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HAMILTONIAN POTENTIAL FUNCTIONS FOR DIFFERENTIAL GAMES

Hamiltonian Potential Functions for Differential Games

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Abstract

We introduce the concept of Hamiltonian potential functions for non-cooperative open-loop differential games and we characterise sufficient conditions for their existence. We also identify a class of games admitting a Hamiltonian potential and illustrate the related properties of their dynamic structure. Possible similarities with the theory of quasi-aggregative games are discussed. As an illustration, we consider an asymmetric oligopoly game with process innovation.

Keywords: Differential games, Potential function, Optimal control, Quasi-aggregative games

1 Introduction

Following Monderer and Shapley [15], a relatively large literature has been devoted to investigating potential functions for static games. In a potential game, the information about Nash equilibria is nested into a single real-valued function (the potential function) over the strategy space. The specific feature of a potential function defined for a given game is that its gradient coincides with the vector of first derivatives of the individual payoff functions of the original game itself. As stressed by Slade [17], the interest of this line of research is that, in a game admitting a potential function, it is as if players were jointly maximising that single function instead of competing to maximise their respective payoffs.

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In this paper we define the requirements of a potential function for differential games and we provide sufficient conditions for its existence. In a differential game setup[1], given that the necessary conditions for the solution contains also information on the impact of the state variables dynamics, verifying the existence of a potential function is essentially different from carrying out the same task for a static game. Focusing on the noncooperative open-loop solution of a differential game, we require the potential function to reproduce the same dynamic system (state and costate equations) and achieve the same open-loop solution(s) the original game yields. When this function (called Hamiltonian potential) exists, the original differential game can be represented as a single-agent optimal control problem.[2]

We show that the existence of a Hamiltonian potential function in a generic noncooperative open-loop differential game requires two properties. The first one, conservativity, implies that there exists a potential function, unique up to an additive constant, whose first order partial derivatives with respect to all controls and all states coincide with the first order partial derivatives of the Hamiltonian functions associated to the original differential open-loop game. The second property relies on the fact that certain costate variables, i.e. the shadow prices attached by players to the accumulation of state variables, evolve separately from all the remaining quantities, forming a decoupled dynamic system. In other words, the existence of the Hamiltonian potential functions relies on the presence of redundant adjoint variables (see Dockner *et al.* [5]). Specifically, Proposition 8 adopts an approach which is similar to the one introduced by Dockner *et al.* in [5].

An interesting feature of the Hamiltonian potential is that it does not require the original game to be symmetric. We show it by considering a class of games with linear state dynamics and admitting a Hamiltonian potential function. This allows to make clear the relation between conservativity and the properties of quasi-aggregative static games.

Although this paper mainly focuses on the open-loop equilibrium of a differential game, we will also present an exploratory analysis on how to construct a tool like the Hamiltonian potential for the feedback solution of the game. In analogy with the Hamiltonian potential, we will show that, for some classes of games, it is possible to derive a Hamilton-Jacobi-Bellman (HJB) equation whose solution is an optimal value potential function which conveys the same information as the standard optimal value function. We illustrate this possibility considering a simple example with one state variable. This also allows showing that the HJB potential function approach does not convey relevant information as to the inefficiency of the non-cooperative game as compared to the Pareto-optimal

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[1] A recent contribution concerning potential in discrete-time is due to González-Sánchez and Hernández-Lerma [9].

[2] Our paper is related to, but different from [7], where the construction of a best-response potential function for a Cournot differential game is implemented.

outcome. This topic certainly needs further and deeper exploration.

Finally, we consider an example belonging to the theory of industrial organization, aimed at helping the readers' intuitive understanding of the Hamiltonian potential function and its role in the dynamic analysis. In this simple differential duopoly game, two firms produce a homogeneous good and carry out R&D activities to reduce their marginal production costs. Each of them also exploits a fraction of R&D developed by its rival to abate costs. The method of construction of the Hamiltonian potential of the game is laid out and its open-loop information structure is extensively discussed.

The remainder of the paper is structured as follows. The construction of potential functions for static games in continuous strategies is briefly summarised in Section 2. The sufficient conditions for the existence of a potential function in an open-loop differential game are investigated in Section 3, where we also examine the case with linear state motion equations and characterize its dynamic properties. Section 4 is devoted to the investigation of the feedback information structure. In particular, we define a dynamic potential function associated to the optimal value functions of the standard problem, and we outline the existing analogies and differences between the open-loop and the feedback equilibrium structures. Section 5 deals with an illustrative example. Final comments are in Section 6.

2 Preliminaries: potential in static games

We briefly recall the concept of potential in static non-cooperative full information games. We borrow from physics the following well-known results:

Definition 1. *A vector field $F(s) = (F_1(s), \ldots, F_n(s))$ defined on an open convex subset $S \in \mathbb{R}^n$ is conservative if and only if there exists a differentiable function $P : S \longrightarrow \mathbb{R}$ such that:*

$$\frac{\partial P(s)}{\partial s_i} = F_i(s), \qquad \forall\ i = 1, \ldots, n. \tag{1}$$

*$P(s)$ is called a **potential function** for F.*

Theorem 2. *A vector field $F(s)$ defined on an open convex subset of $\mathbb{R}^n$ is conservative if*

$$\frac{\partial F_i}{\partial s_j} = \frac{\partial F_j}{\partial s_i}, \quad \forall\ i, j = 1, \ldots, n, \quad i \neq j. \tag{2}$$

Clearly, (2) corresponds to the existence of a potential function for the vector field $F(s)$.

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2.1 Recent aggregation concepts

In recent years a lively discussion has been taking place on potential and aggregation concepts, especially in static games. In particular, games endowed with aggregative prerogatives have been intensely studied by Dubey *et al.* [8], Kukushkin [12] and Jensen [10] who have characterized non-cooperative, pure strategy games endowed with aggregators as *quasi-aggregative games* (see [10], Section 2) and singled out the conditions under which they are best-reply potential games.

Building on the standard inverse demand function of the Cournot model, in [10] an aggregator is defined. Call $\mathcal{I} = \{1, \ldots, n\}$ the set of players, $\widetilde{\pi}_i(\cdot)$ the payoff function for the ith player, S_i the strategy space of player i and S_{-i} the Cartesian product of all strategy spaces except S_i. If $S := S_1 \times \cdots \times S_n$ is the strategy space of the game, $s = (s_1, \ldots, s_n) \in S$ is a feasible strategy. In this setup, an aggregator $g(s)$ is defined as the sum of all strategies which, by additive separability, can be expressed as $g(s) = B_i(\sigma_i(s_{-i}), s_i) = \sigma_i(s_{-i}) + s_i$ for all i, where $\sigma_i(s_{-i})$ plays the role of an interaction term. What follow are two helpful Definitions taken from [10]:

Definition 3. *The game $G = (\widetilde{\pi}_i, \ S_i)_{i \in \mathcal{I}}$ is said to be a **quasi-aggregative game** with aggregator $g : S \longrightarrow \mathbb{R}$ if there exist continuous functions $B_i : \mathbb{R} \times S_i \longrightarrow \mathbb{R}$ and $\sigma_i : S_{-i} \longrightarrow X_{-i} \subseteq \mathbb{R}$, $i \in \mathcal{I}$, such that each of the payoff functions can be written as $\widetilde{\pi}_i(s) = \pi_i(\sigma_i(s_{-i}), s_i)$, where $\pi_i : X_{-i} \times S_i \longrightarrow \mathbb{R}$, for all $i \in \mathcal{I}$ and $g(s) = B_i(\sigma_i(s_{-i}, s_i))$ for all $s \in S$ and for all $i \in \mathcal{I}$.*

Definition 4. *$G = (\widetilde{\pi}_i, \ S_i)_{i \in \mathcal{I}}$ is a **game with reciprocal interactions** if $\widetilde{\pi}_i(s) = \pi_i(\sigma_i(s_{-i}), s_i)$ for all $i \in \mathcal{I}$, where the σ_i's are real-valued, continuously differentiable interaction functions which satisfy:*

$$\frac{\partial \sigma_i(s_{-i})}{\partial s_j} = \frac{\partial \sigma_j(s_{-j})}{\partial s_i} \tag{3}$$

for all $i \in \mathcal{I}$, for all $s \in \widetilde{S}$, where $\widetilde{S}$ is an open convex subset of $\mathbb{R}^{\mathcal{I}}$ containing S.

2.2 Potential in the Cournot game setup

For illustrative purposes, we now briefly summarise the construction of a potential function for a static Cournot-Nash game, a simplified version of the more general Cournot game considered by Slade [17] and Monderer and Shapley [15].

Consider a static market game where n firms simultaneously set output levels q_i to maximise individual profits $\pi_i(q_1, \ldots, q_n)$. The linear market inverse demand function is given by $p(q_1, \ldots, q_n) = a - \sum_i^n q_i$, and firms share the same productive technology, described by the cost function $C_i(q_i) = cq_i$, where $c > 0$

is the constant marginal cost of production. Therefore, the individual profit function can be written as

$$\pi_i\left(q_1,\ldots,q_n\right)=\left(a-q_i-\sum_{j\neq i}q_j-c\right)q_i. \tag{4}$$

The i-th first order condition for non-cooperative profit maximisation is

$$\frac{\partial\pi_i}{\partial q_i}=a-2q_i-\sum_{j\neq i}q_j-c=0. \tag{5}$$

Since for all $i,j=1,\ldots,n$, where $i\neq j$:

$$\frac{\partial}{\partial q_j}\left(\frac{\partial\pi_i}{\partial q_i}\right)=-1=\frac{\partial}{\partial q_i}\left(\frac{\partial\pi_j}{\partial q_j}\right), \tag{6}$$

the vector field $\left(\dfrac{\partial\pi_1}{\partial q_1},\ldots,\dfrac{\partial\pi_n}{\partial q_n}\right)$ is conservative and it admits the potential functions:

$$\widehat{P}(q_1,\ldots,q_n)=\sum_{i=1}^{n}\left[\left(a-\sum_{j\neq i}q_j-c\right)q_i-q_i^2\right]+\sum_{j\neq i}q_iq_j+Z, \tag{7}$$

where Z is a constant originating from integration[3]. It is easy to check that $\widehat{P}(q_1,\ldots,q_n)$ is a potential function of the Cournot-Nash game because its gradient coincides with the vector field $\left(\dfrac{\partial\pi_1}{\partial q_1},\ldots,\dfrac{\partial\pi_n}{\partial q_n}\right)$.

A potential in a static framework contains all the relevant information of the original static n-player game. Analogously, our intuition is that, if it exists, a Hamiltonian potential for differential games must indeed contain all the relevant information of the original dynamic n-player game. In the remainder of the paper, we will consider a class of differential games and identify the requirements allowing the construction of a Hamiltonian potential. Note that this definition will necessarily differ from the one provided by Monderer and Shapley [15] and in the rest of the related literature, as state variables do not appear in static games.

An additional interesting feature of the ensuing construction is the following. The state of the art concerning the existence of potential functions in non-cooperative games (in continuous strategies) is currently confined to one-stage games like the Cournot model outlined above. That is, as yet we have no theoretical results clarifying whether potential functions exist or not in multistage

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[3]In operative terms, the construction of the potential function requires integrating the first-order conditions of the players on the choice variables, summing up the integrals and checking whether what results from this procedure is indeed a conservative field (as in [17]).

static games where, e.g., firms first invest in R&D for some type of innovation (w.r.t. processes or products) and then compete on the market either in output levels or in prices. The crucial difficulty in this respect appears to be generated by the backward induction method usually associated with the way subgame perfect equilibria are generated in such games, or equivalently by the fact that players do not take the first order conditions w.r.t. all of their strategic variables simultaneously.

As we shall see, tackling this issue in dynamic games allows us to bypass this obstacle and investigate the existence of potential functions for games whose static counterparts would be multistage structures, precisely because in dynamic games players take all the first order conditions at the same time, at every instant of the time span along which the game unravels itself.

Clearly, a Cournot game with payoffs such as in (4) is a quasi-aggregative game with reciprocal interactions.

Remark 5. *When the players' payoff structure in a Cournot game is additively separable, so that* $\pi_i(\cdot) = \widetilde{D}(g(\cdot))u_i - C_i(\mathbf{x})$*, where* $g(\cdot)$ *is the aggregator and* $\widetilde{D}(\cdot)$ *indicates the inverse demand function, we can reconstruct some properties of a quasi-aggregative game theory in our setting. For example, the conditions on payoffs read as:*

$$\frac{\partial^2 \pi_i}{\partial u_i \partial u_j} = \frac{\partial^2 \pi_j}{\partial u_i \partial u_j} \iff \frac{\partial \widetilde{D}(g(\mathbf{u}))}{\partial u_j} + u_i \frac{\partial^2 \widetilde{D}(g(\mathbf{u}))}{\partial u_i \partial u_j} = \frac{\partial \widetilde{D}(g(\mathbf{u}))}{\partial u_i} + u_j \frac{\partial^2 \widetilde{D}(g(\mathbf{u}))}{\partial u_j \partial u_i},$$

which is verified if $\widetilde{D}(\cdot)$ *is linear and additively separable in strategies and if all strategies' weights coincide in the inverse demand. This highlights, for example, that in a 2-strategy setting* $\widetilde{D}(g(u_1, u_2)) = a - u_1 - u_2$ *is a suitable inverse demand, whereas* $\widetilde{D}(g(u_1, u_2)) = \dfrac{a}{u_1 + u_2}$ *is not.*

3 A potential for differential games

In this Section we define the concept of Hamiltonian potential function for differential games and we provide necessary conditions for its existence. The goal is to determine the requirements under which a non-cooperative differential game solved under the open-loop information structure can be represented as a single optimal control problem whose solution has the same dynamic properties of the original game.

Consider a non-cooperative differential game Γ over an infinite time-horizon $t \in [0, +\infty)$ with the following features:

- $\widetilde{N}$ is the number of players.

- $\mathbf{u}(t) = (u_1(t), \ldots, u_N(t)) \in U_1 \times \cdots \times U_N := U \subseteq \mathbb{R}^N$ is the vector of control variables. Each player has at least one control variable and $N \geq \widetilde{N}$. The set U is bounded and open;[4]

- $\mathbf{x}(t) = (x_1(t), \ldots, x_M(t)) \in X_1 \times \cdots \times X_M := X \subseteq \mathbb{R}^M$ is the vector of state variables. As a standard assumption in differential games, $M \leq N$. The set X is bounded and open;

- Each player is endowed with the instantaneous payoff function $\pi_i(\mathbf{x}(t), \mathbf{u}(t), t)$.

- Given a set of initial conditions $\mathbf{x}(0) = (x_1(0), \ldots, x_M(0))$, the goal of player i is to maximize the discounted profit flow:

$$J_i \equiv \int_0^\infty e^{-\rho t} \pi_i(\mathbf{x}(t), \mathbf{u}(t), t) dt \tag{8}$$

subject to the dynamic constraints

$$\begin{cases} \dot{x}_k(t) = G_k(\mathbf{x}(t), \mathbf{u}(t), t) \\ x_k(0) = x_{k0} \in X_k \end{cases}, \tag{9}$$

where $\rho > 0$ is the intertemporal discount rate, constant and common to all agents, and $G_k(\cdot) \in C^2(U \times X \times [0, \infty))$ is the transition function of the state variable $x_k(t)$, for $k = 1, \ldots, M$.

To find the open-loop solution of Γ, we construct for each player i, where $i = 1, \ldots, \widetilde{N}$, the following current-value Hamiltonian function (hereinafter referred to as Hamiltonian):

$$H_i(\mathbf{x}(t), \mathbf{u}(t), t) = \pi_i(\mathbf{x}(t), \mathbf{u}(t), t) + \sum_{k=1}^M \lambda_{ik}(t) G_k(\mathbf{x}(t), \mathbf{u}(t), t). \tag{10}$$

Hence the Hamiltonian function of player i can be written as follows:

$$H_i(\mathbf{x}(t), \mathbf{u}(t), t) = \pi_i(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda_{ii}(t) G_i(\mathbf{x}(t), \mathbf{u}(t), t) + \sum_{j \neq i} \lambda_{ij}(t) G_j(\mathbf{x}(t), \mathbf{u}(t), t), \tag{11}$$

where $\lambda_{ii}(t)$ can be interpreted as the current-value costate variable associated by player i to the dynamics of her own state variable, and $\lambda_{ij}(t)$ as the current-value costate variable associated by player i to the dynamics of another state variable j (hereinafter referred to as the cross-multiplier).

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[4] These topological assumptions are not standard in differential game settings, in that assuming a compact U would ensure the existence of interior solutions. We are thus implicitly focusing on cases in which there exists at least one open-loop solution belonging to $int(U)$.

Suppose $H_i \in C^2(X \times U \times \mathbb{R}^M \times [0, +\infty))$. To find the open-loop Nash equilibrium of the differential game, the following necessary conditions are to be satisfied for an interior solution (see [4]):[5]

$$\frac{\partial H_i}{\partial u_i}(\mathbf{x}(t), \mathbf{u}(t), t) = 0 \iff$$

$$\iff \frac{\partial \pi_i}{\partial u_i}(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda_{ii}(t)\frac{\partial G_i}{\partial u_i}(\mathbf{x}(t), \mathbf{u}(t), t) + \sum_{j \neq i} \lambda_{ij}(t)\frac{\partial G_j}{\partial u_i}(\mathbf{x}(t), \mathbf{u}(t), t) = 0,$$
(12)

$$\dot{\lambda}_{ii}(t) = \rho\lambda_{ii}(t) - \frac{\partial H_i}{\partial x_i}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{13}$$

$$\dot{\lambda}_{ij}(t) = \rho\lambda_{ij}(t) - \frac{\partial H_i}{\partial x_j}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{14}$$

$$\dot{x}_k(t) = G_k(\mathbf{x}(t), \mathbf{u}(t), t), \tag{15}$$

for all $i = 1, \ldots, \widetilde{N}$, $k = 1, \ldots, M$ and $j \neq i$, plus the transversality conditions[6]

$$\lim_{t \to \infty} e^{-\rho t}\lambda_{ik}(t)x_k(t) = 0, \tag{16}$$

for all $i = 1, \ldots, \widetilde{N}$, for all $k = 1, ..., M$.

As shown in the previous Section, in a static framework a potential function is a single function whose first order conditions reproduce the first order conditions of all players in the original game. Given the differential game under examination, we require the potential function to accomplish similar requirements. More specifically, we look for a Hamiltonian potential function H_P with the following features:

1. the set of first order partial derivatives of H_P with respect to states and controls must coincide with the first order partial derivatives of the standard Hamiltonian functions.

2. H_P must have a Hamiltonian structure and can be written as the sum of a function P and the scalar product between the M transition functions of the original game and the corresponding costate variables:

$$H_P(\mathbf{x}, \mathbf{u}, \widetilde{\Lambda}, \widehat{\Lambda}, t) = P(\mathbf{x}, \mathbf{u}, \Lambda_{ii}, \Lambda_{ij}, t) + \sum_{i=1}^{M} \lambda_{ii}(t)G_i(\mathbf{x}(t), \mathbf{u}(t), t), \quad (17)$$

where
$$\widetilde{\Lambda} = (\lambda_{11}(t), \dots, \lambda_{MM}(t)) \in \mathbb{R}^M \tag{18}$$

is the vector of current-value costate variables associated by player i to her own state dynamics $\dot{x}_i$, and

$$\widehat{\Lambda} = \left(\lambda_{12}(t), \dots, \lambda_{1M}(t), \dots, \lambda_{\widetilde{N}1}(t), \dots, \lambda_{\widetilde{N}M}(t)\right) \in \mathbb{R}^{M\widetilde{N}-M} \tag{19}$$

is the vector of cross-multipliers.

The Hamiltonian potential is not a Hamiltonian function because the costate variables also appear as arguments of the objective functional. The name "Hamiltonian" is maintained in the potential structure because the solution procedure we propose is similar to the one used to solve an optimal control problem. We will show that the Hamiltonian potential can be, under certain circumstances (e.g. $\lambda_{ij} \equiv 0$) the representation of the original differential game *as if* it were played by a single agent. In the remainder of the exposition we will drop the arguments whenever it does not generate confusion.

Definition 6. *Given a differential game Γ with Hamiltonians H_i, the **Hamiltonian potential function for game** Γ is a function $H_P : X \times U \times \mathbb{R}^{\widetilde{N} \times M} \times [0, +\infty) \to \mathbb{R}$ such that:*

$$\left(\frac{\partial H_P}{\partial x_1}, \dots, \frac{\partial H_P}{\partial x_M}, \frac{\partial H_P}{\partial u_1}, \dots, \frac{\partial H_P}{\partial u_N}\right) = \left(\frac{\partial H_1}{\partial x_1}, \dots, \frac{\partial H_M}{\partial x_M}, \frac{\partial H_1}{\partial u_1}, \dots, \frac{\partial H_N}{\partial u_N}\right)$$

over $X \times U$.

If a differential game admits a Hamiltonian potential function, it is a potential differential game.[7] For the existence of a Hamiltonian potential, the vector field in the r.h.s. of the identity in Definition 6 must be conservative, i.e. it must admit a potential in compliance with Definition 2 and (2). Under such circumstances, the following sufficient condition for the existence of a Hamiltonian potential can be stated:

Theorem 7. *If there exist two functions $\widehat{P}(\mathbf{x}, \mathbf{u}, t)$ and $R(\mathbf{x}, \mathbf{u}, \widetilde{\Lambda}, \widehat{\Lambda}, t)$ such that:*

$$\frac{\partial \widehat{P}}{\partial u_i} + \lambda_{ii}\frac{\partial G_i}{\partial u_i} + \sum_{j \neq i} \lambda_{ij}\frac{\partial R}{\partial u_i} = \frac{\partial H_i}{\partial u_i} \tag{20}$$

for all $i = 1, \dots, N$ and

$$\frac{\partial \widehat{P}}{\partial x_i} + \lambda_{ii}\frac{\partial G_i}{\partial x_i} + \sum_{j \neq i} \lambda_{ij}\frac{\partial R}{\partial x_i} = \frac{\partial H_i}{\partial x_i} \tag{21}$$

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[7]Note that we refer to *the* Hamiltonian potential but, since it results from an integration procedure, we might obtain infinitely many Hamiltonian potential functions up to a constant of integration.

for all $i = 1, \ldots, M$, then the differential game Γ with Hamiltonians H_i admits the following Hamiltonian potential function:

$$H_P(\mathbf{x}, \mathbf{u}, \widetilde{\Lambda}, \widehat{\Lambda}, t) = \widehat{P}(\mathbf{x}, \mathbf{u}, t) + R(\mathbf{x}, \mathbf{u}, \widetilde{\Lambda}, \widehat{\Lambda}, t) + \sum_{i=1}^{N} \lambda_{ii}(t) \, G_i(\mathbf{x}, \mathbf{u}, t). \quad (22)$$

Proof. It immediately follows from Definition 6 applied to structure (22). $\qquad \square$

The above Theorem allows to express the Hamiltonian potential function $H_P(\mathbf{x}, \mathbf{u}, \widetilde{\Lambda}, \widehat{\Lambda}, t)$ as the sum of two functions, where $\widehat{P}(\mathbf{x}, \mathbf{u}, t)$ corresponds to the potential function defined in (7) for the static counterpart of the differential game, and $R(\mathbf{u}, \mathbf{x}, \widetilde{\Lambda}, \widehat{\Lambda}, t)$ is a function which contains the costates $\boldsymbol{\lambda}_{ii}, \boldsymbol{\lambda}_{ij}$ of the original game and therefore contains information on how changes in the state variables affect the objective functions of the players in the original game. The special case $\dfrac{\partial R}{\partial \lambda_{ij}} = 0$ represents the limit case in which the first order conditions of player i do not include information on the rivals' state dynamics. This may occur, e.g., when the state dynamics are decoupled, i.e., when each function G_i only depends on x_i and u_i. In general, however, $\dfrac{\partial R}{\partial \lambda_{ij}}$ is not nil, in which case the function R contains relevant information on the asymmetric structure of the game and on the degree of interdependence between the players.

The set of necessary conditions (12)-(16) determines the open-loop solution of the original differential game. By taking the derivative with respect to time of the first order conditions (12) and replacing $\dot{\lambda}_{ii}(t)$, $\dot{\lambda}_{ij}(t)$ and the value of $\lambda_{ii}(t)$ obtained from (12), the open-loop solution can be equivalently expressed as a system of differential equations describing the dynamics of the optimal controls, states and cross-multipliers. The procedure we follow using the Hamiltonian Potential is a similar one. We first consider the following conditions:

$$\frac{\partial H_P}{\partial u_i} = 0 \iff \frac{\partial \widehat{P}}{\partial u_i} + \lambda_{ii} \frac{\partial G_i}{\partial u_i} + \sum_{j \neq i} \lambda_{ij} \frac{\partial R}{\partial u_i} = 0, \quad (23)$$

$$\dot{\lambda}_{ii}(t) = \rho \lambda_{ii}(t) - \frac{\partial H_P}{\partial x_i}, \quad (24)$$

$$\dot{x}_k(t) = G_k(\mathbf{x}(t), \mathbf{u}(t), t), \quad (25)$$

for all $i = 1, \ldots, \widetilde{N}$ and $k = 1, \ldots, M$, plus the transversality conditions:

$$\lim_{t \to \infty} e^{-\rho t} \lambda_{ik}(t) x_k(t) = 0, \quad (26)$$

for all $i = 1, \ldots, \widetilde{N}$ and for all admissible $x_i(t)$, with $x_i^*(t)$ belonging to an admissible path. Then we take the derivative with respect to time of (23) and replace $\dot{\lambda}_{ii}(t)$ and $\lambda_{ii}(t)$ to obtain a system of differential equations describing the dynamics of the controls, the states and the cross-multipliers.

As we show in the next Subsection, there exist conditions under which the dynamics of the cross-multipliers does not affect the dynamics of optimal controls and states. So neglecting the cross-multipliers still allows to achieve the same optimal paths and the same steady state of the original game.

3.1 Hamilton potential and the dynamic properties of the differential game

In this Section we investigate under what conditions the Hamiltonian potential can be viewed as the Hamiltonian function of a single player replacing the original $\widetilde{N}$ players involved in a game Γ. The final outcome of this procedure must be the same dynamic system of state and control equations as in game Γ.

Given that in the original game also the dynamics of the cross-multipliers λ_{ij} may affect the choice of the optimal control by each player, it is in general not true that the equilibrium structure of a single-agent optimal control problem coincides with the equilibrium structure of the original differential game where $\widetilde{N}$ optimal control problems must be simultaneously solved. These dynamics play an important role, as they take care of the asymmetric cross interactions among the state variables. The original game need not to be symmetric, because it is possible to obtain a Hamiltonian potential from an asymmetric game, as it will be shown in the final example. Under such circumstances, the Hamiltonian potential allows to separate the direct effects (represented by the dynamics of the costate variables λ_{ii}) from the cross effects represented by the dynamics of the costate variables λ_{ij}.

In the following Proposition we show sufficient conditions under which the dynamic properties (i.e. the state-costate system of differential equations that represents the optimal solution) of the original game Γ and those of the corresponding potential game are the same.

Proposition 8. *If all first order conditions (12) of Γ have the following form:*

$$\lambda_{ii} = \sum_{j \neq i} A_{ij}\lambda_{ij} + K_i(\mathbf{u}) + L_i(\mathbf{x}), \tag{27}$$

where A_{ij} are real constants, $K_i(\cdot)$ are differentiable functions in all control variables, and $L_i(\cdot)$ are differentiable functions in all the state variables, and

$$\frac{\partial G_i}{\partial x_i} = \frac{\partial G_j}{\partial x_j}, \qquad\qquad \frac{\partial G_i}{\partial x_j} = 0, \tag{28}$$

for all $i, j = 1, \ldots, M$, $i \neq j$, then equations (14) are decoupled with respect to the state and control dynamics of the game Γ.

Proof. If in all first order conditions (12) the costates λ_{ii} may be expressed as a linear combination of the cross-multipliers λ_{ij} plus two differentiable separate

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functions of controls and states, i.e. there exist $M\widetilde{N} - M$ real constants A_{ij} and N functions $K_i : U \longrightarrow \mathbb{R}$, and N functions $L_i : X \longrightarrow \mathbb{R}$, and all of them are C^2 with respect to all variables such that

$$\lambda_{ii} = \sum_{j \neq i} A_{ij} \lambda_{ij} + K_i(\mathbf{u}) + L_i(\mathbf{x}),$$

for $i = 1, \ldots, N$, then we can differentiate such relation with respect to time:

$$\dot{\lambda}_{ii} = \sum_{j \neq i} A_{ij} \dot{\lambda}_{ij} + \sum_{k=1}^{N} \frac{\partial K_i(\mathbf{u})}{\partial u_k} \dot{u}_k + \sum_{k=1}^{M} \frac{\partial L_i(\mathbf{x})}{\partial x_k} \dot{x}_k.$$

Plugging both λ_{ii} and $\dot{\lambda}_{ii}$ into (13) yields:

$$\sum_{j \neq i} A_{ij} \dot{\lambda}_{ij} + \sum_{k=1}^{N} \frac{\partial K_i(\mathbf{u})}{\partial u_k} \dot{u}_k + \sum_{k=1}^{M} \frac{\partial L_i(\mathbf{x})}{\partial x_k} \dot{x}_k = \rho \left(\sum_{j \neq i} A_{ij} \lambda_{ij} + K_i(\mathbf{u}) + L_i(\mathbf{x}) \right) - \frac{\partial H_i}{\partial x_i},$$

$$(29)$$

for $i = 1, \ldots, M$. Using (14) and rearranging (29) amounts to:

$$-\sum_{j \neq i} A_{ij} \frac{\partial H_i}{\partial x_j} + \sum_{k=1}^{N} \frac{\partial K_i(\mathbf{u})}{\partial u_k} \dot{u}_k + \sum_{k=1}^{M} \frac{\partial L_i(\mathbf{x})}{\partial x_k} \dot{x}_k = \rho \left(K_i(\mathbf{u}) + L_i(\mathbf{x}) \right) - \frac{\partial H_i}{\partial x_i}. \quad (30)$$

Now, if we expand $\dfrac{\partial H_i}{\partial x_j}$ and $\dfrac{\partial H_i}{\partial x_j}$, (30) becomes:

$$-\sum_{j \neq i} A_{ij} \left(\frac{\partial \pi_i}{\partial x_j} + \lambda_{ii} \frac{\partial G_i}{\partial x_j} + \sum_{l \neq i} \lambda_{il} \frac{\partial G_l}{\partial x_j} \right) + \sum_{k=1}^{N} \frac{\partial K_i(\mathbf{u})}{\partial u_k} \dot{u}_k + \sum_{k=1}^{M} \frac{\partial L_i(\mathbf{x})}{\partial x_k} \dot{x}_k =$$

$$= \rho \left(K_i(\mathbf{u}) + L_i(\mathbf{x}) \right) - \frac{\partial \pi_i}{\partial x_i} - \lambda_{ii} \frac{\partial G_i}{\partial x_i} - \sum_{j \neq i} \lambda_{ij} \frac{\partial G_j}{\partial x_i}.$$

Then we can plug the expression of λ_{ii} provided by the assumption (27) to obtain:

$$-\sum_{j \neq i} A_{ij} \left(\frac{\partial \pi_i}{\partial x_j} + \left(\sum_{m \neq i} A_{im} \lambda_{im} + K_i(\mathbf{u}) + L_i(\mathbf{x}) \right) \frac{\partial G_i}{\partial x_j} + \sum_{l \neq i} \lambda_{il} \frac{\partial G_l}{\partial x_j} \right) +$$

$$+ \sum_{k=1}^{N} \frac{\partial K_i(\mathbf{u})}{\partial u_k} \dot{u}_k + \sum_{k=1}^{M} \frac{\partial L_i(\mathbf{x})}{\partial x_k} \dot{x}_k = \rho \left(K_i(\mathbf{u}) + \right.$$

$$+ L_i(\mathbf{x})) - \frac{\partial \pi_i}{\partial x_i} - \left(\sum_{m \neq i} A_{im} \lambda_{im} + K_i(\mathbf{u}) + L_i(\mathbf{x}) \right) \frac{\partial G_i}{\partial x_i} - \sum_{j \neq i} \lambda_{ij} \frac{\partial G_j}{\partial x_i}.$$

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Collecting in a unique function $\mathcal{A}(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}})$ all terms where λ_{ij}'s do not appear yields:

$$\mathcal{A}(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}}) =$$

$$= -\sum_{j \neq i} A_{ij} \left(\sum_{m \neq i} A_{im} \lambda_{im} \frac{\partial G_i}{\partial x_j} + \sum_{l \neq i} \lambda_{il} \frac{\partial G_l}{\partial x_j} \right) + \left(\sum_{m \neq i} A_{im} \lambda_{im} \right) \frac{\partial G_i}{\partial x_i} + \sum_{j \neq i} \lambda_{ij} \frac{\partial G_j}{\partial x_i}. \tag{31}$$

Now, condition (28) implies that(31) becomes:

$$\mathcal{A}(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}}) = -\sum_{j \neq i} A_{ij} \lambda_{ij} \frac{\partial G_j}{\partial x_j} + \sum_{m \neq i} A_{im} \lambda_{im} \frac{\partial G_i}{\partial x_i} = 0,$$

i.e. an equation which does not include any cross-multiplier. Hence, the dynamics of all u_i and x_i is not affected by the costate variables λ_{ij}, for $i \neq j$. □

When Proposition 8 is verified, the dynamics $\dot{\lambda}_{ij}$ are decoupled with respect to the dynamic system of costates u_i and states x_i. It is important to observe that the above result includes the situation in which $\lambda_{ij} \equiv 0$ as a special case. The results also holds if λ_{ij} is not symmetric, which implies that the concept of Hamiltonian potential can be employed (when it exists) for the investigation of differential games with non-symmetric characteristics. This holds, for instance, in some linear state games where the open-loop Nash equilibrium is strongly time consistent[8].

The next Proposition intends to show that the solution of the dynamic system obtained from the Hamiltonian potential coincides with the solution of the original differential game. Call $(\mathbf{x}^P, \mathbf{u}^P, \widetilde{\Lambda}^P) \in X \times U \times \mathbb{R}^M$ the solution vector of the dynamic system obtained from the Hamiltonian potential and $(\mathbf{x}^*, \mathbf{u}^*, \widetilde{\Lambda}^*) \in X \times U \times \mathbb{R}^M$ the one of the original differential game.

Proposition 9. *If Theorem 7 and Proposition 8 hold, then*

$$(\mathbf{x}^P, \mathbf{u}^P, \widetilde{\Lambda}^P) = (\mathbf{x}^*, \mathbf{u}^*, \widetilde{\Lambda}^*).$$

Proof. Theorem 7 ensures the existence of a Hamiltonian potential, whereas Proposition 8 guarantees that the cross-multipliers do not affect the dynamic system of optimal states and controls. Since the differentiation with respect to time of conditions (12) and (23) yield the same dynamic system of states and controls, and since (13) and (24) coincide, the equilibrium trajectories are the same at all $t \in (0, \infty)$ and the related steady states coincide. □

---

[8]For an exhaustive overview of linear state games, see Chapter 4 in [14] and Chapter 7 in [4].

4 Hamiltonian potential and the feedback solution

In this section we explore the connection between the open-loop solution obtained with a Hamiltonian potential function and the feed-back solution obtained through a potential representation of the original game. As a preliminary step in this direction, we will focus on the simple case in which the two solutions coincide in the original game (see [4], [14]).

Consider a simple separable case with one state variable $x(t)$ which evolves according to the following dynamics:

$$\dot{x}(t) = \sum_{j=1}^{N} u_j(t) - \delta x(t). \tag{32}$$

Let the initial value be $x(0) = x_0 > 0$. The N agents' objective functions are:

$$J_i \equiv \int_0^{+\infty} e^{-\rho t} \left[S(x(t)) + T(u_i(t)) \right] dt. \tag{33}$$

It is straightforward to check that this game admits the following Hamiltonian potential:

$$
\begin{aligned}
H_P(x, u_1, \ldots, u_N, \lambda, t) &= \widehat{P}(x, u_1, \ldots, u_n) + \lambda \left(\sum_{j=1}^{N} u_j - \delta x \right) \\
&= \sum_{j=1}^{N} T(u_j) + S(x) + \lambda \left(\sum_{j=1}^{N} u_j - \delta x \right). \quad (34)
\end{aligned}
$$

whose associated first order conditions are

$$T'(u_i) + \lambda = 0$$

Note that expression of (34) exhibits a unique costate variable λ (whose value is not identically equal to zero), that $\widehat{P}(\cdot) = \sum_{j=1}^{N} T(u_j) + S(x)$, and that $R(\cdot) = 0$.

We now consider the feedback solution of the same game. In analogy to the approach followed for the open-loop solution, our goal is to find a function $V_p(x)$ which collects all relevant information contained in the optimal value functions $V_1(x), \ldots, V_N(x)$ of the N players of the game. We start by considering the Hamilton-Jacobi-Bellman equation of each player i (see [4]):

$$\rho V_i(x) = \max_{u_i} \left\{ S(x) + T(u_i) + \frac{\partial V_i}{\partial x} \left[\sum_{j=1}^{N} u_j - \delta x \right] \right\}. \tag{35}$$

whose associated FOCs are, for each player i:

$$T'(u_i) + V_i'(x) = 0. \tag{36}$$

Note that the property $V_i'(x) = V_j'(x)$ holds in view of the fact that there exists just one state variable. Since the game has an additively separable structure and the open-loop and feedback equilibria coincide (see [4], [14], where such cases are extensively treated), the costate variable λ coincides with $V_i'(x)$. Hence the first order conditions of the open-loop and the feedback solution coincide.

If $T(u_i)$ is a linear-quadratic function, say $T(u_i) = au_i^2 + bu_i + c$, the FOCs have a unique solution in u_i:

$$u_i = -\frac{V_i'(x) + b}{2a}. \tag{37}$$

Hence, for all $i = 1, \ldots, N$, (35) becomes:

$$\rho V_i(x) = S(x) + T\left(-\frac{V_i'(x) + b}{2a}\right) + V_i'(x)\left[-\frac{1}{2a}\sum_{j=1}^{N}V_j'(x) - \delta x\right] - \frac{Nb}{2a}. \tag{38}$$

In order construct the value function $V_P(x)$ for the feedback solution of the game, we write the following (unique) Hamilton-Jacobi-Bellman potential equation:

$$\rho V_p(x) = \max_{u_1,\ldots,u_N}\left\{\widehat{P}(x, u_1, \ldots, u_n) + \frac{\partial V_p}{\partial x}\left[\sum_{j=1}^{N}u_j - \delta x\right]\right\}. \tag{39}$$

The FOCs associated with (39) are

$$T'(u_i) + V_P'(x) = 0. \tag{40}$$

If $V_P'(x) = V_i'(x)$ for all $i = 1, \ldots, N$, the feedback solution of the potential representation coincides with the feedback solution of the original game. Note that, although this requires the derivatives of the value functions to be equal, the primitives (the optimal value functions) can differ, as it will be shown in the following example.

Consider a linear-quadratic game in an oligopoly setup. This simple framework allows relating the potential representations obtained when considering open-loop and feedback solutions. It also makes possible to compare the efficiency of the non-cooperative solution with the efficiency of the cooperative solution, based on the fact that value functions measure, by definition, the value of the intertemporal flow of profits associated to the optimal dynamic program.

Example 10. *Consider a game where N players are profit maximizers. Each of them solves the following dynamic program:*

$$\max_{u_i} J_i \equiv \int_0^{+\infty} e^{-\rho t}\left[(A - u_i(t))u_i(t) - \varphi x(t)\right]dt, \tag{41}$$

subject to the accumulation dynamics (32) and the initial constraint $x(0) = x_0$. The parameter $A > 0$ is the market reservation price and $\varphi > 0$ is the marginal cost of the stock. The game admits a Hamiltonian potential of the form (34):

$$H_P(x, u_1, \ldots, u_N, \lambda, t) = \sum_{j=1}^{N}(A - u_j)u_j - \varphi x + \lambda \left(\sum_{j=1}^{N} u_j - \delta x \right).$$

Call $V_1(x) = \cdots = V_N(x) = V(x)$ the optimal value function associated to each player i. Since the game is linear in its unique state variable, the players' optimal value functions are linear: $V_i(x) = \varepsilon x + \xi$ for $i = 1, \ldots N$. The corresponding coefficients are

$$
\begin{cases}
\varepsilon = -\dfrac{\varphi}{\delta + \rho} \\[4mm]
\xi = \dfrac{[-\varphi + A(\delta + \rho)]\,[-3\varphi + A(\delta + \rho)]}{4\rho(\delta + \rho)^2}.
\end{cases}
\tag{42}
$$

To match the requirement $V_P'(x) = V_i'(x)$, the optimal value potential function must be $V_P(x) = \varepsilon x + \xi_P$, with

$$\xi_P = \frac{[-\varphi + A(\delta + \rho)]^2}{2\rho(\delta + \rho)^2}. \tag{43}$$

As expected, the unique steady state equilibrium is the same under open-loop and feedback rules, and it is identified by

$$u^* = \frac{1}{2}\left(A - \frac{\varphi}{\delta + \rho} \right); \ x^* = \frac{N\left[A\,(\delta + \rho) - \varphi \right]}{2\delta\,(\delta + \rho)} \tag{44}$$

In other words, the steady state equilibrium of the game can be indifferently obtained solving the original game, or through its potential representation. Yet, as anticipated above, the individual optimal value functions $V_i(x)$ and the optimal value potential function $V_P(x)$ are different, although they have the same structure and the same first order derivative. In particular, $V_P(x)$ does not measure the value associated to the optimal solution of the individual problem, but it is the optimal value function of an optimal control problem where the static potential $\widehat{P}(\cdot)$ replaces the profit functions of the players.

One may therefore wonder whether the optimal value potential function relates to the optimal value function associated to the cooperative solution obtained when firms cooperate to maximise the sum of their individual payoff functions, defined in (41), subject to (32) and the initial condition specified above. The cooperative solution is Pareto-efficient, in the sense that the value associated to the corresponding optimal dynamic program cannot be lower than the solution obtained in the non-cooperative setup.

Consider a cartel of N firms whose associated Hamilton-Jacobi-Bellman equation (denoted with the subscript C) is

$$\rho V_C(x) = \max_{u_1,\ldots,u_N} \left\{ \sum_{i=1}^{N} [(A - u_i(t))u_i(t) - \varphi x(t)] + \frac{\partial V_C}{\partial x} \left[\sum_{j=1}^{N} u_j - \delta x \right] \right\} \quad (45)$$

The Pareto-efficient equilibrium, omitting the details of the solution for brevity, is identified by

$$u_C = \frac{1}{2} \left(A - \frac{N\varphi}{\delta + \rho} \right); \ x_C = \frac{N[A(\delta + \rho) - N\varphi]}{2\delta(\delta + \rho)}, \quad (46)$$

and the associated optimal value function is

$$V_C(x_C) = \frac{N[A(\delta + \rho) - N\varphi][\delta A(\delta + \rho) - N\varphi(\delta + 2\rho)]}{4\delta\rho(\delta + \rho)^2}. \quad (47)$$

Let $A > N\varphi/(\delta + \rho)$ to ensure the positivity of state and controls at the equilibrium (note that this also ensures that the Nash equilibrium state and controls are positive, since $N \geq 1$). We can therefore compare the value function of the cartel, $V_C(x_C)$, with the value function of the potential game at the equilibrium x^*

$$V_P(x^*) = \frac{N[A(\delta + \rho) - \varphi][\delta A(\delta + \rho) - \varphi(\delta + 2\rho)]}{4\delta\rho(\delta + \rho)^2}. \quad (48)$$

The two solutions coincide when $N = 1$. Otherwise, the following interesting result holds:

$$V_C(x_C) - V_P(x^*) = \frac{N(N-1)\varphi}{4\delta\rho(\delta + \rho)^2} \left[\varphi(\delta(N+1) + 2N\rho) - 2A(\delta + \rho)^2 \right] > 0 \quad (49)$$

for all

$$A \in \left(0, \frac{(\delta(N+1) + 2N\rho)}{2(\delta + \rho)^2} \right) \quad (50)$$

Since

$$\frac{(\delta(N+1) + 2N\rho)}{2(\delta + \rho)^2} < \frac{N\varphi}{\delta + \rho}, \quad (51)$$

we can conclude that $V_C(x_C) < V_P(x^*)$ for all $A > N\varphi/(\delta + \rho)$. Hence, the maximised value function obtained through the procedure used to construct the potential cannot, in general, be taken as a measure of the inefficiency usually associated with the noncooperative solution of a variable sum game.

5 Application: Asymmetric duopoly with process innovation

In this Section we illustrate the use of the Hamiltonian potential structure introduced in the previous sections by analysing a differential game between firms that invest in process innovation (for analogous models, see [1], [2], [3], [11], [13], [18]) for the case where $\widetilde{N} = 2$ players, $N = 4$ controls and $M = 2$ state variables.

Consider a Cournot oligopoly where two single-product firms sell a homogeneous good. Let the inverse demand function faced by firm $i = 1, 2$ be

$$p_i(t) = a - q_i(t) - sq_j(t), \qquad i \neq j, \tag{52}$$

where $q_i(t)$ and $q_j(t)$ is the output of firm i and j, respectively, $s \in [0, 1]$ is a parameter that captures the exogenous degree of substitutability between the two goods, and $a > 0$ is the reservation price. The production costs are linear and correspond to the functions $C_i(q_i(t)) = c_i(t)q_i(t)$. The marginal production cost $c_i(t)$ is a state variable that changes over time due to depreciation of the production technology and investment in R&D. Each firm i invests $k_i \geq 0$ in R&D at a quadratic cost $\Gamma_i(k_i) = k_i^2(t)$, for $i = 1, 2$. We consider the following dynamics for the marginal production costs

$$\dot{c}_1(t) = -k_1(t) - \beta_1 k_2(t) + \gamma c_1(t) \tag{53}$$
$$\dot{c}_2(t) = -k_2(t) - \beta_2 k_1(t) + \gamma c_2(t) \tag{54}$$

where the parameter $\gamma > 0$ represents the rate of obsolescence. The parameters $\beta_1, \beta_2 \geq 0$ represent possible technological spillovers that each firm enjoys from the R&D investment of the rival firm. To allow for asymmetries, we assume $\beta_1 \geq \beta_2$, which implies that firm 1 is able to exploit the rival's investment in R&D more intensively as opposed to firm 2.

Omitting arguments whenever possible, the i-th firm's profit function writes as

$$\pi_i(q_i, q_j, c, k_i) = (a - q_i - sq_j - c_i)q_i - k_i^2. \tag{55}$$

Given the initial conditions $c_1(0) = c_{10}$, $c_2(0) = c_{20}$, the goal of firm i is to choose q_i and k_i to maximize the following function (omitting the time argument)

$$J_i \equiv \int_0^\infty e^{-\rho t}[(a - q_i - sq_j - c_i)q_i - k_i^2]dt, \tag{56}$$

s.t. the motion equations of c_1 and c_2.

The Hamiltonian functions are:

$$H_1(\cdot) = (a - q_1 - sq_2 - c_1)q_1 - k_1^2 + \lambda_{11}(-k_1 - \beta_1 k_2 + \gamma c_1) + \lambda_{12}(-k_2 - \beta_2 k_1 + \gamma c_2),$$

$$H_2(\cdot) = (a - q_2 - sq_1 - c_2)q_2 - k_2^2 + \lambda_{21}(-k_1 - \beta_1 k_2 + \gamma c_1) + \lambda_{22}(-k_2 - \beta_2 k_1 + \gamma c_2).$$

For internal solutions, the following first order conditions must hold for firm 1:

$$\frac{\partial H_1}{\partial q_1} = a - 2q_1 - sq_2 - c_1 = 0, \tag{57}$$

$$\frac{\partial H_1}{\partial k_1} = -2k_1 - \lambda_{11} - \beta_2\lambda_{12} = 0, \tag{58}$$

$$\dot{\lambda}_{11} = \rho\lambda_{11} - \frac{\partial H_1}{\partial c_1} = (\rho - \gamma)\lambda_{11} + q_1, \tag{59}$$

$$\dot{\lambda}_{12} = \rho\lambda_{12} - \frac{\partial H_1}{\partial c_2} = (\rho - \gamma)\lambda_{12}. \tag{60}$$

Analogously, for firm 2 the following must hold:

$$\frac{\partial H_2}{\partial q_2} = a - 2q_2 - sq_1 - c_2 = 0, \tag{61}$$

$$\frac{\partial H_2}{\partial k_2} = -2k_2 - \lambda_{22} - \beta_1\lambda_{21} = 0 \tag{62}$$

$$\dot{\lambda}_{21} = \rho\lambda_{21} - \frac{\partial H_2}{\partial c_1} = (\rho - \gamma)\lambda_{21}, \tag{63}$$

$$\dot{\lambda}_{22} = \rho\lambda_{22} - \frac{\partial H_2}{\partial c_2} = (\rho - \gamma)\lambda_{22} + q_2. \tag{64}$$

The differential game under consideration admits the following Hamiltonian potential function:

$$H_P(q_1, q_2, k_1, k_2, c_1, c_2, \lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, t) =$$

$$\widehat{P}(q_1, q_2, k_1, k_2) + R(k_1, k_2, \lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) + \lambda_{11}\dot{c}_1 + \lambda_{22}\dot{c}_2, \tag{65}$$

where

$$\widehat{P}(q_1, q_2, k_1, k_2) = \sum_{j=1}^{2} \left(-q_j^2 + (a - c_j)q_j - k_j^2\right) - sq_1q_2, \tag{66}$$

$$R(k_1, k_2, \lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) = \beta_1(\lambda_{11} - \lambda_{12})k_2 + \beta_2(\lambda_{22} - \lambda_{21})k_1.$$

Note that, since q_1 and q_2 do not affect the kinematic equations (53) and (54), $R(\cdot)$ does not depend on the control variables q_1 and q_2.

It is simple to check that (65) is a Hamiltonian potential for this duopoly game as

$$\frac{\partial H_P}{\partial q_1} = \frac{\partial H_1}{\partial q_1}, \qquad \frac{\partial H_P}{\partial q_2} = \frac{\partial H_2}{\partial q_2}, \qquad \frac{\partial H_P}{\partial k_1} = \frac{\partial H_1}{\partial k_1},$$

$$\frac{\partial H_P}{\partial k_2} = \frac{\partial H_2}{\partial k_2}, \qquad \frac{\partial H_P}{\partial c_1} = \frac{\partial H_1}{\partial c_1}, \qquad \frac{\partial H_P}{\partial c_2} = \frac{\partial H_2}{\partial c_2}. \tag{67}$$

The solution of the Hamiltonian potential can be described by a system of differential equations of control and state variables, as it is commonly done in

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differential game modelling. The first order conditions allow us to express the optimal controls as functions of the state and costates of the potential problem:[9]

$$
\begin{cases}
\dfrac{\partial H_P}{\partial q_1} = 0 \\[2mm]
\dfrac{\partial H_P}{\partial q_2} = 0 \\[2mm]
\dfrac{\partial H_P}{\partial k_1} = 0 \\[2mm]
\dfrac{\partial H_P}{\partial k_2} = 0
\end{cases}
\iff \cdots \iff
\begin{cases}
q_1^* = \dfrac{(s-2)a + 2c_1 - sc_2}{s^2 - 4} \\[2mm]
q_2^* = \dfrac{(s-2)a - sc_1 + 2c_2}{s^2 - 4} \\[2mm]
k_1^* = -\dfrac{\lambda_{11} + \beta_2 \lambda_{12}}{2} \\[2mm]
k_2^* = -\dfrac{\lambda_{22} + \beta_1 \lambda_{21}}{2}
\end{cases}
. \tag{68}
$$

Differentiating (57) and (61) with respect to time yields the following pair of ODEs for the output choices q_i:

$$
\begin{cases}
\dot{q}_1 = \dfrac{(\beta_2 s - 2)k_1 + (s - 2\beta_1)k_2 + \gamma(2c_1 - sc_2)}{s^2 - 4} \\[3mm]
\dot{q}_2 = \dfrac{(s - 2\beta_2)k_1 + (\beta_1 s - 2)k_2 + \gamma(-sc_1 + 2c_2)}{s^2 - 4}
\end{cases}
. \tag{69}
$$

If we also differentiate (58) and (62) and then substitute them into (59) and (64) we obtain a pair of dynamic equations for the investments k_i:

$$
\begin{cases}
\dot{\lambda}_{11} = -2\dot{k}_1 - \beta_2 \dot{\lambda}_{12} \\
\dot{\lambda}_{22} = -2\dot{k}_2 - \beta_1 \dot{\lambda}_{21}
\end{cases}
\iff
\begin{cases}
-2\dot{k}_1 - \beta_2 \dot{\lambda}_{12} = (\rho - \gamma)(-2k_1 - \beta_2 \lambda_{12}) + q_1 \\
-2\dot{k}_2 - \beta_1 \dot{\lambda}_{21} = (\rho - \gamma)(-2k_2 - \beta_1 \lambda_{22}) + q_2
\end{cases}
,
$$

which can be simplified using (60) and (63) to achieve:

$$
\begin{cases}
\dot{k}_1 = (\rho - \gamma)k_1 - \dfrac{q_1}{2} \\[3mm]
\dot{k}_2 = (\rho - \gamma)k_2 - \dfrac{q_2}{2}
\end{cases}
. \tag{70}
$$

Note that (70) do not contain any cross-multiplier because Proposition 8 is verified (namely, (27) corresponds to (58) and (62), whereas $\dfrac{\partial \dot{c}_i}{\partial c_i} = \gamma$ for all i, and $\dfrac{\partial \dot{c}_i}{\partial c_j} = 0$ if $i \neq j$).

---

[9] The concavity of H_P with respect to all control variables and its linearity with respect to all state variables can be easily ascertained. In particular, the Hessian matrix of H_P is negative definite if $4 - s^2 > 0$, which holds because $s \in [0,1]$.

After substituting q_1^* and q_2^*, the state-control dynamic system describing the dynamics of the differential game is the following:

$$
\begin{cases}
\dot{c}_1 = -k_1 - \beta_1 k_2 + \gamma c_1 \\[2mm]
\dot{c}_2 = -k_2 - \beta_2 k_1 + \gamma c_2 \\[2mm]
\dot{k}_1 = (\rho - \gamma)k_1 - \dfrac{(s-2)a + 2c_1 - sc_2}{2(s^2 - 4)} \\[3mm]
\dot{k}_2 = (\rho - \gamma)k_2 - \dfrac{(s-2)a - sc_1 + 2c_2}{2(s^2 - 4)}
\end{cases}
\qquad , \qquad (71)
$$

whose unique steady state is given by:

$$
\begin{cases}
c_1^{ss} = \Gamma \left\{ \beta_1 \beta_2 + 2(\beta_1 + 1)\gamma(2-s)(\gamma - \rho) - 1 \right\} \\[2mm]
c_2^{ss} = \Gamma \left\{ \beta_1 \beta_2 + 2(\beta_2 + 1)\gamma(2-s)(\gamma - \rho) - 1 \right\} \\[2mm]
k_1^{ss} = \Gamma \gamma \left[\beta_1 + 2\gamma(2-s)(\gamma - \rho) - 1 \right] \\[2mm]
k_2^{ss} = \Gamma \gamma \left[\beta_2 + 2\gamma(2-s)(\gamma - \rho) - 1 \right]
\end{cases}
\qquad , \qquad (72)
$$

where

$$
\Gamma = \frac{a}{2\gamma(\gamma - \rho)\left[8\gamma(\rho - \gamma) + 2\gamma s^2(\gamma - \rho) - s\beta_2 + 4 \right] + \beta_1 \left[\beta_2 + 2\gamma s(\rho - \gamma) \right] - 1}.
\tag{73}
$$

The Jacobian matrix in the steady state reads as

$$
J =
\begin{bmatrix}
\rho - \gamma & 0 & \frac{1}{4-s^2} & -\frac{s}{2(4-s^2)} \\[2mm]
0 & \gamma + \rho & -\frac{s}{2(4-s^2)} & \frac{1}{4-s^2} \\[2mm]
-1 & -\beta_1 & \gamma & 0 \\[2mm]
-\beta_2 & -1 & 0 & \gamma
\end{bmatrix}.
$$

The corresponding eigenvalues are

$$
e_{1,2,3,4} = \frac{\rho}{2} \pm \frac{1}{2}\sqrt{\frac{\Theta \pm \sqrt{\Omega}}{4 - s^2}}
\tag{74}
$$

where

$$\begin{aligned}
\Theta &= s(\beta_1 + \beta_2) - s^2(\rho - 2\gamma)^2 + 4(\rho - 2\gamma)^2 - 4 \quad (75) \\
\Omega &= 16\beta_1\beta_2 + s^2\left[(\beta_1 - \beta_2)^2 + 4\right] - 8s(\beta_1 + \beta_2)
\end{aligned}$$

The conditions $\Theta > 0$, $\Omega > 0$ and $|J| > 0$ are necessary and sufficient for all eigenvalues to be real, two being positive and two being negative. In such a case the steady state that represents the Nash equilibrium of the original differential game has saddle point stability with real roots (see Dockner and Feichtinger [6]).

Alternatively, the solution of the Hamiltonian potential can be expressed as a system of differential equations of costates and states given by (53), (54), (59) and (64). The remaining costate equations involving the cross-multipliers, i.e. (60) and (63), admit the following solutions:

$$\lambda_{12}^*(t) = \lambda_{12}^*(0)e^{(\rho-\gamma)t}, \qquad \lambda_{21}^*(t) = \lambda_{21}^*(0)e^{(\rho-\gamma)t}. \quad (76)$$

The above equations (76) allow for the special case in which the cross-multipliers are identically zero, $\lambda_{12}^* = \lambda_{12}^* = 0$, in which case the Hamiltonian potential is given by:

$$H_P^*(q_1, q_2, k_1, k_2, c_1, c_2, \lambda_{11}, 0, 0, \lambda_{22}) =$$

$$= \widehat{P}(q_1, q_2, k_1, k_2) + \beta_1\lambda_{11}k_2 + \beta_2\lambda_{22}k_1 + \lambda_{11}\dot{c}_1 + \lambda_{22}\dot{c}_2. \quad (77)$$

In such a case the function (77) can be rewritten as:

$$H_P^*(q_1, q_2, k_1, k_2, c_1, c_2, \lambda_{11}, \lambda_{22}) = \widehat{P}(q_1, q_2, k_1, k_2) + \lambda_{11}(-k_1 + \gamma c_1) + \lambda_{22}(-k_2 + \gamma c_2), \quad (78)$$

which is equivalent to the Hamiltonian function of a single agent that chooses 4 strategic variables in order to maximize the objective (potential) function $\widehat{P}(q_1, q_2, k_1, k_2)$ subject to two decoupled dynamic constraints. In terms of the model under examination, this would be the case of a single firm operating with multiple plants.

6   Concluding remarks and extensions

In a static potential game, the potential function contains all relevant information of the original n-player game. Analogously, a Hamiltonian potential for differential games must contain all relevant information of the original dynamic n-player game. This notion will necessarily differ from the one given in Monderer and Shapley [15] and the rest of the literature on static potential games, as state variables do not appear in static games.

In this paper we have introduced the concept of potential functions for differential games and we have determined sufficient conditions for the existence of

a Hamiltonian potential representing the original differential game as a single-agent optimal control problem. As an illustration, we have applied the concept of potential to differential games with linear state equations.

As a first attempt to explore the concept of potential in differential games, we have focused on open-loop Nash equilibria in simple settings. This leaves a perspective for future research toward two directions. First, one can study the existence of a Hamiltonian potential for differential games involving, e.g., non linear-quadratic games or non additively separable state equations. Second, as is basically introduced in Section 4, the existence and properties of potential functions for differential games solved under the feedback information structure, are still an open issue. Both directions are left for future research.

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