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OPTION PRICING UNDER LÉVY PROCESSES: A UNIFYING FORMULA

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OPTION PRICING UNDER LÉVY PROCESSES: A UNIFYING FORMULA.

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ABSTRACT. A new option pricing formula is presented that unifies several results of the existing literature on pricing exotic options under Lévy processes. To demonstrate the flexibility of the formula a few examples are given which provide new valuation formulas within the Lévy framework.

1. INTRODUCTION

The aim of this paper is to prove a comprehensive option pricing formula within a Lévy framework. Lévy processes have attracted much interest for financial applications in the last decades, because they allow to capture certain features of the market prices that the classical models cannot describe accurately. In particular, the log returns in financial time series exhibit a non-Gaussian character, in that they have a significant skewness and the leptokurtic property, and contain jump components. Therefore a number of option pricing models have been developed adopting more flexible distributions than the Normal distribution. Some remarkable examples of alternative distributions which have been proposed are the Variance Gamma (VG) (Madan and Seneta [21]), the Hyperbolic (H) (Eberlein and Keller [11]), the Normal Inverse Gaussian (NIG) (Barndorff-Nielsen [4]), the more general Generalized Hyperbolic (GH) (Eberlein and Prause [13]), the Meixner process (Schoutens [25]), the four-parameter distribution named CGMY after the names of Carr, German, Madan and Yor [9], which was generalized to a six-parameter distribution in [10]. Since most of the proposed processes belong to the mathematically attractive family of the Lévy processes, a vast literature on option pricing has been developed replacing the traditional underlying source of randomness, the Brownian motion, by a Lévy process. As a consequence new mathematical challenges have been issued with respect to exotic option pricing (see [17], for a compendium of recent research on the topic).

In this paper we adopt the class of regular Lévy processes of exponential type (RLPE) as the driving processes, following [5]. As [5] points out, it is the most tractable subclass of Lévy process from the analytical point of view if the Brownian motion is not available. Their characteristic exponents $\psi(\xi)$ enjoy very favourable properties as symbols of pseudo differential operators, since their real part behave as $c|\xi|^v$, with positive c and v , as $|\xi| \rightarrow \infty$ in the strip of regularity. Thus the integrals appearing in the pricing formulas are absolutely convergent thanks to the terms of the form $e^{-\tau\psi(\xi)}$. Moreover, one can differentiate under the integral sign or

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shift the line of integration by using the Cauchy theorem for holomorphic functions. Such properties allow for a great flexibility of the method when working out the analytical pricing formulas for several exotic option. The aim of this paper is to prove a "quintessential" valuation formula which applies to a wide range of options, thus providing a generalization of the idea of [8] to a Lévy framework. The idea that a broad class of financial derivatives can be evaluated in terms of elementary contracts such as digital options traces back to [20] and has been developed in [8] in a Gaussian environment. By providing the non-Gaussian counterpart of this view, we are able to throw a new insight into some known pricing expressions and, more importantly, to obtain new valuation formulas which have not yet been written down in a non-Gaussian setting. The main result is given in section 3. Then several examples are provided as an application to illustrate the flexibility of the approach. For example, by employing multi-period digitals we price chooser options and compound options; moreover, an extension of the result in [2] for digital options is proved in order to incorporate discrete Asian barrier options. While the latter options have already been priced within a Lévy framework, although through different methods (see the references quoted in Section 4, Ex. 6 and 7), the valuation of chooser and compound options has never been presented by other authors, to the author's knowledge. However the list of exotics studied in this paper is by no means exhaustive with respect to the potentialities of the approach and the author trusts that several other pricing formulas may be obtained from the main formula, thanks to its comprehensiveness.

2. NOTATION

This section outlines the main definitions and the general notation to be used throughout the paper. As usual in a Lévy setting, the classical modeling of stock prices as a geometric Brownian motion is replaced with a geometric Lévy motion, that is, the stock price S_t is assumed to be e^{X_t} , where X_t is a Lévy process. Following [5] we suppose that X_t is one-dimensional RLPE of order $v \in]0, 2]$ and exponential type $[\lambda_-, \lambda_+]$, $\lambda_- < 0 < \lambda_+$, that is, it has a characteristic exponent $\psi(\xi)$ which admits a representation of the form:

$$(2.1) \quad \psi(\xi) = -i\mu\xi + \phi(\xi)$$

where ϕ is holomorphic in the strip $\text{Im } \xi \in]\lambda_-, \lambda_+[$, continuous up to the boundary of the strip, and $\phi(\xi) = C|\xi|^v + O(|\xi|^{v_1})$ for $\xi \rightarrow \infty$ and $|\phi(\xi)| \leq C(1 + |\xi|^{v_2})$ in $\text{Im } \xi \in [\lambda_-, \lambda_+]$ with $v_1, v_2 < v$.

Clearly a Brownian motion is a RLPE of order 2 and any exponential type. We consider a Lévy market, i.e. a model of financial market with a deterministic saving account e^{rt} , $r \geq 0$, and a stock following a stochastic process $S_t = e^{X_t}$. Since we want to price contingent claims on the stock, it is convenient to consider an equivalent martingale measure (EMM) Q which makes the discounted price process $e^{-rt}S_t$ a martingale. Let ψ_P (respectively ψ_Q) denote the characteristic exponent with respect to the historic measure P (an EMM Q , respectively), i.e. $E_P(e^{i\xi X_t}) = e^{-t\psi_P(\xi)}$ ($E_Q(e^{i\xi X_t}) = e^{-t\psi_Q(\xi)}$, respectively). If the discounted stock price is a martingale under Q , then $S_0 = E^Q(e^{-rt}S_t) = S_0 e^{-t[r + \psi_Q(-i)]}$, which holds under the EMM-condition $r + \psi_Q(-i) = 0$. Note that the additional

condition $\lambda_- < -1$, which is usually assumed, is needed to price the stock. Finally, we recall that Q can be constructed by means of Esscher transform, by solving the following equation:

$$(2.2) \quad \psi_P(-ih) - \psi_P(-ih - i) = r$$

for h , and then letting $\psi_Q(\xi) = \psi_P(\xi - ih) - \psi_P(-ih)$. In [5] it is shown that while (2.2) may have no real solution for an arbitrary Lévy process, uniqueness holds for RLPEs and sufficient conditions for existence can be proved. (See Lemma 4.1 in [5]). In what follows we assume that an EMM Q is chosen so that X_t is a RLPE under it and, for simplicity's sake, we will omit the subscript Q both in E and ψ . If $g(X_T)$ denotes the terminal payoff of an option on S_t at the expiry date T , then the no-arbitrage price of the option at the current time t ($t < T$) is given by:

$$F(S_t, t) = E[e^{-r(T-t)}g(X_T) \mid X_t = \ln(S_t)].$$

If $e^{\omega x}g(x) \in L^1(\mathbb{R})$ for some $\omega \in]\lambda_-, \lambda_+[$ then the Fourier transform $\widehat{g}(\xi)$ of g can be defined as usual on $\text{Im } \xi = \omega$ and finally one can write:

$$(2.3) \quad F(S_t, t) = \frac{1}{2\pi} \int_{-\infty+i\omega}^{+\infty+i\omega} e^{i\xi \ln S_t - (T-t)(r+\psi(\xi))} \widehat{g}(\xi) d\xi$$

The arguments above can be extended to multi-dimensional RLPE. An n -valued Lévy process X_t is a RLPE of order $v \in]0, 2]$ if its characteristic exponent $\psi(\xi)$ is of the form:

$$(2.4) \quad \psi(\xi) = -i\mu \cdot \xi + \phi(\xi)$$

where $\mu \in \mathbb{R}^n$, ϕ admits the analytic continuation into a tube domain $\mathbb{R}^n + iU$ (U is a bounded open set with $0 \in U \subset \mathbb{R}^n$), the continuation of ϕ is continuous up to the boundary of the tube domain, $\phi(\xi) = |\xi|^v C(\xi/|\xi|) + O(|\xi|^{v-1})$ for $\xi \rightarrow \infty$ where C is a positively homogeneous degree zero function.

The characteristic exponent under the EMM Q is $\psi_Q(\xi) = \psi_P(\xi - ih) - \psi_P(-ih)$, where $h \in \mathbb{R}^n$ is determined by solving the system $\psi_P(-ih) - \psi_P(-ih - ie_j) = r$, $j = 1, \dots, n$, for h . (Here e_j denotes the vectors whose components are all 0 except the j^{th} one which is 1). If $g(X_T)$ denote the terminal payoff of an option on a basket of assets $S_t = (S_{t1}, \dots, S_{tn})$, then the price of the option at the current time t ($t < T$) is given by

$$F(S_t, t) = \frac{1}{(2\pi)^n} \int_{-\infty+i\omega_1}^{+\infty+i\omega_1} \dots \int_{-\infty+i\omega_n}^{+\infty+i\omega_n} e^{i\xi \cdot \ln S_t - (T-t)(r+\psi(\xi))} \widehat{g}(\xi) d\xi_1 \dots d\xi_n$$

where $(\omega_1, \dots, \omega_n) \in U$ are chosen in order to avoid the points where the analyticity of the integrand fails.

3. SET-UP AND MAIN RESULT

In this section an almost universal option pricing formula is proved within the Lévy framework described in the previous section. In particular we derive the arbitrage-free price for generalised multi-period exotic power digital options. Such options are the building blocks for a broad class of exotic options with a single underlying asset, because several exotic options can be expressed as static portfolios of these multi-period power digitals. The idea of pricing several exotic options

by a single universal formula is due to [8], where the classical Gaussian case was studied. Thus this Section provides the generalisation to the Lévy framework and the resulting formula is a most comprehensive one, both in terms of the kind of option and of the stochastic process driving the underlying asset. Since the aim is to establish a unifying set-up, the formal notation is a bit involved and is laid down along the lines of [8].

Assume that the payoff of an option depends on M fixed asset price monitoring times, $T_1 < \dots < T_M$, where, for simplicity's sake, $T_M = T$, the expiry date of the option. Let \mathbf{T} denote the set of times $[t, T_1, \dots, T_M]$, where t is the current time, $t < T_1$. Let S_k denote the price of the underlying asset at the monitoring time T_k and let \mathbf{S} denote the M -dimensional vector assembling all the components S_k which are relevant for the option under study. If γ is M -dimensional, then \mathbf{S}^γ denotes $S_1^{\gamma_1} \dots S_M^{\gamma_M}$ and γ is referred to as the payoff index vector. Since $S_t = e^{X_t}$, where X_t is a Lévy process, in most cases it will be convenient to work directly with X_k , the value of X at T_k , and with the vector $\mathbf{X} = (X_1, \dots, X_M)$. Since we want to treat call and put options together, we introduce W , a diagonal matrix with all the diagonal entries w_{ii} equal to ± 1 . To simplify notation, w_{ii} will be denoted by w_i . Then both the indicator functions $I_{[K_i, +\infty)}(X_i)$ and $I_{(-\infty, K_i]}(X_i)$ may be encompassed in a unique notation, $\mathbf{1}_1(w_i X_i \geq w_i K_i)$, depending on the sign of w_i . Let \mathbf{K} denote the exercise price vector (of dimension N) and let $\mathbf{1}_N(\mathbf{Y} \geq \mathbf{K})$ denote the N -dimensional indicator function $\prod_{i=1}^N \mathbf{1}_1(Y_i \geq K_i)$. In order to give a greater

flexibility to the approach, the exercise condition matrix is introduced. Such term will denote any $N \times M$ matrix $A = (a_{nk})$ involved in the payoff, where N is the exercise dimension. For example, the payoff of discrete mean Asian options depends on $\sqrt[M]{\prod_{k=1, \dots, M} S_k} \leq \tilde{K}$ or, equivalently, $\sum_{k=1, \dots, M} \frac{1}{M} X_k \leq \ln \tilde{K} = K$. Thus their payoff can be expressed in terms of $\mathbf{1}_1(w_1 A \mathbf{X} \geq w_1 K)$, where $A = [\frac{1}{M}, \dots, \frac{1}{M}]$ is a $1 \times M$ matrix. On the other hand, the payoff of a discretely monitored digital option $\mathbf{1}_M(WX \geq WK)$ can be expressed in the general form $\mathbf{1}_M(WA\mathbf{X} \geq W\mathbf{K})$ by taking A as the $M \times M$ identity matrix. All the useful parameters are summarized in the payoff parameter set $\mathbf{P} = [(\gamma_1, \dots, \gamma_M), \mathbf{K}, W, A]$.

Let $F(S_t, t; \mathbf{T}, \mathbf{P})$ denote the value of a multi-period binary whose payoff is specified in terms of \mathbf{T} and \mathbf{P} and where S_t denotes the value of the underlying asset at the current time t . More specifically, the expiry T payoff function is $\exp[\sum_{k=1}^M \gamma_k X_k] \cdot \mathbf{1}_N(WA\mathbf{X} \geq W\mathbf{K})$. In most applications $\gamma_k = 0$ for $k \neq M$. The following Proposition gives the arbitrage-free current value of this generalised multi-period option under the assumption that the value S_t of the underlying asset S_t is e^{X_t} , where X_t is a RLPE of order $v \in]0, 2]$ and exponential type $[\lambda_-, \lambda_+]$. The main result of the paper is the following:

Proposition 1. *The following valuation formula holds:*

$$F(S_t, t; \mathbf{T}, \mathbf{P}) = \frac{e^{-r(T_M - t)}}{(2\pi i)^N} S_t^{\sum_{k=1}^M \gamma_k} \prod_{n=1}^N w_n \int_{-\infty - iw_N \omega_N}^{+\infty - iw_N \omega_N} \dots \int_{-\infty - iw_1 \omega_1}^{+\infty - iw_1 \omega_1} \frac{1}{\prod_{n=1, \dots, N} \xi_n} \exp\left[i \sum_{n=1}^N \xi_n \left(\sum_{k=1}^M a_{nk} \ln S_t - K_n\right) - \Psi(t, \xi_1, \dots, \xi_N)\right] d\xi_1 \dots d\xi_N$$

where $\Psi(t, \xi_1, \dots, \xi_N) = \sum_{j=1}^M (T_j - T_{j-1}) \psi \left(\sum_{n=1}^N \sum_{k=j}^M a_{nk} \xi_n - i \sum_{k=j}^M \gamma_k \right)$ with $T_0 = t$, $\omega_n > 0$ for $n = 1, \dots, N$ and $\sum_{k=j}^M \left[\sum_{n=1}^N w_n \omega_n a_{nk} + \gamma_k \right] \in] - \lambda_+, -\lambda_- [$ for $j = 1, \dots, M$.

Before proving Proposition 1 we give a pricing formula for the simple case of a power digital option, which is slightly more general than Proposition 1 in [2].

Lemma 1. *Let $F(S_t, t)$ denote the current arbitrage-free price of the power option with expiry T payoff function $S_T^\gamma \mathbf{1}_1(waX_T \geq wK)$, $a \neq 0$, $w = \pm 1$, $X_t = \ln S_t$. Then for any $\omega > 0$ such that $a\omega w + \gamma \in] - \lambda_+, -\lambda_- [$, one can write*

$$(3.1) \quad F(S_t, t) = \frac{wS_t^\gamma}{2\pi i} \int_{-\infty - i\omega}^{+\infty - i\omega} e^{i\xi[a \ln S_t - K] - (T-t)[r + \psi(a\xi - i\gamma)]} \frac{1}{\xi} d\xi$$

Proof. The Fourier transform of the payoff function $g(X) = e^{\gamma X} \mathbf{1}_1(waX_T \geq wK)$ is $\hat{g}(\eta) = \frac{w \cdot \text{sgn}(a)}{i(\eta + i\gamma)} \exp[(\gamma - i\eta) \frac{K}{a}]$ with $\text{Im } \eta = -w\omega_\gamma$ for any ω_γ such that $\gamma - w\omega_\gamma \geq 0$ whenever $aw \leq 0$ and $w\omega_\gamma \in] - \lambda_+, -\lambda_- [$. Then:
 $F(S_t, t) = \frac{w \cdot \text{sgn}(a)}{2\pi i} \int_{-\infty - i\omega_\gamma}^{+\infty - i\omega_\gamma} e^{i\eta[\ln(S_t) - \frac{K}{a}] - (T-t)(r + \psi(\eta))} \frac{1}{\eta + i\gamma} d\eta$
 which gives (3.1) changing to variables $\eta + i\gamma = a\xi$ and letting $\omega_\gamma - w\gamma = a\omega$.

Remark 1. *The expression (3.1) can be rewritten in terms of pseudo differential operators as follows:*

$F(S_t, t) = wS_t^\gamma \exp[-(T-t)(r + \psi(aD_x - i\gamma))] I^{(w)}(\ln \frac{S_t}{K})$
 where $I^{(w)}$ denotes the indicator function $I^{(w)}(x) = I_{[0, +\infty)}(wx)$ and the notation $P(D_x)$ denotes a pseudo differential operator whose symbol is $P(\xi)$. Alternatively, $F(S_t, t) = f(X_t, t)$ can be viewed as the solution to the pseudo differential equation: $[\partial_t - (r + \psi(D_x))]f(X_t, t) = 0$ with the final condition $f(X_T, T) = e^{\gamma X_T} \mathbf{1}_1(waX_T \geq wK)$.

Lemma 2. *Let $F(S_t, t)$ denote the current arbitrage-free price of the power option with expiry T payoff function $S_T^\gamma \mathbf{1}_N(w_n a_n X_T \geq w_n K_n; n = 1, \dots, N)$, $a_n \neq 0$, $w_n = \pm 1$, $X_t = \ln S_t$. Then, for any $\omega_n > 0$ such that $\sum_{n=1}^N w_n \omega_n a_{nk} + \gamma \in] - \lambda_+, -\lambda_- [$, one can write*

$$(3.2) \quad F(S_t, t) = \frac{e^{-r(T-t)}}{(2\pi i)^N} S_t^\gamma \prod_{n=1}^N w_n \int_{-\infty - i\omega_n \omega_n}^{+\infty - i\omega_n \omega_n} \dots \int_{-\infty - i\omega_1 \omega_1}^{+\infty - i\omega_1 \omega_1} \frac{1}{\prod_{n=1, \dots, N} \xi_n}.$$

$$(3.3) \quad \exp\left[i \sum_{n=1}^N \xi_n (a_n X_t - K_n) - \Psi(t, \xi_1, \dots, \xi_N)\right] d\xi_1 \dots d\xi_N$$

where $\Psi(t, \xi_1, \dots, \xi_N) = (T-t) \psi \left(\sum_{n=1}^N a_n \xi_n - i\gamma \right)$

Proof. The result follows by writing the Fourier transform of the payoff as a convolution of N terms and by arguing as in Lemma 1.

Proof of Proposition 1. Let us first prove the case $N = 1$. Let $\mathbf{P} = [(\gamma_1, \dots, \gamma_M), K, w, (a_1, \dots, a_M)]$ with $w = \pm 1$. For $m = 1, \dots, M$ let $K_m^* = K - \sum_{k=1}^{m-1} a_k X_k$, $\gamma_m^* = \sum_{k=1}^{m-1} \gamma_k X_k$ ($K_1^* = K$, $\gamma_1^* = 0$). Let $f_m(X_t, t)$ solve $\partial_t f_m - (r + \psi(D_x))f_m = 0$ for $t \in [T_{m-1}, T_m]$, with $f_m(T_m, X_m) = f_{m+1}(T_m, X_m)$ for $m < M$, and $f_M(T_M, X_M) = e^{\gamma_M X_M + \gamma_M^*} \mathbf{1}_{(w a_M X_M \geq w K_M^*)}$. In view of Lemma 1 and Remark 1 one has:

$$f_M(X_t, t) = \frac{w e^{\gamma_M X_t + \gamma_M^*}}{2\pi i} \int_{-\infty - i\omega w}^{+\infty - i\omega w} e^{i\xi [a_M X_t - K_M^*] - (T_M - t)[r + \psi(a_M \xi - i\gamma_M)]} \frac{1}{\xi} d\xi.$$

Then one can prove recursively that for any m :

$$f_m(X_t, t) = \frac{w e^{-r(T_M - t)} \exp(\gamma_h^* + \sum_{k=m}^M \gamma_k)}{2\pi i} \int_{-\infty - i\omega w}^{+\infty - i\omega w} \exp[i\xi (\sum_{k=m}^M a_k X_t - K_m^*) - \Psi_m(t, \xi)] \frac{1}{\xi} d\xi$$

where $\Psi_m(t, \xi) = \sum_{j=m}^M (T_j - T_{j-1}) \psi(\xi \sum_{k=j}^M a_k - i \sum_{k=j}^M \gamma_k)$ with $T_{m-1} = t$, $\omega > 0$ and

$$\sum_{k=j}^M [w \omega a_k + \gamma_k] \in] - \lambda_+, -\lambda_- [\quad \text{for } j = m, \dots, M. \quad \text{Thus } h = 1 \text{ yields the result}$$

in the case $N = 1$. Finally, the general case is proved along the same lines, by employing Lemma 2 and arguing recursively.

4. Examples

This section presents some option pricing formulas which are obtained as applications of Proposition 1. While Examples 1, 2 and 3 refer to simple options which have been already priced within a Lévy model, although throughout a slightly different method, the analytical valuation expressions presented in Examples 4 and 5 are new. The options studied in Examples 6 and 7 are included in the existing literature on Lévy processes, but the argument we employ to prove the valuation formulas is different from those used in the quoted literature. The aim of this section is to show how several kinds of options can be priced directly within a single framework and thus one does not need to devise an *ad hoc* method for each exotic option.

1) *Vanilla European options.* The pricing formula is well-known in the literature on Lévy processes and can be obtained by straightforward application of Lemma 1. Indeed, the payoff is $\max(wS_T - wK, 0) = wS_T I_{[wK, +\infty)}(wS_T) - wK I_{[wK, +\infty)}(wS_T)$ and Lemma 1, with $a = 1$ and $\gamma = 0$ or $\gamma = 1$, yields the following expression for the current price of the option:

$$F(S_t, t) = \frac{wS_t}{2\pi i} \int_{-\infty - i\omega w}^{+\infty - i\omega w} e^{i\xi \ln(S_t/K) - (T-t)(r + \psi(\xi - i))} \frac{1}{\xi} d\xi - \frac{wK}{2\pi} \int_{-\infty - i\omega w}^{+\infty - i\omega w} e^{i\xi \ln(S_t/K) - (T-t)(r + \psi(\xi))} \frac{1}{\xi} d\xi$$

for any $\omega \in]0, -\lambda_- - 1[$ if $w = 1$ ($\omega \in]0, \lambda_+[$ if $w = -1$). Changing to variables $\xi - i = \xi'$ in the first integral and then shifting upwards (from $\text{Im } \xi = -w\omega - 1$ to $\text{Im } \xi = -w\omega$) the line of integration, one gets

$$F(S_t, t) = -\frac{K}{2\pi} \int_{-\infty - i\omega w}^{+\infty - i\omega w} e^{i\xi \ln(S_t/K) - (T-t)(r + \psi(\xi))} \frac{1}{\xi(\xi + i)} d\xi$$

which are the known formulas (compare (4.31) and (4.32) in [5]).

2) *Power digitals.* Power digital are valued taking $a = 1$ in Lemma 1. Let us see how a Ingersoll's remark on power digitals (see [20]) generalizes to a Lévy environment. Ingersoll observes that a power option in the Gaussian framework can be valued by simply replacing S_t with S_t^γ , the volatility σ with $\gamma\sigma$ and subtracting $(1 - \gamma)(r + \gamma\sigma^2/2)$ from the drift term in the corresponding asset-or-nothing formula

(see (16) in [20]). This remark can be transferred to the Lévy setting as follows. Fix $\gamma \geq 1$. Changing to variables $\xi = \gamma\eta$ in the integral (3.1) with $a = 1$ and denoting ω/γ by ω' , the pricing formula for the power digital can be written as:

$$F_\gamma(S_t, t) = \frac{wS_t^\gamma}{2\pi} \int_{-\infty - i\omega\omega'}^{+\infty - i\omega\omega'} e^{i\eta \ln(S_t^\gamma/K) - (T-t)(r + \psi_\gamma(\eta - i))} \frac{1}{i\eta} d\eta$$

where $\psi_\gamma(\eta) = \psi(\gamma\eta)$ and $\omega' \in]0, -\lambda_- - \gamma[$ ($]0, \lambda_+[$, respectively) if $w = 1$ ($w = -1$, respectively). In other words, the value of the power digital, $F_\gamma(S_t, t)$, can be straightforwardly obtained from the price of the corresponding asset-or-nothing option by simply replacing S_t with S_t^γ and ψ with ψ_γ . Note that, in the Gaussian case $\psi(\xi) = -i(r - \frac{\sigma^2}{2})\xi + \frac{\sigma^2\xi^2}{2}$, one has $\psi_\gamma(\xi) = -i(r - q_\gamma - \frac{\gamma^2\sigma^2}{2})\xi + \frac{\gamma^2\sigma^2\xi^2}{2}$ with $q_\gamma = (1 - \gamma)(r + \gamma\sigma^2/2)$, in keeping with [20].

3) *Self-quanto options.* The payoff is $\max[w(S_T - K, 0)]S_T$, where the binary indicator w represents a call (put) for $w = 1$ ($w = -1$). From Lemma 1 one gets:

$$F(S_t, t) = \frac{wS_t^2}{2\pi i} \int_{-\infty - i\omega'w}^{+\infty - i\omega'w} e^{i\xi' \ln(S_t/K) - (T-t)(r + \psi(\xi' - 2i))} \frac{1}{\xi'} d\xi' - \frac{wS_t K}{2\pi} \int_{-\infty - i\omega''w}^{+\infty - i\omega''w} e^{i\xi'' \ln(S_t/K) - (T-t)(r + \psi(\xi'' - i))} \frac{1}{\xi''} d\xi''$$

for any positive ω', ω'' such that $\omega'w + 2$ and $\omega''w + 1 \in]-\lambda_+, -\lambda_-[$. Changing to variables $\xi' - 2i = \xi$ in the first integral and $\xi'' - i = \xi$ in the second integral, and then shifting upwards the line of integration, one gets

$$F(S_t, t) = -\frac{K^2}{2\pi} \int_{-\infty - i\omega}^{+\infty - i\omega} e^{i\xi[\ln(S_t) - \ln(K)] - (T-t)(r + \psi(\xi))} \frac{1}{(\xi+i)(\xi+2i)} d\xi,$$

where $\omega \in]2, \lambda_w[$, $\lambda_w = -\lambda_-$ for $w = 1$ and $\lambda_w = \lambda_+ + 2$ for $w = -1$. This is in keeping with the relationship between the formula for a general strike price K and that for $K = 1$ obtained in [22], p.68.

4) Chooser options

A chooser option gives its holder the right to decide at a prespecified time (choice date = T_1) before the maturity T whether he/she would like the option to be a call or a put option. As a straightforward application of our main Proposition (with $A = I$) we give a valuation formula for simple chooser options, i.e. the call and the put have the same strike price K and maturity date T . Note that the decision whether the option is a call or a put depends on the value of:

$$\max(C(S_{T_1}; K, T), P(S_{T_1}; K, T)) = C(S_{T_1}; K, T) + \max(Ke^{-r(T-T_1)} - S_{T_1}, 0).$$

In other words the choice is:

$$\text{Call} \iff Ke^{-r(T-T_1)} < S_{T_1}; \quad \text{Put} \iff Ke^{-r(T-T_1)} > S_{T_1}.$$

The payoff can be expressed as:

$$(S_T - K)\mathbf{1}_2(S_T > K, S_{T_1} > Ke^{-r(T-T_1)}) + (K - S_T)\mathbf{1}_2(S_T < K, S_{T_1} < Ke^{-r(T-T_1)})$$

Then, in view of Proposition 1, with $N = 1$ and $M = 2$, the price for the simple chooser option at time $t < T_1$ can be written in the form:

$$F(S_t, t) = A_1 - K * A_2 + K * A_3 - A_4$$

where the following choice are made in Proposition 1 for each term:

$$\begin{aligned} \text{for } A_1 : \mathbf{K} &= (Ke^{-r(T-T_1)}, K) & \gamma &= (0, 1) & (w_1, w_2) &= (1, 1) \\ \text{for } A_2 : \mathbf{K} &= (Ke^{-r(T-T_1)}, K) & \gamma &= (0, 0) & (w_1, w_2) &= (1, 1) \\ \text{for } A_3 : \mathbf{K} &= (Ke^{-r(T-T_1)}, K) & \gamma &= (0, 0) & (w_1, w_2) &= (-1, -1) \\ \text{for } A_4 : \mathbf{K} &= (Ke^{-r(T-T_1)}, K) & \gamma &= (0, 1) & (w_1, w_2) &= (-1, -1). \end{aligned}$$

Then

$$A_1 = \frac{S_t e^{-r(T-t)}}{(2\pi i)^2} \int_{-\infty - i\omega_2}^{+\infty - i\omega_2} \int_{-\infty - i\omega_1}^{+\infty - i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K} + r(T-T_1) + i\xi_2 \ln \frac{S_t}{K} - \Psi(t, \xi_1, \xi_2 - i))} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2$$

In view of the residue theorem A_1 becomes:

$$= A_4 + \frac{S_t e^{-r(T-t)}}{2\pi i} \int_{-\infty-i\omega_2}^{+\infty-i\omega_2} e^{i\xi_2 \ln \frac{S_t}{K} - (T-t)\psi(\xi_2-i)} \frac{1}{\xi_2} d\xi_2 +$$

$$+ \frac{S_t e^{-r(T-t)}}{2\pi i} \int_{-\infty+i\omega_1}^{+\infty+i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K} + r(T-T_1) - (T_1-t)\psi(\xi_1-i))} \frac{1}{\xi_1} d\xi_1$$

On the other hand

$$A_2 = \frac{e^{-r(T-t)}}{(2\pi i)^2} \int_{-\infty+i\omega_2}^{+\infty+i\omega_2} \int_{-\infty+i\omega_1}^{+\infty+i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K} + r(T-T_1) + i\xi_2 \ln \frac{S_t}{K} - \Psi(t, \xi_1, \xi_2))} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2$$

which under the residue theorem is transformed into:

$$= A_3 + \frac{e^{-r(T-t)}}{2\pi i} \int_{-\infty+i\omega_2}^{+\infty+i\omega_2} e^{i\xi_2 \ln \frac{S_t}{K} - (T-t)\psi(\xi_2)} \frac{1}{\xi_2} d\xi_2 +$$

$$+ \frac{e^{-r(T-t)}}{2\pi i} \int_{-\infty-i\omega_1}^{+\infty-i\omega_1} e^{i\xi_1 \ln(\frac{S_t}{K} + r(T-T_1) - (T_1-t)\psi(\xi_1))} \frac{1}{\xi_1} d\xi_1$$

Thus the formula is simplified because the double integrals cancel out. In the Gaussian case the formula becomes

$$F(S_t, t) = S[N(d_T^+) - N(-d_{T, T_1}^+)] - K e^{-r(T-t)} [N(d_T^-) - N(-d_{T, T_1}^-)]$$

with $d_T^\pm = (\ln(S_t/K) + (r \pm \frac{\sigma^2}{2})(T-t))/(\sigma\sqrt{T-t})$
 $d_{T, T_1}^\pm = d_{T_1}^\pm + r(T-T_1)/(\sigma\sqrt{T_1-t})$

which is the price for a chooser option obtained by Rubinstein [24].

5) *Compound options.* Let $F_2(S_t, t; w_1, w_2; T_1, T_2; K_1, K_2)$ denote the current value of a European compound option. In particular, for $w_1 = w_2 = 1$, we obtain a call on call, for $w_1 = w_2 = -1$ a put on put, for $w_1 = 1, w_2 = -1$ a call on put and for $w_1 = -1, w_2 = 1$ a put on a call. Suppose that the expiration time of the underlying option is $T_2 > t$, its strike price K_2 and its underlying asset follows a RLPE, as it is assumed throughout the whole paper. Let $T_1 < T_2$ be the expiration time of the compound option and K_1 its strike price. Let $F_1(S_t, t; w_2; T_2; K_2)$ denote the value of the underlying option. In view of Example 3 in [2] one can state that the solution S^* to $F_1(S_t, t; w_2; T_2; K_2) = K_1$ is unique whenever exists. The payoff of the compound option can be written as:

$$w_1 w_2 S_{T_2} \mathbf{1}_2(w_2 S_{T_2} \geq w_2 K_2, w_1 S_{T_1} \geq w_1 S^*) - w_1 w_2 K_2 \mathbf{1}_2(w_2 S_{T_2} \geq w_2 K_2, w_1 S_{T_1} \geq w_1 S^*) - w_1 K_1 \mathbf{1}_1(w_1 S_{T_1} \geq w_1 S^*)$$

and therefore our method applies. The current value of the last term is obtained by straightforward application of Lemma 1 with $a = 1$, while the current values of the other terms require Proposition 1 with $N = 1$, $M = 2$, $A = I$, $\mathbf{K} = (S^*, K_2)$, $\gamma = (0, 1)$ for the first term and $\gamma = (0, 0)$ for the second term. Then we obtain:

$$F(S_t, t) = \frac{S_t e^{-r(T_2-t)}}{(2\pi i)^2} \int_{-\infty-iw_2\omega_2}^{+\infty-iw_2\omega_2} \int_{-\infty-iw_1\omega_1}^{+\infty-iw_1\omega_1} e^{i\xi_1 \ln \frac{S_t}{S^*} + i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, \xi_1, \xi_2 - i)} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2 -$$

$$\frac{K_2 e^{-r(T_2-t)}}{(2\pi i)^2} \int_{-\infty-iw_2\omega_2}^{+\infty-iw_2\omega_2} \int_{-\infty-iw_1\omega_1}^{+\infty-iw_1\omega_1} e^{i\xi_1 \ln \frac{S_t}{S^*} + i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, \xi_1, \xi_2)} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2 -$$

$$\frac{K_1 e^{-r(T_1-t)}}{2\pi} \int_{-\infty-iw_1\omega_1}^{+\infty-iw_1\omega_1} e^{i\xi_1 \ln \frac{S_t}{S^*} - (T_1-t)\psi(\xi_1)} \frac{1}{i\xi_1} d\xi_1$$

for $\omega_1, \omega_2 \in]0, -\lambda_- - 1[$. In the Gaussian case, if we change variables $\xi_j \sigma \sqrt{T_j - t} = \eta_j$, let ω_1, ω_2 be any positive value and denote $(\ln \frac{S_t}{S^*} + (r \pm \frac{\sigma^2}{2})(T_1 - t))/(\sigma\sqrt{T_1 - t})$ by d_1^\pm , $(\ln \frac{S_t}{K_2} + (r \pm \frac{\sigma^2}{2})(T_2 - t))/(\sigma\sqrt{T_2 - t})$ by d_2^\pm and $\sqrt{\frac{T_1-t}{T_2-t}}$ by ρ , our formula collapses into:

$$F(S_t, t) = \frac{S_t}{(2\pi i)^2} \int_{-\infty-iw_2\omega_2}^{+\infty-iw_2\omega_2} \int_{-\infty-iw_1\omega_1}^{+\infty-iw_1\omega_1} e^{i\eta_1 d_1^+ + i\eta_2 d_2^+ - \frac{1}{2}(\eta_1^2 + \eta_2^2 + 2\rho\eta_1\eta_2)} \frac{1}{\eta_1\eta_2} d\eta_1 d\eta_2 -$$

$$\frac{K_2 e^{-r(T_2-t)}}{(2\pi i)^2} \int_{-\infty-iw_2\omega_2}^{+\infty-iw_2\omega_2} \int_{-\infty-iw_1\omega_1}^{+\infty-iw_1\omega_1} e^{i\eta_1 d_1^- + i\eta_2 d_2^- - \frac{1}{2}(\eta_1^2 + \eta_2^2 + 2\rho\eta_1\eta_2)} \frac{1}{\eta_1\eta_2} d\eta_1 d\eta_2 -$$

$$w_1 K_1 e^{-r(T_1-t)} N(w_1 d_1^-).$$

Finally we get Geske's formula for compound options:

$$w_1 w_2 S_t N_2(w_1 d_1^+, w_2 d_2^+, w_1 w_2 \rho) - w_1 w_2 K_2 e^{-r(T_2-t)} N_2(w_1 d_1^-, w_2 d_2^-, w_1 w_2 \rho) - w_1 K_1 e^{-r(T_1-t)} N(w_1 d_1^-)$$

if we apply the following:

Lemma 3. *The following identity holds for $w_i = \pm 1$ and for any $\omega_i > 0$, $i = 1, 2$:*

$$\frac{1}{(2\pi i)^2} \int_{-\infty - iw_2 \omega_2}^{+\infty - iw_2 \omega_2} \int_{-\infty - iw_1 \omega_1}^{+\infty - iw_1 \omega_1} e^{i\eta_1 h_1 + i\eta_2 h_2 - \frac{1}{2}(\eta_1^2 + \eta_2^2 + 2\rho\eta_1\eta_2)} \frac{1}{\eta_1 \eta_2} d\eta_1 d\eta_2 = w_1 w_2 N_2(w_1 h_1, w_2 h_2, w_1 w_2 \rho).$$

Proof. The term on the left can be written as

$$\frac{w_1 w_2}{(2\pi i)^2} \int_{-\infty - iw_2}^{+\infty - iw_2} \int_{-\infty - iw_1}^{+\infty - iw_1} e^{i\xi_1 \widehat{h}_1^- + i\xi_2 \widehat{h}_2^- - \frac{1}{2}(\xi_1^2 + \xi_2^2 + 2\widehat{\rho}\xi_1\xi_2)} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2$$

with $\widehat{h}_i = w_i h_i$ and $\widehat{\rho} = w_1 w_2 \rho$. Recall that the Fourier transform of $\exp(-\frac{1}{2} \langle AX, X \rangle)$, where A is a symmetric non singular $n \times n$ matrix and $\langle \cdot, \cdot \rangle$ is the inner product in \mathbb{R}^n , is $(2\pi)^{\frac{n}{2}} (\det B)^{\frac{1}{2}} \exp(-\frac{1}{2} \langle B\xi, \xi \rangle)$ with $B = A^{-1}$. Thus the term can be also written as:

$$\frac{w_1 w_2}{2\pi \sqrt{1-\widehat{\rho}^2}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\frac{1}{2} \langle AX, X \rangle} 1_{[0,+\infty)}(\widehat{h}_1 - x_1) 1_{[0,+\infty)}(\widehat{h}_2 - x_2) dx_1 dx_2$$

with $A = \frac{1}{1-\widehat{\rho}^2} \begin{pmatrix} 1 & -\widehat{\rho} \\ -\widehat{\rho} & 1 \end{pmatrix}$, which is $w_1 w_2 N_2(\widehat{h}_1, \widehat{h}_2, \widehat{\rho})$.

Now we are going to obtain the put-call parity relationship starting from our formula.

$$\begin{aligned} F_2(S_t, t; 1, K_1, T_1; w_2, K_2, T_2) &= \\ &= \frac{S_t e^{-r(T_2-t)}}{(2\pi i)^2} \int_{-\infty - iw_2 \omega_2}^{+\infty - iw_2 \omega_2} \int_{-\infty - iw_1}^{+\infty - iw_1} e^{i\xi_1 \ln \frac{S_t}{S_*} + i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, \xi_1, \xi_2 - i)} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2 - \\ &\frac{K_2 e^{-r(T_2-t)}}{(2\pi i)^2} \int_{-\infty - iw_2 \omega_2}^{+\infty - iw_2 \omega_2} \int_{-\infty - iw_1}^{+\infty - iw_1} e^{i\xi_1 \ln \frac{S_t}{S_*} + i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, \xi_1, \xi_2)} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2 - \\ &\frac{K_1 e^{-r(T_1-t)}}{2\pi} \int_{-\infty - iw_1}^{+\infty - iw_1} e^{i\xi_1 \ln \frac{S_t}{S_*} - (T_1-t)\psi(\xi_1)} \frac{1}{i\xi_1} d\xi_1. \end{aligned}$$

Let us shift the line of integration $\text{Im } \xi_1 = \omega_1$ up. Since we cross the pole at $\xi_1 = 0$, the residue theorem gives:

$$\begin{aligned} F_2(S_t, t; 1, K_1, T_1; w_2, K_2, T_2) &= \\ &\frac{S_t e^{-r(T_2-t)}}{(2\pi i)^2} \int_{-\infty - iw_2 \omega_2}^{+\infty - iw_2 \omega_2} \int_{-\infty + i\omega_1}^{+\infty + i\omega_1} e^{i\xi_1 \ln \frac{S_t}{S_*} + i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, \xi_1, \xi_2 - i)} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2 - \\ &\frac{K_2 e^{-r(T_2-t)}}{(2\pi i)^2} \int_{-\infty - iw_2 \omega_2}^{+\infty - iw_2 \omega_2} \int_{-\infty + i\omega_1}^{+\infty + i\omega_1} e^{i\xi_1 \ln \frac{S_t}{S_*} + i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, \xi_1, \xi_2)} \frac{1}{\xi_1 \xi_2} d\xi_1 d\xi_2 - \\ &\frac{K_1 e^{-r(T_1-t)}}{2\pi} \int_{-\infty + i\omega_1}^{+\infty + i\omega_1} e^{i\xi_1 \ln \frac{S_t}{S_*} - (T_1-t)\psi(\xi_1)} \frac{1}{i\xi_1} d\xi_1 + \\ &+ 2\pi i \left[\frac{S_t e^{-r(T_2-t)}}{(2\pi i)^2} \int_{-\infty - iw_2 \omega_2}^{+\infty - iw_2 \omega_2} e^{i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, 0, \xi_2 - i)} \frac{1}{\xi_2} d\xi_2 - \right. \\ &\left. \frac{K_2 e^{-r(T_2-t)}}{(2\pi i)^2} \int_{-\infty - iw_2 \omega_2}^{+\infty - iw_2 \omega_2} e^{i\xi_2 \ln \frac{S_t}{K_2} - \Psi(t, 0, \xi_2)} \frac{1}{\xi_2} d\xi_2 - \frac{K_1 e^{-r(T_1-t)}}{2\pi i} e^{-(T_1-t)\psi(0)} \right] = \\ &F_2(S_t, t; -1, K_1, T_1; w_2, K_2, T_2) + F_1(S_t, t; w_2, K_2, T_2) - K_1 e^{-r(T_1-t)}. \end{aligned}$$

We can summarize the result on the put-call parity for compound options as follows. The value of a call on an option equals the value of the corresponding put on the same option plus the value of the underlying option diminished by $K_1 e^{-r(T_1-t)}$, where K_1 and T_1 are the strike price and the maturity date of the compound option.

6) *Asian options.* Asian options under Lèvy processes have been priced in a number of papers (see [3], [15], [17]). In this subsection we show how a pricing formula for geometric Asian options easily obtains from our general result. At first we consider discrete Asian options - whose payoff depends on a discrete average of the asset price at N monitoring times, $T_1 < \dots < T_M$ - which are the most popular in the trading practice. The continuous average case is obtained as a limit. Consider a forward-start fixed strike Asian option with strike price K . Note that a pricing expression for floating Asian options is easily obtained in view of the symmetry relationship proved in [12]. Let $T = T_M$ be the maturity date and let $T' = T_1$ be the time at which the averaging starts. The payoff is $\max(w[\Sigma_M - K], 0)$, where $\Sigma_M = (\prod_{j=1, \dots, M} S_{T_j})^{\frac{1}{M}}$ and w is the binary indicator. In terms of $X_t = \ln(S_t)$ the payoff is:

$$w \prod_{j=1, \dots, M} e^{\frac{1}{M} X_{T_j}} \mathbf{1}_1(w \mathbf{A} \mathbf{X} \geq w \ln(K)) - K \mathbf{1}_1(w \mathbf{A} \mathbf{X} \geq w \ln(K)), \text{ where } \mathbf{A} = [\frac{1}{M}, \dots, \frac{1}{M}].$$

Thus Proposition 1 applies with $N = 1$ and yields the following valuation formula, after some algebraic manipulation:

$$F(S_t, t) = -\frac{wKe^{-r(T_M-t)}}{2\pi} \int_{-\infty-iw\omega}^{+\infty-iw\omega} \frac{1}{\xi(\xi+i)} \exp[i\xi \ln \frac{S_t}{K} - \sum_{j=1}^M (T_j - T_{j-1})\psi(\xi \frac{M-j}{M})] d\xi$$

with $T_0 = t$, $\omega \in]1, -\lambda_-[$ ($]0, \lambda_+[$) if $w = 1$ ($w = -1$). Note that the analogous expression obtained in [15], (10), is derived throughout a different argument, that is considering the distribution of $\ln(\Sigma_M)$.

The pricing formula for the continuous-time monitoring case, where the geometric average is $\exp[\frac{1}{T-T'} \int_{T'}^T \ln(S_t) dt]$, follows from the discrete pricing formula just letting $M \rightarrow \infty$. Note that the limit can be computed under the integral sign in view of the nice behaviour of ψ . In particular, for the continuous case, one has:

$$F(S_t, t) = -\frac{wKe^{-r(T-t)}}{2\pi} \int_{-\infty-iw\omega}^{+\infty-iw\omega} \frac{1}{\xi(\xi+i)} \exp[i\xi \ln \frac{S_t}{K} - \int_0^1 \psi(\xi(1-y)) dy] d\xi.$$

Let us now see that our formula collapses to the know valuation formula for discretely monitored Asian options in the Gaussian case (see [27]). Let h denote the averaging frequency, that is, $T_j = T - (M-j)h$, $j = 1, \dots, M$. Then, in the Gaussian case, $\sum_{j=1}^M (T_j - T_{j-1})\psi(\xi \frac{M-j}{M}) = -ih(r - \frac{\sigma^2}{2})\xi \frac{M-1}{2} + h\frac{\sigma^2}{2}\xi^2 \frac{(M-1)(2M-1)}{6M}$,

because $\sum_{j=1}^M (M-j) = \frac{M(M-1)}{2}$ and $\sum_{j=1}^M (M-j)^2 = \frac{M(M-1)(2M-1)}{6}$. Thus

$$F(S_t, t) = \frac{wKe^{-r(T-t)}}{2\pi i} \int_{-\infty-iw\omega}^{+\infty-iw\omega} \exp[i\xi(\ln \frac{S_t}{K} + \frac{1}{2}(r - \frac{\sigma^2}{2})(T-T')) - (T-T')\frac{\sigma^2}{2}\xi^2 \frac{2M-1}{6M}] \cdot (\frac{1}{\xi+i} - \frac{1}{\xi}) d\xi$$

for any $\omega > 0$. Splitting the integral into two integrals and changing variables, one gets:

$$F(S_t, t) = \frac{wS_t e^{-\beta}}{2\pi i} \int_{-\infty-iw\omega}^{+\infty-iw\omega} \exp[i\eta D^+ - \frac{\sigma^2}{2}\eta^2] \frac{1}{\eta} d\eta - \frac{wKe^{-r(T-t)}}{2\pi i} \int_{-\infty-iw\omega}^{+\infty-iw\omega} \exp[i\eta D^- - \frac{\sigma^2}{2}\eta^2] \frac{1}{\eta} d\eta$$

where $D^- = [\ln \frac{S_t}{K} + \frac{1}{2}(r - \frac{\sigma^2}{2})(T-T')]/(\sigma\sqrt{T-T'}\sqrt{\frac{2M-1}{6M}})$,

$D^+ = D^- + \sigma\sqrt{T-T'}\sqrt{\frac{2M-1}{6M}}$ and $\beta = r(T-t) + (r + \sigma^2(\frac{1}{2} - \frac{2M-1}{6M}))\frac{T-T'}{2}$.

Finally we point out that Proposition 1 straightforwardly applies to the more general flexible geometric Asian options, where the flexible geometric average is

$\prod_{j=1, \dots, M} S_{T_j}^{\theta_j}$ with $\theta_j = \theta(j) / \sum_{j=1}^M \theta(j)$ and θ any non-negative function (see [27] for the Gaussian case). The following expression obtains:

$$F(S_t, t) = -\frac{wKe^{-r(T_M-t)}}{2\pi} \int_{-\infty-iw\omega}^{+\infty-iw\omega} \frac{1}{\xi(\xi+i)} \exp[i\xi \ln \frac{S_t}{K} - \sum_{j=1}^M (T_j - T_{j-1})\psi(\xi \sum_{k=j}^M \theta_k)] d\xi$$

with $T_0 = t$, $\omega \in]1, -\lambda_-[$ ($]0, \lambda_+[$) if $w = 1$ ($w = -1$).

7) *Discrete barrier options.* Most analytical pricing formulas for barrier options assume continuous monitoring of the barrier, while in practice the barrier is normally monitored only at discrete points in time (e.g., at the close of the market). A discrete barrier option is either knocked in or knocked out if the price of the underlying asset is across the barrier at the time it is monitored. In the Gaussian case the pricing formulas have been studied by [7], [16], [18], [8]; in the Lévy process models an interesting survey is presented in [19], where the novel method of [14] is also discussed. (See also [23] for a numerical approach). In this subsection we derive a valuation formula for a discrete barrier option as a further straightforward application of Proposition 1. While there exists eight barrier options types, depending on the barrier knocking in or out, on the barrier being above or below the initial value of spot (up or down) and on the call/put attribute, we confine ourselves to a down-and-out call without rebate. The other cases can be treated similarly.

Let B denote the level of the barrier and suppose that the underlying asset is monitored at times T_j , $j = 1, \dots, M-1$ before the option expiry T_M . The payoff is $(S_{T_M} - K)\mathbf{1}_M(S_{T_j} > B, j = 1, \dots, M-1; S_{T_M} \geq K)$. Therefore Proposition 1 with $N = M$ and $A = I$, the $M \times M$ identity matrix, yields:

$$F(S_t, t) = \frac{Ke^{-r(T_M-t)}}{(2\pi i)^M} \int_{-\infty-i\omega_N}^{+\infty-i\omega_N} \dots \int_{-\infty-i\omega_1}^{+\infty-i\omega_1} \frac{1}{(\xi_M+i) \prod_{k=1, \dots, M-1} \xi_k}$$

$$\cdot \exp[i \sum_{j=1}^{M-1} \xi_j \ln \frac{S_t}{B} + i\xi_M \ln \frac{S_t}{K} - \sum_{j=1}^M (T_j - T_{j-1})\psi(\sum_{k=j}^M \xi_k)] d\xi_1 \dots d\xi_M$$

with $T_0 = t$, $\omega_M \in]1, -\lambda_-[$, $\omega_j > 0$ and $\sum_{j=1}^M \omega_j < -\lambda_-$.

5. CONCLUDING REMARKS

This paper presents a unified formula to price several exotic options in Lévy process-based models. Our method is based on the approach in [5] which employs Fourier transform in the complex plane. In this paper the method is extended to the sequential evaluation of such integrals. Alternatively, one may view the result as the solution to sequential pseudo-differential equations whose symbol is related to the characteristic exponent of the underlying process. As [5] shows, the analytical method gives rise to a new numerical method (integration-along-cut method) which performs better than the Fast Fourier Transform in many cases. Thus our result is of interest also from a numerical point of view.

Thanks to the unifying spirit of our approach, several types of options can be priced directly and do not need to devise a specific method for each of them. The formula is tailored to encompass discretely monitored options, which are of importance due to regulatory and practical issues. However, the continuous counterpart can be deduced in some cases (see Example 6). We have chosen to present some examples which are already known in the literature, though in a different form,

and to give only two new valuation formulas (Examples 4 and 5). Several other exotic options might be priced throughout our main formula, for example, discrete lookback options, one-clique options, installment options, complex chooser options and all options with a compound feature (see [1], for example). Thus we hope that our approach might be useful for several applications.

A generalisation of the model to multi-asset options is left to future research.

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