t, this ridge regression prior implies the penalized least squares representation

$$\beta = \left(X'X + \frac{1}{\tau^2} I_p\right)^{-1} X'Y$$

where $X = \left(x_1', ..., x_T'\right)'$ and $Y = \left(y_{1+h}', ..., y_{T+h}'\right)'$ .The dependence of all parameters β_j, $j = 1, ..., p$, on the unknown parameter τ^2 can reduce the risk over the traditional LS estimator. For $\tau \to \infty$ we can see that $\beta = \left(X'X\right)^{-1} X'Y = \beta^{LS}$.

Following a different path, Judge and Bock (1978) suggested an Empirical Bayes (i.e. data-based) estimator of V, of the form $V = \tau^2 \left(X'X\right)^{-1}$ where $\tau^2 = \frac{\widehat{\sigma}^2}{\widehat{\xi}^2}$, and

$$\begin{aligned}
\widehat{\sigma}^2 &= \left(Y - X\beta^{LS\prime}\right)' \left(Y - X\beta^{LS\prime}\right)/T \\
\widehat{\xi}^2 &= \frac{\beta^{LS\prime}\beta^{LS}}{tr\left(X'X\right)^{-1}} - \widehat{\sigma}^2
\end{aligned}$$

This empirical Bayes rule is Stein-like, shrinking β^{LS} towards 0, since the posterior mean writes

$$\widetilde{\beta} = \left(1 - \frac{\widehat{\sigma}^2}{\widehat{\sigma}^2 + \widehat{\xi}^2}\right) \beta^{LS}.$$

Nowadays, priors of the form $V = \sigma^2 \tau^2 \left(X'X\right)^{-1}$, which are called g-priors or Zellner's prior (Zellner, 1986), tend to be very popular in economics; see for instance Koop and Potter (2004) and references therein. Over the course of the years there have been many connections between values of τ^2 and information criteria; see for example Fernandez, Ley and Steel (2001) for a review. Nevertheless, Zellner originally proposed this prior to provide a shrinkage representation since the posterior mean writes

$$\beta = \frac{\tau^2}{1+\tau^2}\beta^{LS}$$

This formulation implies a shrinkage factor $\delta = \tau^2/\left(1+\tau^2\right)$ which regulates the proportion (0%-100%) of shrinkage over the unrestricted OLS estimator. Note that priors which are data-based have desirable shrinkage properties, compared to the weak shrinkage of a ridge-regression prior.

We can immediately observe that these two typical examples of Bayesian shrinkage have undesirable properties for very demanding problems with many predictors. The ridge regression prior is based on a global shrinkage parameter τ^2 for all p regressors. In sparse regression problems, i.e. when p is very large and we expect that only a tiny proportion of regressors are relevant for prediction, weighting a-priori all regression coefficients by the same factor τ^2 is guaranteed not to work well. Empirical Bayes priors partly solve this problem since τ^2 is scaled by the Information Matrix, giving a varying degree of prior weight to each regression coefficient based on the information in the likelihood. Nevertheless, the Information Matrix cannot be estimated precisely (for large p), or cannot be estimated at all (for $p > T$)[1]. Thus, the next subsection develops on shrinkage representations which are automatic (i.e. they allow minimal input by the researcher about the expected shrinkage factor) and can be applied in sparse regressions within the "large p, small T" paradigm.

II.2 Full Bayes (hierarchical) priors for adaptive shrinkage

Modern computational methods allow to estimate the parameter(s) in the prior covariance matrix V in a formal way, by placing hyper-prior distributions on these parameters. Moreover, adding an extra layer of hierarchy on the prior covariance matrix (and hence treating this matrix as a parameter to be estimated from the likelihood) allows to implement many popular formulations of adaptive shrinkage. For that reason, the prior covariance matrix on the coefficients β is formulated as $V = diag\left\{\tau_1^2, ..., \tau_p^2\right\}$. This formulation allows the individual elements τ_j^2, $j = 1, ..., p$, to be *independently* updated towards 0 which eventually results in shrinkage of the coefficient β_j to a point mass at zero. All hierarchical priors presented below are special cases of a Normal-Gamma prior, i.e. a Normal prior for β, and a Gamma prior for τ_j^2 of the form

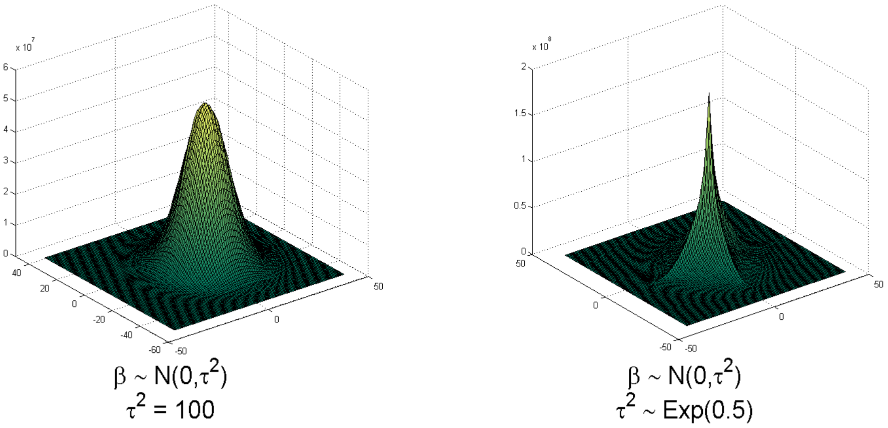
$$\begin{aligned} \pi\left(\beta|\tau^2\right) &\sim N_p\left(0, V\right) \\ \pi\left(\tau_j^2\right) &\sim Gamma\left(a, b\right) \end{aligned}$$

This formulation is very flexible and nests many cases used previously in the shrinkage literature. Given the properties of the Gamma distribution I will give special attention to the cases

1. $\tau_j^2 \sim Gamma\left(a = 0^+, b = 0^+\right)$ which is equivalent to $\log\left(\tau_j^2\right) \sim Uniform^* \left[0, +\infty\right)$

2. $1/\tau_j^2 \sim Gamma\left(a = \rho, b = \xi\right)$ which is equivalent to $\tau_j^2 \sim iGamma\left(\rho, \xi\right)$, and

[1] It is only recently that Maruyama and George (2010) derived a particular decomposition of Zellner's g-prior that can be used when more predictors than observations are present.



$\beta \sim N(0,\tau^2)$
$\tau^2 = 100$

$\beta \sim N(0,\tau^2)$
$\tau^2 \sim Exp(0.5)$

Figure 1: The left panel of the figure plots the ridge regression prior which is informative on the support of the parameters β. For large τ, this prior is locally uninformative (flat) around zero and has no shrinkage properties. The right panel shows how the shrinkage towards zero is achieved using the Laplace prior.

3. $\tau_j^2 \sim Gamma\left(a = 1, b = \frac{2}{\lambda^2}\right)$ which is equivalent to $\tau_j^2 \sim Exponential\left(\frac{\lambda^2}{2}\right)$

where $Uniform^*$ denotes the unnormalized (and hence improper) Uniform distribution, and $iGamma$ is the inverse Gamma distribution.

These hierarchical priors basically transform the typical independent Normal prior in (2) into a scale mixture of Normals prior. These examples can basically be generalized to even more cases. However, for the specific choices made above, various known distributions can be recovered. For instance, case 2 (the Normal-inverse Gamma density) is a mixture representation for the t-density (Geweke, 1993), while case 3 (the Normal-Exponential density, also called "double-exponential" density) is a mixture representation of the Laplace density. Figure 1 shows intuitively why a mixture prior (right panel) is to be preferred over the traditional ridge regression prior (left panel). While the ridge regression prior is informative on the support of the parameters (it is bell-shaped, as opposed to being completely flat like a uniform prior), it is locally uninformative in a neighborhood zero (the point of shrinkage). In contrast, the Laplace prior (which is constructed in such a way that it has similar support to the ridge regression prior) provides faster rates of shrinkage in the neighborhood of zero.

Adaptive shrinkage Jeffreys' prior Hobert and Casella (1996) studied first the shrinkage properties of the Jeffreys' prior on the covariance matrix of the regression coefficients. One can think of Jeffreys' prior as the simplest, default choice because it is not dependent upon further hyperparameters.

Let $V = diag\left\{\tau_1^2, ..., \tau_p^2\right\}$, then the scale-invariant, improper Jeffrey's (hyper-)prior on each τ_j^2 takes the form

$$\pi\left(\tau_j^2\right) \sim 1/\tau_j^2, \text{ for } j = 1, ..., p \tag{3}$$

Adaptive shrinkage t-priors For a covariance matrix $V = diag\left\{\tau_1^2, ..., \tau_p^2\right\}$ we can consider a specific form of a Gamma prior on τ_j^2, $j = 1, ..., p$, i.e. the inverse Gamma prior. Following Geweke (1993) we can show that this Normal-inverse Gamma mixture prior is equivalent to a Student-t prior on β. The t-density has heavy tails and is more leptokurtic around the origin, but in general is much "smoother" than the Laplace density plotted in panel B of Figure 1. The priors on τ_j^2 are of the form

$$\pi\left(\tau_j^2\right) \sim iGamma\left(\rho, \xi\right), \text{for } j = 1, ..., p \tag{4}$$

where ρ is the *shape* parameter and ξ the *scale* parameter of the inverse Gamma density; see also Armagan and Zaretzki (2010). Once the τ_j^2's are integrated out from the joint posterior, this prior is analogous to the regularized least squares problem which solves (ignoring once again the regressors z_t for simplicity)

$$\arg\min_{\beta} \frac{1}{2\sigma^2} \sum_{t=1}^{T} \left(y_{t+h} - x_t\beta\right)^2 + \left(\rho + \frac{1}{2}\right) \sum_{j=1}^{p} \log\left(2\xi + \beta_j\right)$$

Finally, notice that this formula also applies for the Jeffrey's prior case (for $\rho, \xi \to 0$).

Hierarchical LASSO Tibshirani (1996) proposed the Lasso algorithm which can be viewed as a L_1-penalized least squares estimate which solves

$$\arg\min_{\beta} \frac{1}{2\sigma^2} \sum_{t=1}^{T} \left(y_{t+h} - x_t\beta\right)^2 + \lambda \sum_{j=1}^{p} \left|\beta_j\right|$$

Tibshirani (1996) also noted that this form of penalty is equivalent to the posterior mode of the Bayes estimate under the Laplace prior

$$\pi\left(\beta|\sigma^2\right) \sim \prod\nolimits_{j=1}^{p} \frac{\lambda}{2\sqrt{\sigma^2}} e^{-\frac{\lambda}{\sqrt{\sigma^2}}\left|\beta_j\right|}$$

One can take advantage of the fact that the Laplace density can be written as a scaled mixture of Normals (see Park and Casella, 2008). Notice that the formulation above implies that for the Bayesian LASSO prior (as well as the Fused LASSO and the Elastic Net) we need to condition on the error variance σ^2. Park and Casella (2008) underline that this conditioning ensures that the posterior of the regression coefficients β is unimodal, otherwise expensive simulation methods would be needed to handle multimodal posteriors (for instance, simulated tempering). Subsequently, assume for this case a diagonal prior covariance matrix of the form $V = \sigma^2 \times diag\left\{\tau_1^2, ..., \tau_p^2\right\}$. The hierarchical version of the LASSO uses a normal prior for β of the form in eq. (2) augmented with the hyperprior

$$\pi\left(\tau_j^2\right) \sim Exponential\left(\frac{\lambda^2}{2}\right), \text{for } j = 1, ..., p \tag{5}$$

where λ is a hyperparameter, which is the *rate* parameter of the Exponential distribution.

Hierarchical Fused LASSO The Fused LASSO was proposed by Tibshirani et al. (2005) as a means to account for a possible meaningful ordering of variables[2]. Thus, this estimator penalizes the L_1-norm of both the coefficients and their differences

$$\arg\min_{\beta} \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(y_{t+h}-x_t\beta\right)^2+\lambda_1\sum_{j=1}^{p}\left|\beta_j\right|+\lambda_2\sum_{j=1}^{p-1}\left|\beta_{j+1}-\beta_j\right|$$

The representation of the Bayesian prior for β in the penalized regression using the Fused LASSO is

$$\pi\left(\beta|\sigma^2\right)\sim e^{-\frac{\lambda_1}{\sigma}\sum_{j=1}^{p}\left|\beta_j\right|-\frac{\lambda_2}{\sigma}\sum_{j=1}^{p-1}\left|\beta_{j+1}-\beta_j\right|}$$

Kyung et al. (2010) show that the hierarchical representation of this prior is

$$\pi\left(\tau_j^2\right) \sim Exponential\left(\frac{\lambda_1^2}{2}\right), \text{ for } j=1,...,p \tag{6a}$$

$$\pi\left(\omega_j^2\right) \sim Exponential\left(\frac{\lambda_2^2}{2}\right), \text{ for } j=1,...,p-1 \tag{6b}$$

where the correlation between β_{j+1} and β_j enters through the prior covariance matrix V. In this case V is a tridiagonal matrix with main diagonal $\left\{\tau_i^2+\omega_{i-1}^2+\omega_i^2\right\}$ for $i=1,..,p$ and off-diagonal elements $\left\{-\omega_i^2\right\}$, and for simplicity we can set $\omega_0=\omega_p=0$.

Hierarchical Elastic Net Zou and Hastie (2005) proposed the Elastic Net as a more stabilized version of the LASSO that also allows grouping effects and is particularly useful when $p>T$. The Elastic Net estimator is the solution to the minimization problem

$$\arg\min_{\beta} \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(y_{t+h}-x_t\beta\right)^2+\lambda_1\sum_{j=1}^{p}\left|\beta_j\right|+\lambda_2\sum_{j=1}^{p}\beta_j^2$$

A Bayesian prior for β in the penalized regression using this estimator is

$$\pi\left(\beta|\sigma^2\right)\sim e^{-\frac{\lambda_1}{\sqrt{\sigma^2}}\sum_{j=1}^{p}\left|\beta_j\right|-\frac{\lambda_2}{2\sigma^2}\sum_{j=1}^{p}\beta_j^2}$$

Kyung et al. (2010) show that a hierarchical representation of this density exists, and it is of double-exponential form, as in the simple LASSO. This means that the hyperprior on τ_j^2 is

$$\pi\left(\tau_j^2|\lambda_1^2\right)\sim Exponential\left(\frac{\lambda_1^2}{2}\right), \text{ for } j=1,...,p \tag{7a}$$

where in this case the difference with the standard LASSO prior is that the covariance matrix is of the form $V=\sigma^2\times diag\left\{\left(\tau_1^{-2}+\lambda_2\right)^{-1},...,\left(\tau_p^{-2}+\lambda_2\right)^{-1}\right\}$.

[2] The data set in this paper implies such an ordering. Many disaggregated and component series of the same aggregated series appear in order. Additionally, all variables in this dataset are ordered according to statistical releases.

As opposed to maximizing the likelihood using no prior information, estimation for the Bayesian means that the likelihood function has to be averaged using each of the five priors presented above. This weighted average is the posterior distribution of the regression coefficients β, and the mode of the posterior is identical to frequentist shrinkage estimators. In Appendix B I give all the necessary details on how to get samples from the posterior distribution of all regression coefficients $\theta = \left\{\alpha, \beta, \sigma^2\right\}$ by sampling from their conditional posteriors using Markov Chain Monte Carlo (MCMC) methods. These are fairly easy to implement and computationally efficient.

It should be noted here that other extensions of these priors are possible, although many of these extensions can become cumbersome computationally. For instance one can use the fact that a $Gamma\left(a = v/2, b = 2\right)$ distribution is equivalent to a $\chi^2 \sim (v)$ distribution, with the cost that a Normal-Chi-square mixture is not a representation of any known distribution with known desirable shrinkage properties. Similarly, Park and Casella (2008) discuss some alternative priors based specifically on the LASSO, such as the extension proposed by Rosset and Zhu (2004). These authors propose to robustify the LASSO by considering a quadratic Huber-type loss function H which has the property that the coefficients β are shrunk quadratically around zero, while outside the neighborhood of zero this function becomes piecewise linear. This "Huberized LASSO" takes the form

$$\min_{\beta} \sum_{t=1}^{T} H\left(y_{t+h} - x_t \beta\right) + \lambda \sum_{j=1}^{p} \left|\beta_j\right|$$

but Park and Casella (2008) note that in this case it is not straightforward to marginalize over αz_t (which was purposely ignored in our discussion so far, since as Appendix B shows, it is easy to marginalize over αz_t when assuming the five priors presented above). Lastly, Hobert and Geyer (1998) proved geometric ergodicity of the two-stage Gibbs sampler from hierarchical models of a general Normal-Gamma form, a result which can be generalized to the LASSO, Fused LASSO and Elastic Net priors (see Kyung et al., 2010).

II.2.1 Tuning the hyperparameters

Hierarchical priors provide the advantage of allowing the data to determine the prior hyperparameter of interest (covariance of the Normal prior in our case). However from the formulations above we can observe that introducing a second layer of hierarchy (the Gamma-type densities) means that at least one new hyperparameter is introduced; it is only for the Normal-Uniform prior that this is obviously not the case. For the Normal-inverse Gamma prior (Student-t) we need to select values for the hyperparameters (ρ, ξ) of the inverse Gamma density. Although one can easily set a prior on the scale parameter ξ[3], typical uninformative values for the inverse Gamma distribution in Bayesian analysis are usually $\rho = \xi = 0.01$ or $\rho = \xi = 0.001$ (see Gelman, 2006). Since for these very low values of (ρ, ξ) the inverse Gamma becomes equivalent to a Jeffrey's prior for τ_j^2 (which is the first shrinkage prior examined), I will examine the more informative prior $iGamma\left(\rho = 3, \xi = 0.001\right)$ which concentrates τ_j^2 around the neighborhood of zero (note that for $\rho \leq 2$ the variance of the inverse Gamma does not exist).

[3] A conjugate prior on ξ is the $Gamma\left(\alpha_0, \beta_0\right)$ density. Then the posterior of the inverse Gamma prior is again an inverse Gamma density with parameters $\left(\alpha_0 + p\rho, \beta_0 + \sum_{i=1}^{p} \tau_i^{-2}\right)$.

In the Hierarchical LASSO prior case, in the absence of other information, we should find uninformative values for the Exponential prior distribution of τ_j^2. Therefore we would want to make a specific choice of the rate parameter λ that would give a combination of a low prior mean value for τ_j^2 (ideally zero) and a quite large prior variance. Given that the mean and variance of an $Exponential\left(\lambda\right)$ distribution are λ^{-1} and λ^{-2} respectively, this is not a straightforward combination to achieve (both mean and variance increase or decrease at the same time). In that respect, one can introduce an additional hierarchical layer for the parameters λ. A conjugate prior which would facilitate posterior computations when using the Exponential prior, is the Gamma prior on λ^2 (not λ) of the form

$$\pi\left(\lambda^2\right) \sim Gamma\left(r, \delta\right)$$

Similarly, an additional layer on the hyperparameters λ_1, λ_2 of the Fused Lasso and Elastic Net priors is of the form

$$\begin{aligned} \pi\left(\lambda_1^2\right) &\sim Gamma\left(r_1, \delta_1\right) \\ \pi\left(\lambda_2^2\right) &\sim Gamma\left(r_2, \delta_2\right) \end{aligned}$$

and hence now it easy to verify that setting $r = \delta = 0.01$ (similarly $r_1 = \delta_1 = r_2 = \delta_2 = 0.01$) we have a near-Uniform (noninformative) prior on the hyperparameters λ, λ_1, λ_2.

III Empirical Results

The data-set consists of 129 quarterly U.S. macroeconomic time series spanning the period 1959:Q1 to 2010:Q2 (the effective sample size, after converting to stationary and taking lags is 1960:Q1-2010:Q2). The series were downloaded from the St. Louis Fed FRED database (http://research.stlouisfed.org/fred2/) and a complete description is given in Table A.1 in the Appendix. The whole dataset is quite standard for this type of application, and includes among others data releases like personal income and outlays, GDP and components, assets and liabilities of commercial banks in the United States, productivity and costs measures, exchange rates and selected interest rates. All series are seasonally adjusted, where this is applicable, and transformed to be approximately stationary. All transformations are summarized in column "T" in Table A.1 and explained in detail in Appendix A. Bottom line is that when the series are used as predictors in x_t, standard stationarity transformations are applied, like first and second (log) differences. In contrast, when the series are used as the series to be predicted (y_{t+h}), then h-quarter growth or differences transformations are used.

In the dataset there are series which are higher level aggregates (mainly sums) of individual disaggregated series. There are 14 series like that in the dataset which are excluded when extracting factors, as it is not sensible to extract a common factor between, say, two series and their sum. Column "F" in Table A.1 denotes with 1 only the 115 disaggregated variables which are used to extract factors. This restriction does not hold when using the shrinkage priors and all series are used as predictors.

III.1 Forecasting with many predictors

All forecasts are from the univariate regression (1) where iteratively I use one of the 129 variables as the dependent variable (y_{t+h}) and the remaining 128 variables enter the regression as the matrix of standardized exogenous predictors (x_t). Then the five Bayesian shrinkage priors are applied to estimate $\widehat{\beta}^j$, j =*Jeffreys, Student-t, Lasso, Fused Lasso, Elastic Net*, and forecasts are produced using the original, unstandardized matrix of predictors x_t^*. In order to forecast with the dynamic factor model (*DFM*), x_t is replaced with the first five principal components of the 115 disaggregated series in x_t and $\widehat{\beta}^{DFM}$ is estimated with simple OLS. The variables which are always included in each of the six forecasting models (z_t) are the intercept and two lags of the one-quarter growth rates or differences of the dependent variable (i.e. lags y_t and y_{t-1} using the same stationarity transformations as in the variables in x_t).

The first estimation period is 1960:Q1 (after taking lags and transforming to stationarity) to 1984:Q4 and the sample 1985:Q1 to 2010:Q2 (last 102 observations) is kept for evaluation of h-step ahead forecasts, $h = 1, 2, 4$. In particular, using the initial sample (where y_{t+h} is observed from 1960:Q1+h to 1984:Q4 and (z_t, x_t) is observed from 1960:Q1 to 1984:Q4-h) estimation of the regression (1) provides parameter estimates $\widehat{\alpha}, \widehat{\beta}, \widehat{\sigma}^2$, and then forecasts can be computed for $y_{1984:Q4+h}$ by plugging-in the regression the realization of the predictors in 1984:Q4, i.e. the values $(z_{1984:Q4}, x_{1984:Q4})$. Then one data point is added and the same procedure is followed until the sample is exhausted. Since the models with shrinkage priors are estimated using MCMC (see Appendix B), which provides draws from the whole posterior density of the parameters, predictive simulation is used to obtain the whole predictive density. For each of the 129 dependent variables, the five prior distributions, the three forecast horizon, and the 102-h out-of-sample observations, 7.000 post-burn in draws from the conditional posteriors of the regression parameters $\left(\alpha, \beta, \sigma^2\right)$ are saved (see Appendix B for exact formulæ), and using each parameter draw 10 forecasts are generated leading in 70.000 draws from the predictive density of each of the 129 variables.

In a similar comparison of shrinkage estimators for regressions with many predictors, Stock and Watson (2011) use 4 lags in each of their 143 univariate regressions and report all their results relative to an AR(4) model. Del Mol, Giannone and Reichlin (2008) consider only an unrestricted intercept in their shrinkage regressions and report results relative to a random walk. In this paper, since the effects of an intercept and two lags are partialled out in each forecasting model, it is natural to consider forecast performance statistics relative to an AR(2) model. In this paper Mean Absolute Forecast Errors (MAFE) and Mean Squared Forecast Errors (MSFE) are considered. Unless stated otherwise, all results are based on the MAFE and MSFE statistics of model j relative to the MAFE and MSFE of the AR(2) model (i.e. $MAFE_j = mafe_j / mafe_{AR(2)}$ and $MSFE_j = msfe_j / msfe_{AR(2)}$). Consequently, $MAFE_j > 1$ means that the AR(2) dominates in terms of absolute forecast error, while the opposite is true when $MAFE_j < 1$.

Tables 1 to 3 present the average absolute forecast errors for 1,2 and 4 quarters ahead. Since the relative MAFE results are averaged over many series, three decimals are used in this table because otherwise the differences are quite small (see also Stock and Watson, 2011). First, based on the median MAFE using the total number of series, the simple LASSO and the Elastic Net give the smallest forecast errors in all cases (note that for $h = 2$ there is a difference but this is minimal). This might suggest that taking into account the correlation among the predictors, which is what the Elastic Net algorithm adds to the simple LASSO algorithm, is not that important with these data. However, the Elastic Net consistently has the smallest maximum MAFE, and consequently has the smallest variance across all 129 series. The same shrinkage algorithm clearly dominates in most of the 17 data categories for $h = 1$, and on average. For $h = 4$ all three srhinkage estimators (LASSO, Elastic Net and the DFM) are doing equally well.

Hierarchical shrinkage priors based on the Uniform and inverse Gamma distributions are doing very poorly on average, although for some data categories they provide the smallesat MAFE among all shrinkage estimators. Looking at the MAFE descriptives based on the total number of series, Student-t shrinkage is always doing better than Jeffreys shrinkage in lowering the median MAFE. Note however that for the Student-t prior, a single default choice of hyperparameters applied to all 129 series. Although this choice works well on average (median MAFEs), in some series it completely collapses. For example, there are cases where this estimate leads to MAFEs up to ten times higher (see the maximum MAFE based on total number of series in Table 3) than the benchmark model. On the other hand, Jeffrey's prior is not dependent on a choice of hyperparameters, and we can safely say that its shrinkage and forecasting performance is very unsatisfactory for the specific design of this study. Finally, the idea behind the fused LASSO, i.e. taking into account the correlation among consecutive predictors, does not help improve forecasting performance at all. In fact forecasts from this estimator are always dominated from the LASSO and the Elastic Net.

Once we turn to Tables 4 to 6 with the MSFE results based on the total number of series, it is obvious that the DFM is dominating all Bayesian shrinkage estimators at all three forecast horizons. Although the LASSO and the Elastic Net improve over the benchmark AR(2) forecasts, they are still not as good as the DFM. Nevertheless, by looking at the individual data categories, the Elastic Net is the best in forecasting GDP and its components at horizons $h = 1, 2$, as well as the various Consumer Price Indexes at all forecast horizons. Note that this pattern was also true for the MAFE results in Tables 1 to 3. Therefore, summarizing the results in Tables 1 to 6, from a mean forecast error point of view the Elastic Net and the LASSO are the best Bayesian shrinkage estimators. However, these might not improve too much over principal component shrinkage using a factor model and the final result is dependent on the series being forecasted each time.

Table 7 gives a better view of the total performance of each shrinkage estimator. Hit rates are calculated based on MAFEs, MSFEs and predictive likelihoods. These are estimated as the proportion of times (among the 129 series) a specific shrinkage estimator had the lowest MAFE, the lowest MSFE and the highest average predictive likelihood (APL). The average predictive likelihood can be used to evaluate the whole predictive density of each regression model; see Geweke and Amisano (2010) and references therein. Although in Tables 1 to 6 we saw that based on the total number of series, the Elastic Net had exactly the same median MAFE and MSFE as the LASSO, Table 7 shows that the LASSO has better hit rates for all three measures. In terms of mean forecasting, the LASSO always does better in MSFE and MAFE hit rates by 8 to 15% compared to the Elastic Net. In terms of density forecasting, the LASSO improves even more the density forecasts from the Elastic Net (an average improvement of 25% at all forecast horizons). This is because parameter uncertainty feeds in the predictive likelihood evaluation. Thus the Elastic Net having two regularization parameters λ_1 and λ_2, the uncertainty (posterior variance) about both parameters feeds in the density forecasts of y_{t+h}. The LASSO, having only one regularization parameter, i.e. $\lambda_1 = \lambda$ and $\lambda_2 = 0$, has less forecast uncertainty/variance (given that for this specific case-study, forecasts of the *mean* coming from both estimators are more or less identical).

The similarity among the five shrinkage forecasts is assessed in Table 8. The lower triangular entries in this table show the correlation coefficients of all MSFEs for all 129 variables for horizon $h = 1$. The correlations among all shrinkage forecast errors is one, except for the Student-t forecasts which are less correlated to the other four shrinkage methods. This is simply because the other four hierarchical shrinkage priors (Jeffreys, LASSO, Fused LASSO and Elastic Net) are based on noninformative priors on the lower level of their hierarchies. Entries above the diagonal of Table 8 are the mean absolute difference between the row/column method RMSEs, averaged across series. The results confirm that the Student-t forecasts, which were the worst according to Tables 1-6, are the most distant from the forecasts generated from the other four priors. In contrast, the LASSO and Elastic Net forecasts have the smallest difference among any other method, something also confirmed by their equal forecasting performance shown in Tables 1-6.

No matter how correlated on average are the forecasts from the different shrinkage priors, we saw in Tables 1-6 that their differences were substantial across different data releases and across forecasts horizons. The LASSO and Elastic Net priors have a better ability to take into account the correlation patterns in the predictor variables, while the Fused LASSO is less good at this task (because of the very specific correlation pattern it has to find, i.e. penalize less/more consecutive predictors as a group). The Jeffreys and Student-t priors do not explicitly account for correlation in the predictor variables and, hence, their performance can be very risky sometimes, with forecast errors which are multiples of those produces by the other three methods.

If correlation among the predictors is a crucial determinant of the performance of these algorithms, then a natural question to ask is what happens if we forecast with orthogonal predictors (the case that Stock and Watson, 2011, examine). Table 9 presents MAFE, MSFE and APL descriptive statistics based on all 129 series for $h = 1$, when the exogenous predictors are orthogonalized. For that reason the MATLAB function ORTH is used, which creates an orthonormal basis for the range of the matrix of exogenous predictors x_t, and which is based on simply taking the singular value decomposition of x_t. Consequently, this orthogonalization is like taking all possible principal components from the 128 exogenous predictors and then apply each of the five shrinkage algorithms to select the number of components to forecast with (while the rest are shrunk to zero). In fact, as seen on Table 9, orthogonalization of the data amounts to almost identical forecasting performance of the five Bayesian algorithms. Additionally their performance is equal to the best performing method, the LASSO, when using correlated predictors (compare the total MAFE and total MSFE results in Tables 1 and 4). This shows that orthogonalization is enough to guarantee that any of these shrinkage priors will always perform well in forecasting. However the reader should note that this happens due to the effect of the default, uninformative priors used in this paper. For informative choices on the regularization parameters the shrinkage penalty induced will - in general - be different among the five shrinkage priors (see the discussion in the following subsection).

III.2 Forecasting one year ahead US GDP growth using the LASSO

The previous subsection focused on evaluating default semi-automatic shrinkage priors using 129 variables. In practical situations, the applied macroeconomist will most probably want to focus on a few variables of interest (like inflation, an output-gap or stock prices). Additionally, the previous subsection does not answer the question if other hyperparameter choices exist that could possibly make Bayesian shrinkage perform even better. Subsequently, here I focus on forecasting U.S. GDP using only the simple hierarchical LASSO prior, with various choices on the regularization parameter λ. In particular, following Del Mol et al. (2008) I use a regression model with an intercept and the 128 remaining variables as predictors (no own lags used).

The main difference with the previous subsection is that I compare four choices for λ

1. $\lambda^2 \sim Gamma\left(r, \delta\right)$ with $r = \delta = 0.01$ (as in the benchmark case examined so far)
2. $\lambda^2 \sim Gamma\left(r, \delta\right)$ with $r = 1$ and $\delta = 0.1$
3. $\lambda^2 \sim Gamma\left(r, \delta\right)$ with $r = 3$ and $\delta = 1$
4. λ estimated by finding the maximum marginal likelihood (MML) using the Monte Carlo EM algorithm described in Park and Casella (2008)

Forecasts are generated for $h = 4$ steps ahead, and MAFE and MSFE statistics relative to the random walk model are reported in Table 10. The benchmark prior is the best performing for US GDP and in fact forecasts are highly correlated with the DFM model. Using the full sample, estimates of the posterior median estimate of the regularization parameter λ in the four LASSO models are 87.2, 33.7, 10.8 and 10.6, respectively. The prior choice $\lambda^2 \sim Gamma\left(3, 1\right)$ gives posterior parameter estimates (and hence forecasts) identical to MML estimation of λ, and this actually occurs for a wide range of choices of $r \geq 3$.

As $\lambda \rightarrow \infty$ all coefficients are penalized heavily, i.e. $\beta_{\lambda \rightarrow \infty}^{LASSO} = 0$ which further implies that in the limit the dynamic regression model with many predictors reduces to $y_{t+h} = \alpha z_t + \varepsilon_{t+h}$. In this case, the scale invariant prior (benchmark case) provides the largest posterior estimate of λ which implies posterior estimates of β which are heavily penalized (but not exactly zero). As we allow informative priors (cases 2 and 3), more and more variables are left unrestricted and the results resemble the case of selection of regressors. For the third case, 14 coefficients are "sufficiently" different than zero, while the remaining 115 are very "low" (remember that the regressors are standardized, so it makes some sense to talk about "large" and "small" coefficients as being important or not). Nevertheless, one-year ahead forecasts of GDP growth are not improved when forecasting with these "14 predictors", and hence the benchmark case which penalizes heavily all predictors performs better than using an informative prior on λ. This result is robust at other forecast horizons as well (results not presented here). The only difference is that as the forecast horizon increases (for $h = 8$ for instance) more predictors are relevant for forecasting GDP, so that the $\lambda^2 \sim Gamma\left(3, 1\right)$ prior leads to forecasts much closer to (but still dominated from) the choice $\lambda^2 \sim Gamma\left(0.01, 0.01\right)$.

IV Concluding remarks

This paper has investigated the properties of Bayesian shrinkage using hierarchical priors. A general shrinkage representation is provided using Normal-Gamma distributions and five special cases of interest have been evaluated in forecasting using a large macroeconomic dataset. A default semi-automatic approach using noninformative, near-improper priors was given special attention in this paper, but also a sensitivity analysis with more informative priors has been carried out for forecasting US GDP.

The results suggest that Bayesian shrinkage can compete favorably with dynamic factor models, although it is not straightforward to say whether one method clearly dominates over the other. Both methods are efficient in reducing the dimension of large datasets and help achieve smaller forecast errors (especially for long–run forecasts), however extra care has to be taken when selecting a prior for Bayesian shrinkage. From an applied econometrician's point of view (whether "frequentist" or "Bayesian"), the form of Bayesian shrinkage analyzed in this paper can be seen as a pragmatic tool useful for out-of-sample forecasting in the presence of many possible predictor variables (a typical every-day task for a researcher at the Fed, where thousands of series are available) or when time series are short (what is part of the life of a researcher in the ECB, with most Euro-Area macro series beginning around 1995). Subsequently, this paper argues that, similarly to the very popular Bayesian Model Averaging (BMA) and the empirical Bayes Minnesota prior for vector autoregressions, "formal" (i.e. hierarchical) Bayesian treatment of the shrinkage problem should also become a standard technique for handling modern medium to large amounts of information.

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Appendices

A. Data and transformations

All series were downloaded from St. Louis' FRED database in December 2010 and cover the quarters Q1:1959 to Q2:2010. All series were seasonally adjusted: either taken adjusted from FRED or by applying to the unadjusted series a quarterly X11 filter based on an AR(4) model (after testing for seasonality). Some series in the database were observed only on a monthly basis and quarterly values were computed by averaging the monthly values over the quarter (as opposed to keeping the mid-month of the quarter). All variables are transformed to be approximately stationary and the transformation codes for each variable appear in the column 'T' on the table below.

In particular, if $w_{i,t}$ is the original un-transformed series in levels, when the series is used as a predictor (R.H.S. of equation (1)) the transformation codes are: 1 - no transformation (levels), $x_{i,t} = w_{i,t}$; 2 - first difference, $x_{i,t} = w_{i,t} - w_{i,t-1}$; 3- second difference, $x_{i,t} = \Delta w_{i,t} - \Delta w_{i,t-1}$ 4 - logarithm, $x_{i,t} = \log w_{i,t}$; 5 - first difference of logarithm, $x_{i,t} = \log w_{i,t} - \log w_{i,t-1}$; 6 - second difference of logarithm, $x_{i,t} = \Delta \log w_{i,t} - \Delta \log w_{i,t-1}$.

When the series is used as the variable to be predicted (L.H.S. of equation (1)) the transformation codes are: 1 - no transformation (levels), $y_{i,t+h} = w_{i,t+h}$; 2 - first difference, $y_{i,t+h} = w_{i,t+h} - w_{i,t}$; 3- second difference, $y_{i,t+h} = \frac{1}{h}\Delta^h w_{i,t+h} - \Delta w_{i,t}$ 4 - logarithm, $y_{i,t+h} = \log w_{i,t+h}$; 5 - first difference of logarithm, $y_{i,t+h} = \log w_{i,t+h} - \log w_{i,t}$; 6 - second difference of logarithm, $y_{i,t+h} = \frac{1}{h}\Delta^h \log w_{i,t+h} - \Delta \log w_{i,t}$. In the transformations above, I define $\Delta w_t = w_t - w_{t-1}$ and $\Delta^h w_{t+h} = w_{t+h} - w_t$.

From the 129 series, 14 are higher level aggregates and do not add information when extracting principal components. These series are indicated with a 0 in column 'F' of the table below, and only the rest 115 series are used for estimating factors.

Table A.1: Description of series

No	Series ID	T	F	Title
1	GDPC96	5	1	Real Gross Domestic Product, 3 Decimal
2	GDPDEF	5	1	Gross Domestic Product: Implicit Price Deflator
3	PCECC96	5	1	Real Personal Consumption Expenditures
4	PCECTPI	5	1	Personal Consumption Expenditures: Chain-type Price Index
5	GPDIC96	5	1	Real Gross Private Domestic Investment, 3 Decimal
6	IMPGSC96	5	1	Real Imports of Goods & Services, 3 Decimal
7	EXPGSC96	5	1	Real Exports of Goods & Services, 3 Decimal
8	CBIC96	1	1	Real Change in Private Inventories
9	FINSLC96	5	1	Real Final Sales of Domestic Product
10	GSAVE	5	1	Gross Saving
11	GCEC96	5	1	Real Government Consumption Expenditures & Gross Investment
12	SLEXPND	6	1	State & Local Government Current Expenditures
13	SLINV	6	1	State & Local Government Gross Investment
14	DPIC96	6	1	Real Disposable Personal Income
15	PINCOME	6	1	Personal Income
16	PSAVE	5	1	Personal Saving
17	PRFI	6	1	Private Residential Fixed Investment

No	Series ID	T	F	Title
18	PNFI	6	1	Private Nonresidential Fixed Investment
19	PCDG	5	1	Personal Consumption Expenditures: Durable Goods
20	PCND	5	1	Personal Consumption Expenditures: Nondurable Goods
21	PCESV	5	1	Personal Consumption Expenditures: Services
22	GPDICTPI	6	1	Gross Private Domestic Investment: Chain-type Price Index
23	WASCUR	6	1	Compensation of Employees: Wages & Salary Accruals
24	DIVIDEND	6	1	Net Corporate Dividends
25	CP	6	1	Corporate Profits After Tax
26	CCFC	6	1	Corporate: Consumption of Fixed Capital
27	HOUST	4	0	Housing Starts: Total: New Privately Owned Housing Units Started
28	HOUST1F	4	1	Privately Owned Housing Starts: 1-Unit Structures
29	HOUST5F	4	1	Privately Owned Housing Starts: 5-Unit Structures or More
30	HOUSTW	4	1	Housing Starts in West Census Region
31	HOUSTMW	4	1	Housing Starts in Midwest Census Region
32	HOUSTS	4	1	Housing Starts in South Census Region
33	HOUSTNE	4	1	Housing Starts in Northeast Census Region
34	INDPRO	5	0	Industrial Production Index
35	IPCONGD	5	1	Industrial Production: Consumer Goods
36	IPDCONGD	5	1	Industrial Production: Durable Consumer Goods
37	IPNCONGD	5	1	Industrial Production: Nondurable Consumer Goods
38	IPMAT	5	1	Industrial Production: Materials
39	IPDMAT	5	1	Industrial Production: Durable Materials
40	IPNMAT	5	1	Industrial Production: Nondurable Materials
41	IPBUSEQ	5	1	Industrial Production: Business Equipment
42	IPFINAL	5	1	Industrial Production: Final Products (Market Group)
43	UTL11	1	1	Capacity Utilization: Manufacturing
44	UEMPLT5	5	1	Civilians Unemployed - Less Than 5 Weeks
45	UEMP5TO14	5	1	Civilians Unemployed for 5-14 Weeks
46	UEMP15T26	5	1	Civilians Unemployed for 15-26 Weeks
47	UEMP27OV	5	1	Civilians Unemployed for 27 Weeks and Over
48	UNRATE	2	1	Civilian Unemployment Rate
49	PAYEMS	5	0	Total Nonfarm Payrolls: All Employees
50	NDMANEMP	5	1	All Employees: Nondurable Goods Manufacturing
51	DMANEMP	5	1	All Employees: Durable Goods Manufacturing
52	USCONS	5	1	All Employees: Construction
53	USGOOD	5	0	All Employees: Goods-Producing Industries
54	USFIRE	5	1	All Employees: Financial Activities
55	USWTRADE	5	1	All Employees: Wholesale Trade
56	USTPU	5	1	All Employees: Trade, Transportation & Utilities
57	USTRADE	5	1	All Employees: Retail Trade
58	USMINE	5	1	All Employees: Natural Resources & Mining
59	USPBS	5	1	All Employees: Professional & Business Services
60	USLAH	5	1	All Employees: Leisure & Hospitality
61	USINFO	5	1	All Employees: Information Services
62	USEHS	5	1	All Employees: Education & Health Services
63	SRVPRD	5	1	All Employees: Service-Providing Industries
64	USPRIV	5	0	All Employees: Total Private Industries
65	USGOVT	5	1	All Employees: Government
66	AHEMAN	6	1	Average Hourly Earnings: Manufacturing

No	Series ID	T	F	Title
67	AHECONS	6	1	Average Hourly Earnings: Construction
68	AWHMAN	5	1	Average Weekly Hours of Production: Manufacturing
69	AWOTMAN	5	1	Average Weekly Hours: Overtime: Manufacturing
70	EMRATIO	5	1	Civilian Employment-Population Ratio
71	CIVPART	5	1	Civilian Participation Rate
72	OPHPBS	5	1	Business Sector: Output Per Hour of All Persons
73	ULCNFB	5	1	Nonfarm Business Sector: Unit Labor Cost
74	BUSLOANS	6	1	Commercial and Industrial Loans at All Commercial Banks
75	REALLN	6	1	Real Estate Loans at All Commercial Banks
76	CONSUMER	5	1	Consumer (Individual) Loans at All Commercial Banks
77	INVEST	5	0	Total Investments at All Commercial Banks
78	LOANS	6	0	Total Loans and Leases at Commercial Banks
79	MPRIME	2	1	Bank Prime Loan Rate
80	GS1	2	1	1-Year Treasury Constant Maturity Rate
81	GS3	2	1	3-Year Treasury Constant Maturity Rate
82	GS5	2	1	5-Year Treasury Constant Maturity Rate
83	GS10	2	1	10-Year Treasury Constant Maturity Rate
84	FEDFUNDS	2	1	Effective Federal Funds Rate
85	TB3MS	2	1	3-Month Treasury Bill: Secondary Market Rate
86	TB6MS	2	1	6-Month Treasury Bill: Secondary Market Rate
87	AAA	2	1	Moody’s Seasoned Aaa Corporate Bond Yield
88	BAA	2	1	Moody’s Seasoned Baa Corporate Bond Yield
89	M1SL	6	1	M1 Money Stock
90	M2SL	6	1	M2 Money Stock
91	CURRSL	6	1	Currency Component of M1
92	DEMDEPSL	6	1	Demand Deposits at Commercial Banks
93	SAVINGSL	6	1	Savings Deposits - Total
94	TCDSL	6	0	Total Checkable Deposits
95	TVCKSSL	6	1	Travelers Checks Outstanding
96	CURRCIR	6	1	Currency in Circulation
97	MZMSL	6	1	MZM Money Stock
98	M1V	5	1	Velocity of M1 Money Stock
99	M2V	5	1	Velocity of M2 Money Stock
100	NONREVSL	6	0	Total Nonrevolving Credit Outstanding
101	TOTALSL	6	0	Total Consumer Credit Outstanding
102	CPIAUCSL	6	0	Consumer Price Index for All Urban Consumers: All Items
103	CPILEGSL	6	0	Consumer Price Index for All Urban Consumers: All Items Less Energy
104	CPIULFSL	6	0	Consumer Price Index for All Urban Consumers: All Items Less Food
105	CPIENGSL	6	1	Consumer Price Index for All Urban Consumers: Energy
106	CPIUFDSL	6	1	Consumer Price Index for All Urban Consumers: Food
107	CPIAPPSL	6	1	Consumer Price Index for All Urban Consumers: Apparel
108	CPIMEDSL	6	1	Consumer Price Index for All Urban Consumers: Medical Care
109	CPITRNSL	6	1	Consumer Price Index for All Urban Consumers: Transportation
110	PPIACO	6	0	Producer Price Index: All Commodities
111	PPIFCG	6	1	Producer Price Index: Finished Consumer Goods
112	PPIFCF	6	1	Producer Price Index: Finished Consumer Foods
113	PFCGEF	6	1	Producer Price Index: Finished Consumer Goods Excluding Foods
114	PPIFGS	6	1	Producer Price Index: Finished Goods

No	Series ID	T	F	Title
115	PPICRM	6	1	Producer Price Index: Crude Materials for Further Processing
116	PPICPE	6	1	Producer Price Index Finished Goods: Capital Equipment
117	PPIITM	6	1	Producer Price Index: Intermediate Materials: Supplies & Components
118	SP500	5	1	S&P 500 Index
119	EXUSUK	5	1	U.S. / U.K Foreign Exchange Rate
120	EXSZUS	5	1	Switzerland / U.S. Foreign Exchange Rate
121	EXJPUS	5	1	Japan / U.S. Foreign Exchange Rate
122	EXCAUS	5	1	Canada / U.S. Foreign Exchange Rate
123	PMI	1	1	ISM Manufacturing: PMI Composite Index
124	NAPMNOI	1	1	ISM Manufacturing: New Orders Index
125	NAPMII	1	1	ISM Manufacturing: Inventories Index
126	NAPMEI	1	1	ISM Manufacturing: Employment Index
127	NAPMPRI	1	1	ISM Manufacturing: Prices Index
128	NAPMPI	1	1	ISM Manufacturing: Production Index
129	NAPMSDI	1	1	ISM Manufacturing: Supplier Deliveries Index

Table A.2: Categories of data series based on statistical releases

Group	Release	Number of series
1	Gross Domestic Product	26
2	New Residential Construction	7
3	G.17 Industrial Production and Capacity Utilization	10
4	The Employment Situation	28
5	Productivity and Costs	2
6	H.8 Assets and Liabilities of Commercial Banks in the United States	5
7	H.15 Selected Interest Rates	10
8	H.6 Money Stock Measures	7
9	H.4.1 Factors Affecting Reserve Balances	1
10	Money Zero Maturity (MZM)	1
11	Money Velocity	2
12	G.19 Consumer Credit	2
13	Consumer Price Index	8
14	Producer Price Index	8
15	Standard & Poors	1
16	G.5 Foreign Exchange Rates	4
17	Manufacturing ISM Report on Business	7

B. Bayesian hierarchical shrinkage priors

Note that I denote the inverse Gaussian distribution with parameters c, d as $IG\left(c,d\right)$, while the inverse Gamma with parameters a, b is denoted as $iGamma\left(a,b\right)$. A variable coming from the inverse Gamma distribution is the reciprocal of a variable distributed as gamma, while the same *is not* true for the inverse Gaussian variate (i.e. if $z \sim IG\left(c,d\right)$, then $\left(z^{-1}\right) \nsim N(c,d)$). There are many parametrizations of the Gamma distribution, and the one I am using in this article is

$$Gamma\left(a,b\right) \equiv f\left(y;a,b\right) = C_G y^{a-1} b^a e^{-by}$$

for a, b non-negative, real numbers, where $C_G = \Gamma\left(a\right) = (a-1)!$ is the gamma function.

The parameters on the unrestricted variables z_t are integrated out with the noninformative prior $\pi\left(\alpha\right) \propto 1$, leading to a conditional posterior

$$\alpha|\beta, \sigma^2, data \sim N_q\left(\left(Z'Z\right)^{-1} Z'\widetilde{y}^{\beta}, \sigma^2\left(Z'Z\right)^{-1}\right) \tag{B.1}$$

with $Z = \left(z_1', ..., z_T'\right)'$. That is, in the formulas of the conditional posteriors below, we need to add in each and every case of hierarchical prior the sampling step in equation (B.1) above. For notational convenience, in (B.1) and in the conditional posteriors below some or all of the quantities $\widetilde{y}^{\beta}$, $\widetilde{y}^{a}$ and Ψ show up, which are defined as $\widetilde{y}^{\beta} = y - X\beta$, $\widetilde{y}^{\alpha} = y - Z\alpha$ and $\Psi = \left(y - Z\alpha - X\beta\right)'\left(y - Z\alpha - X\beta\right)$, respectively. *Finally, in the formulas for the conditional posteriors we have to condition on the data matrices* (y, Z, X), *but this is omitted for notational simplicity (to keep the formulas more compact).*

B.1 Adaptive shrinkage Jeffrey's prior

The priors are defined using the following hierarchy

$$\begin{aligned} \pi\left(\beta|\tau_1^2, ..., \tau_p^2\right) &\sim N_p\left(0, V\right) \\ \pi\left(\tau_j^2\right) &\sim 1/\tau_j^2, \text{ for } j = 1, ..., p \end{aligned}$$

where $V = diag\left\{\tau_1^2, ..., \tau_p^2\right\}$. The posteriors of β and τ_j^2 can be obtained by sampling recursively from (B.1) and the full conditionals

$$\beta|a, \sigma^2, \left\{\tau_j^2\right\}_{j=1}^p \sim N_p\left(\left(X'X + \sigma^2 V^{-1}\right)^{-1} X'\widetilde{y}^{a}, \sigma^2\left(X'X + \sigma^2 V^{-1}\right)^{-1}\right) \tag{B.2a}$$

$$\frac{1}{\tau_j^2}|a, \beta_j, \sigma^2 \sim Gamma\left(\frac{1}{2}, \frac{\beta_j^2}{2}\right), \text{ for } j = 1, ..., p \tag{B.2b}$$

$$\sigma^2|a, \beta, \left\{\tau_j^2\right\}_{j=1}^p \sim iGamma\left(\frac{T}{2}, \frac{1}{2}\Psi\right) \tag{B.2c}$$

B.2 Adaptive shrinkage t-prior

The prior for this case is of the hierarchical form

$$\begin{aligned}\pi\left(\beta|\sigma^2,\tau_1^2,...,\tau_p^2\right) &\sim N_p\left(0,\sigma^2 V\right)\\ \pi\left(\tau_j^2\right) &\sim iGamma\left(\rho,\xi\right), \text{ for } j=1,...,p\end{aligned}$$

where $V=diag\left\{\tau_1^2,...,\tau_p^2\right\}$.Draws from the posterior can be obtained by sampling recursively from (B.1) and the full conditionals

$$\beta|a,\sigma^2,\left\{\tau_j^2\right\}_{j=1}^p \sim N_p\left(\left(X'X+\sigma^2V^{-1}\right)^{-1}X'\widetilde{y}^a,\sigma^2\left(X'X+\sigma^2V^{-1}\right)^{-1}\right) \quad \text{(B.3a)}$$

$$\frac{1}{\tau_j^2}|a,\beta_j,\sigma^2 \sim Gamma\left(\rho+\frac{1}{2},\frac{\beta_j^2}{2}+\xi\right), \text{ for } j=1,...,p \quad \text{(B.3b)}$$

$$\sigma^2|a,\beta,\left\{\tau_j^2\right\}_{j=1}^p \sim iGamma\left(\frac{T}{2},\frac{1}{2}\Psi\right) \quad \text{(B.3c)}$$

B.3 Hierarchical Lasso

The full hierarchical representation of the LASSO prior is

$$\pi\left(\beta|\sigma^2,\tau_1^2,...,\tau_p^2\right) \sim N_p\left(0,\sigma^2V\right) \quad \text{(B.4a)}$$

$$\pi\left(\tau_j^2\right) \sim Exponential\left(\frac{\lambda^2}{2}\right), \text{ for } j=1,...,p \quad \text{(B.4b)}$$

$$\pi\left(\lambda^2\right) \sim Gamma\left(r,\delta\right) \quad \text{(B.4c)}$$

where $V=diag\left\{\tau_1^2,...,\tau_p^2\right\}$.

Given these priors, the posterior can be obtained by sampling recursively from (B.1) and the full conditionals

$$\beta|a,\sigma^2,\left\{\tau_j^2\right\}_{j=1}^p \sim N_p\left(\left(X'X+V^{-1}\right)^{-1}X'\widetilde{y}^a,\sigma^2\left(X'X+V^{-1}\right)^{-1}\right) \quad \text{(B.5a)}$$

$$\frac{1}{\tau_j^2}|a,\beta,\sigma^2 \sim IG\left(\sqrt{\frac{\lambda^2\sigma^2}{\beta_j^2}},\lambda^2\right), \text{ for } j=1,...,p \quad \text{(B.5b)}$$

$$\lambda^2|a,\beta,\sigma^2,\left\{\tau_j^2\right\}_{j=1}^p \sim Gamma\left(p+r,\frac{1}{2}\sum\nolimits_{j=1}^p\tau_j^2+\delta\right) \quad \text{(B.5c)}$$

$$\sigma^2|a,\beta,\left\{\tau_j^2\right\}_{j=1}^p \sim iGamma\left(\frac{T-1}{2}+\frac{p}{2},\frac{1}{2}\Psi+\frac{1}{2}\beta'V^{-1}\beta\right) \quad \text{(B.5d)}$$

B.4 Hierarchical Fused Lasso

The hierarchical representation of the Fused Lasso prior is

$$\begin{aligned}
\pi\left(\beta|\sigma^2,\tau_1^2,...,\tau_p^2\right) &\sim N_p\left(0,\sigma^2 V\right) && \text{(B.6a)}\\
\pi\left(\tau_j^2|\lambda_1\right) &\sim Exponential\left(\frac{\lambda_1^2}{2}\right)\text{, for } j=1,...,p && \text{(B.6b)}\\
\pi\left(\omega_j^2|\lambda_2\right) &\sim Exponential\left(\frac{\lambda_2^2}{2}\right)\text{, for } j=1,...,p-1 && \text{(B.6c)}\\
\pi\left(\lambda_1^2\right) &\sim Gamma\left(r_1,\delta_1\right) && \text{(B.6d)}\\
\pi\left(\lambda_2^2\right) &\sim Gamma\left(r_2,\delta_2\right) && \text{(B.6e)}
\end{aligned}$$

where in this case V is the tridiagonal matrix

$$V=\sigma^2\times\begin{bmatrix}
\left(\tau_1^2+\omega_0^2+\omega_1^2\right) & -\omega_1^2 & 0 & \cdots & 0\\
-\omega_1^2 & \left(\tau_2^2+\omega_1^2+\omega_2^2\right) & \ddots & \ddots & \vdots\\
0 & \ddots & \ddots & -\omega_{p-2}^2 & 0\\
\vdots & \ddots & -\omega_{p-2}^2 & \left(\tau_{p-1}^2+\omega_{p-2}^2+\omega_{p-1}^2\right) & -\omega_{p-1}^2\\
0 & \cdots & 0 & -\omega_{p-1}^2 & \left(\tau_p^2+\omega_{p-1}^2+\omega_p^2\right)
\end{bmatrix}.$$

Given these priors, the posteriors can be obtained by sampling recursively from (B.1) and the full conditionals

$$\begin{aligned}
\beta|a,\sigma^2,\left\{\tau_j^2\right\}_{j=1}^p,\left\{\omega_j^2\right\}_{j=1}^{p-1} &\sim N_p\left(\left(X'X+V_{FL}^{-1}\right)^{-1}X'\widetilde{y}^a,\sigma^2\left(X'X+V^{-1}\right)^{-1}\right) && \text{(B.7a)}\\
\frac{1}{\tau_j^2}|a,\beta,\left\{\omega_j^2\right\}_{j=1}^{p-1},\sigma^2 &\sim IG\left(\sqrt{\frac{\lambda_1^2\sigma^2}{\beta_j^2}},\lambda^2\right)\text{, for } j=1,...,p && \text{(B.7b)}\\
\frac{1}{\omega_j^2}|a,\beta,\left\{\tau_j^2\right\}_{j=1}^{p},\sigma^2 &\sim IG\left(\sqrt{\frac{\lambda_2^2\sigma^2}{\left(\beta_{j+1}-\beta_j\right)^2}},\lambda^2\right)\text{, for } j=1,...,p-1 && \text{(B.7c)}\\
\lambda_1^2|a,\beta,\sigma^2,\left\{\tau_j^2\right\}_{j=1}^p,\left\{\omega_j^2\right\}_{j=1}^{p-1} &\sim Gamma\left(p+r_1,\frac{1}{2}\sum\nolimits_{j=1}^p\tau_j^2+\delta_1\right) && \text{(B.7d)}\\
\lambda_2^2|a,\beta,\sigma^2,\left\{\tau_j^2\right\}_{j=1}^p,\left\{\omega_j^2\right\}_{j=1}^{p-1} &\sim Gamma\left(p-1+r_2,\frac{1}{2}\sum\nolimits_{j=1}^{p-1}\omega_j^2+\delta_2\right) && \text{(B.7e)}\\
\sigma^2|a,\beta,\left\{\tau_j^2\right\}_{j=1}^p &\sim iGamma\left(\frac{T-1}{2}+\frac{p}{2},\frac{1}{2}\Psi+\frac{1}{2}\beta'V^{-1}\beta\right) && \text{(B.7f)}
\end{aligned}$$

B.5 Hierarchical Elastic Net

For a covariance $V=\sigma^2\times diag\left\{\left(\tau_1^{-2}+\lambda_2\right)^{-1},...,\left(\tau_p^{-2}+\lambda_2\right)^{-1}\right\}$ matrix the hierarchical elastic net prior is

$$\begin{aligned}
\pi\left(\beta|\sigma^2,\tau_1^2,...,\tau_p^2\right) &\sim N_p\left(0,V\right) \\
\pi\left(\tau_j^2|\lambda_1^2\right) &\sim Exponential\left(\frac{\lambda_1^2}{2}\right)\text{, for } j=1,...,p \\
\pi\left(\lambda_1^2\right) &\sim Gamma\left(r_1,\delta_1\right) \\
\pi\left(\lambda_2^2\right) &\sim Gamma\left(r_2,\delta_2\right)
\end{aligned}$$

Given these priors, the posterior can be obtained by sampling recursively from (B.1) and the full conditionals

$$\beta|a,\sigma^2,\left\{\tau_j^2\right\}_{j=1}^p \sim N_p\left(\left(X'X+V^{-1}\right)^{-1}X'\widetilde{y}^a,\sigma^2\left(X'X+V^{-1}\right)^{-1}\right) \quad \text{(B.8a)}$$

$$\frac{1}{\tau_j^2}|a,\beta,\sigma^2 \sim IG\left(\sqrt{\frac{\lambda_1^2\sigma^2}{\beta_j^2}},\lambda_1^2\right)\text{, for } j=1,...,p \quad \text{(B.8b)}$$

$$\lambda_1^2|a,\beta,\sigma^2,\left\{\tau_j^2\right\}_{j=1}^p \sim Gamma\left(p+r_1,\frac{1}{2}\sum\nolimits_{j=1}^p\tau_j^2+\delta_1\right) \quad \text{(B.8c)}$$

$$\lambda_2^2|a,\beta,\sigma^2,\left\{\tau_j^2\right\}_{j=1}^p \sim Gamma\left(\frac{p}{2}+r_2,\frac{1}{2\sigma^2}\sum\nolimits_{j=1}^p\beta_j^2+\delta_2\right) \quad \text{(B.8d)}$$

$$\sigma^2|a,\beta,\left\{\tau_j^2\right\}_{j=1}^p \sim iGamma\left(\frac{T-1}{2}+\frac{p}{2},\frac{1}{2}\Psi+\frac{1}{2}\beta'V^{-1}\beta\right) \quad \text{(B.8e)}$$

C. Results: Tables

Table 1: MAFE results for the five Bayes shrinkage estimators and the DFM, $h = 1$

	Jef	St-t	LASSO	Fused LASSO	Elastic Net	DFM
Median MAFE's based on statistical releases						
GDP and components	1.064	1.001	0.987	1.016	**0.986**	1.030
Housing	1.255	1.020	1.039	1.019	1.040	**0.924**
IP	1.016	1.027	0.988	1.013	0.988	**0.928**
Employment situation	1.058	1.024	1.012	1.033	1.012	**0.976**
Productivity/Costs	1.018	1.169	0.986	1.043	0.987	**0.959**
Assets/Liabilities of banks	0.966	0.973	0.970	0.973	0.971	**0.957**
Interest rates	1.056	0.970	0.967	0.991	**0.966**	1.066
Money stock	0.917	0.910	0.905	0.908	**0.904**	1.050
Currecny in circulation	0.677	0.680	0.678	**0.671**	0.677	0.678
MZM	0.910	0.894	0.896	0.900	**0.893**	1.294
Money velocity	0.994	0.976	0.981	0.990	**0.979**	1.092
Consumer Credit	1.001	0.985	0.985	0.998	**0.980**	1.043
CPI	1.006	0.914	0.916	0.914	**0.909**	0.963
PPI	**0.913**	0.917	0.916	0.918	0.919	0.931
Stock prices	**1.003**	1.008	1.006	1.004	1.005	1.072
Exchange rates	0.999	1.004	1.003	**0.997**	1.003	0.999
ISM surveys	1.108	1.276	1.031	1.038	1.028	**0.987**
MAFE descriptives based on total number of series						
median	1.017	0.998	**0.987**	1.004	**0.987**	0.993
variance	0.045	0.509	0.008	0.015	0.007	0.015
min	0.677	0.680	0.678	0.671	0.677	0.678
max	1.857	2.342	1.435	1.645	1.286	1.582

Note: Entries are MAFE-based statistics relative to the MAFE of an AR(2) model.

Table 2: MAFE results for the five Bayes shrinkage estimators and the DFM, $h = 2$

	Jef	St-t	LASSO	Fused LASSO	Elastic Net	DFM
Median MAFE's based on statistical releases						
GDP and components	1.100	1.025	**0.988**	1.025	**0.988**	1.005
Housing	1.208	1.045	1.040	1.035	1.043	**1.017**
IP	1.050	1.086	**0.987**	1.026	0.989	0.989
Employment situation	1.089	1.029	1.017	1.039	1.016	**0.982**
Productivity/Costs	1.160	1.386	1.081	1.184	1.082	**0.942**
Assets/Liabilities of banks	**0.967**	1.072	0.973	0.977	0.969	0.968
Interest rates	1.294	0.998	**0.986**	1.082	0.987	1.000
Money stock	**0.929**	0.933	0.932	0.930	0.931	1.002
Currecny in circulation	0.737	0.736	0.737	**0.733**	0.735	0.746
MZM	0.855	0.855	**0.845**	0.847	0.849	1.112
Money velocity	1.000	**0.995**	0.996	1.002	0.998	1.029
Consumer Credit	**0.990**	0.999	0.996	**0.990**	0.996	1.010
CPI	1.080	0.962	0.955	0.979	**0.954**	0.994
PPI	0.958	0.942	**0.941**	0.957	0.945	0.975
Stock prices	**0.987**	0.995	0.997	0.992	0.996	1.049
Exchange rates	1.026	1.007	1.008	**1.005**	1.009	1.030
ISM surveys	1.041	1.415	**1.021**	1.057	**1.021**	1.050
MAFE descriptives based on total number of series						
median	1.038	1.015	**0.990**	1.019	0.991	0.999
variance	0.296	0.625	0.009	0.029	0.008	0.007
min	0.491	0.494	0.487	0.495	0.482	0.488
max	3.230	4.536	1.458	1.931	1.289	1.222

Note: Entries are MAFE-based statistics relative to the MAFE of an AR(2) model.

Table 3: MAFE results for the five Bayes shrinkage estimators and the DFM, $h = 4$

	Jef	St-t	LASSO	Fused LASSO	Elastic Net	DFM
Median MAFE's based on statistical releases						
GDP and components	1.094	1.061	**0.990**	1.022	**0.990**	0.994
Housing	1.043	1.071	1.061	1.062	1.062	**0.979**
IP	1.030	1.040	**0.983**	1.022	0.987	0.997
Employment situation	1.188	1.029	1.004	1.075	1.004	**0.937**
Productivity/Costs	1.320	1.544	1.112	1.397	1.109	**0.891**
Assets/Liabilities of banks	**0.968**	1.128	0.979	0.998	0.970	0.969
Interest rates	1.624	1.050	1.002	1.113	1.000	**0.969**
Money stock	**0.933**	**0.933**	0.935	0.934	0.938	1.019
Currecny in circulation	0.916	0.920	0.918	0.925	0.920	**0.884**
MZM	0.928	0.931	0.930	**0.922**	0.934	1.117
Money velocity	1.000	0.987	**0.985**	0.993	0.987	1.013
Consumer Credit	0.967	1.006	**0.964**	0.990	0.968	1.015
CPI	1.190	1.698	**0.964**	1.051	0.968	1.031
PPI	0.983	0.964	0.964	0.989	**0.962**	1.010
Stock prices	0.968	**0.948**	0.952	0.951	0.959	1.031
Exchange rates	1.093	1.053	1.029	**1.027**	1.033	1.036
ISM surveys	1.045	1.289	**0.982**	1.026	0.984	0.990
MAFE descriptives based on total number of series						
median	1.046	1.029	**0.989**	1.025	**0.989**	**0.989**
variance	0.693	1.776	0.009	0.081	0.007	0.009
min	0.578	0.575	0.576	0.574	0.578	0.610
max	6.707	9.883	1.609	3.386	1.364	1.275

Note: Entries are MAFE-based statistics relative to the MAFE of an AR(2) model.

Table 4: MSFE results for the five Bayes shrinkage estimators and the DFM, $h = 1$

	Jef	St-t	LASSO	Fused LASSO	Elastic Net	DFM
Median MSFE's based on statistical releases						
GDP and components	1.036	0.996	0.993	1.002	**0.994**	1.047
Housing	1.236	1.068	1.059	1.059	1.059	**0.974**
IP	1.017	1.022	0.998	1.024	1.000	**0.843**
Employment situation	1.036	1.026	1.019	1.037	1.019	**0.899**
Productivity-Costs	1.003	1.108	0.986	1.023	0.985	**0.896**
Assets-Liabilities of banks	0.970	0.973	0.969	0.973	0.972	**0.960**
Interest rates	1.045	**0.954**	0.959	0.995	0.956	0.995
Money stock	0.948	0.949	0.945	0.946	**0.943**	1.094
Currecny in circulation	0.746	0.747	0.750	0.748	0.747	**0.563**
MZM	0.920	0.905	0.907	**0.904**	**0.904**	1.668
Money velocity	1.009	0.994	0.996	1.007	**0.994**	1.129
Consumer Credit	0.989	0.978	**0.976**	0.985	**0.976**	1.087
CPI	0.995	0.940	**0.936**	0.945	**0.936**	0.945
PPI	0.939	0.947	0.942	0.945	0.945	**0.906**
Stock prices	1.007	1.007	1.006	1.006	**1.003**	1.133
Exchange rates	0.999	1.002	0.999	1.000	**0.998**	1.016
ISM surveys	1.091	1.236	1.026	1.016	1.025	**0.932**
MSFE descriptives based on total number of series						
median	1.007	0.994	0.991	1.002	0.991	**0.958**
variance	0.041	0.084	0.006	0.010	0.006	0.054
min	0.671	0.747	0.666	0.706	0.664	0.452
max	2.380	2.797	1.474	1.545	1.370	2.498

Note: Entries are MSFE-based statistics relative to the MSFE of an AR(2) model.

Table 5: MSFE results for the five Bayes shrinkage estimators and the DFM, $h = 2$

	Jef	St-t	LASSO	Fused LASSO	Elastic Net	DFM
Median MSFE's based on statistical releases						
GDP and components	1.074	1.028	1.004	1.022	**0.998**	1.004
Housing	1.168	1.082	1.072	1.074	**1.071**	1.161
IP	1.041	1.152	1.012	1.049	**1.011**	1.013
Employment situation	1.055	1.035	1.020	1.036	1.022	**0.985**
Productivity-Costs	1.128	1.327	1.075	1.155	1.077	**0.882**
Assets-Liabilities of banks	0.976	1.037	0.980	0.970	0.977	**0.912**
Interest rates	1.260	1.008	0.994	1.084	0.998	**0.946**
Money stock	0.974	0.974	0.976	**0.972**	0.973	1.062
Currecny in circulation	0.785	0.783	0.786	0.788	0.785	**0.627**
MZM	0.915	0.911	0.907	**0.904**	0.910	1.392
Money velocity	1.021	1.003	**1.002**	1.019	1.005	1.026
Consumer Credit	0.986	0.994	**0.991**	0.996	0.993	0.993
CPI	1.048	0.967	0.966	0.986	**0.965**	0.999
PPI	0.979	0.968	**0.967**	0.976	0.968	0.975
Stock prices	**0.994**	1.000	1.000	1.000	1.000	1.073
Exchange rates	1.020	1.012	1.011	**1.010**	1.011	1.023
ISM surveys	1.036	1.697	**1.007**	1.020	1.010	1.043
MSFE descriptives based on total number of series						
median	1.035	1.018	0.998	1.017	0.999	**0.995**
variance	0.144	0.205	0.007	0.017	0.006	0.026
min	0.563	0.568	0.557	0.562	0.554	0.304
max	3.392	3.536	1.420	1.667	1.313	1.897

Note: Entries are MSFE-based statistics relative to the MSFE of an AR(2) model.

Table 6: MSFE results for the five Bayes shrinkage estimators and the DFM, $h = 4$

	Jef	St-t	LASSO	Fused LASSO	Elastic Net	DFM
Median MSFE's based on statistical releases						
GDP and components	1.061	1.057	**0.997**	1.022	0.999	1.004
Housing	**1.065**	1.097	1.087	1.092	1.087	1.106
IP	1.016	1.025	0.996	1.023	0.998	**0.986**
Employment situation	1.112	1.025	1.006	1.048	1.005	**0.916**
Productivity-Costs	1.222	1.614	1.099	1.289	1.093	**0.815**
Assets-Liabilities of banks	0.975	1.094	0.977	0.988	0.975	**0.924**
Interest rates	1.571	1.084	1.001	1.096	1.003	**0.957**
Money stock	**0.973**	0.974	0.975	**0.973**	0.975	1.114
Currecny in circulation	0.905	0.906	0.905	0.908	0.907	**0.782**
MZM	0.932	0.934	0.934	**0.927**	0.938	1.194
Money velocity	1.020	0.999	0.996	1.013	0.998	**0.987**
Consumer Credit	**0.969**	1.026	0.971	0.987	0.973	1.032
CPI	1.085	1.354	0.969	1.014	**0.968**	1.081
PPI	0.972	**0.962**	0.964	0.977	0.963	0.999
Stock prices	0.984	0.976	0.978	0.979	0.981	1.090
Exchange rates	1.098	1.060	**1.035**	**1.035**	1.036	1.062
ISM surveys	1.052	1.521	0.998	1.032	**0.996**	1.026
MSFE descriptives based on total number of series						
median	1.039	1.022	0.995	1.026	0.996	**0.987**
variance	0.386	0.417	0.008	0.047	0.006	0.028
min	0.620	0.610	0.611	0.612	0.612	0.414
max	5.389	6.784	1.573	2.771	1.378	1.620

Note: Entries are MSFE-based statistics relative to the MSFE of an AR(2) model.

Table 7: Hit-rates of the five Bayes estimators, total no. of series

	Jeffreys'	Student-t	LASSO	Fused LASSO	Elastic Net
Hit rates, $h = 1$					
% of lowest MAFE	14.0	17.1	**35.7**	10.9	22.5
% of lowest MSFE	15.5	20.9	**34.9**	7.8	20.9
% of highest APL	0.8	2.3	**62.0**	7.8	27.1
Hit rates, $h = 2$					
% of lowest MAFE	18.6	13.2	**34.1**	11.6	22.5
% of lowest MSFE	17.1	20.9	**28.7**	11.6	21.7
% of highest APL	1.6	0.8	**60.5**	13.9	23.3
Hit rates, $h = 4$					
% of lowest MAFE	17.8	12.4	**31.8**	10.9	27.1
% of lowest MSFE	13.2	17.1	**34.9**	10.1	24.8
% of highest APL	0.0	0.0	**55.8**	8.5	35.7

Note: This table shows the proportion of times (over the 129 series being forecasted) that each estimator achieved the lowest value of the MAFE and MSFE statistics, and the highest value of the Average Predictive Likelihood (APL).

Table 8: Average similarity of Bayes forecasts, $h = 1$: correlation (lower left) and mean absolute difference of forecasts (upper right)

	Jeffreys'	Student-t	LASSO	Fused LASSO	Elastic Net
Jef		0.088	0.037	0.032	0.037
St-t	0.944		0.080	0.083	0.080
LASSO	1.000	0.944		0.015	0.002
Fused LASSO	1.000	0.945	1.000		0.015
Elastic Net	1.000	0.944	1.000	1.000	

Table 9: MAFE, MSFE and Predictive Likelihoods for all 129 series, orthogonal predictors, $h = 1$

	Jeffreys'	Student-t	LASSO	Fused LASSO	Elastic Net
MAFE descriptives based on total number of series					
median	0.9876	0.9871	0.9873	0.9846	0.9863
25% quantile	0.9539	0.9536	0.9523	0.9551	0.9505
75% quantile	1.0166	1.0128	1.0183	1.0139	1.0167
variance	0.0061	0.0068	0.0060	0.0062	0.0061
min	0.6768	0.6812	0.6813	0.6800	0.6821
max	1.2988	1.3151	1.2797	1.3053	1.2803
MSFE descriptives based on total number of series					
median	0.9901	0.9923	0.9901	0.9903	0.9904
25% quantile	0.9558	0.9558	0.9559	0.9548	0.9510
75% quantile	1.0156	1.0227	1.0182	1.0178	1.0175
variance	0.0046	0.0049	0.0046	0.0048	0.0047
min	0.6714	0.6726	0.6666	0.6676	0.6670
max	1.2197	1.2750	1.2150	1.2397	1.2160
PL descriptives based on total number of series					
median	0.4190	0.4175	0.4711	0.4837	0.4724
25% quantile	0.2241	0.2232	0.2636	0.2717	0.2636
75% quantile	0.7112	0.7050	0.7839	0.7968	0.7854
variance	0.2065	0.1679	0.1867	0.1954	0.1876
min	0.0380	0.0379	0.0487	0.0510	0.0487
max	3.7777	3.2578	3.3826	3.4491	3.3998

Table 10: LASSO forecasts of US GDP, corr. predictors, $h = 4$

	LASSO 1	LASSO 2	LASSO 3	LASSO 4	DFM
MAFE	0.38	0.97	0.88	0.87	0.37
MSFE	0.24	0.93	0.81	0.79	0.22
Corr.with DFM forecasts	0.90	0.27	0.49	0.48	1

Note: The LASSO 1,2,3,4 models are the four univariate regressions described in the text, with estimation of λ using 1) $(r, \delta) = (0.01, 0.01)$, 2) $(r, \delta) = (1, 0.1)$, 3) $(r, \delta) = (3, 1)$ and 4) marginal maximum likelihood.