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"(S,s)-ADJUSTMENT STRATEGIES AND DYNAMIC HEDGING"

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Abstract

We study the destabilising effect of dynamic hedging strategies on the price of the underlying asset in the presence of transaction costs. Once transaction costs are taken into account, continuous portfolio re-hedging is no longer an optimal strategy. Using a non-optimising (local in time) strategy for portfolio rebalancing, explicit dynamics for the price of the underlying asset are derived, focusing in particular on excess volatility and feedback effects of these portfolio insurance strategies. Moreover, it is shown how these latter depend on the heterogeneity of the insured payoffs. Finally, conditions are derived under which it may be still reasonable, from a practical viewpoint, to implement Black-Scholes strategies.

1 Introduction

Standard option pricing literature relies on the hypothesis that the dynamics of the underlying asset are independent of the hedging strategy. Dynamic delta hedging strategies require to sell the underlying asset if its price decreases, while they require to buy if its price increases. The hypothesis of independency between strategies and price dynamics of the underlying asset corresponds to the assumption that the market for the underlying asset is perfectly liquid.

Positive feedback effects from dynamic delta hedging strategies have been studied recently assuming that the asset market for the underlying asset is only "finitely liquid", that is, relaxing one of the major assumptions of the Black-Scholes model that the market in the underlying asset is perfectly elastic[1] (see, for example, Frey and Stremme (1997), Schönbucher and Wilmott (2000), Sircar and Papanicolaou (1998), Gennotte and Leland (1990), Donaldson and Uhlig (1993)). It has been shown that in this case portfolio insurance activity has a destabilising effect on the dynamics of the price of the underlying asset. In particular, it increases the volatility of the price of the underlying asset. Frey and Stremme (1997) study the feedback effects of dynamic hedging strategies on the volatility of the market equilibrium price of the underlying asset. They derive the tracking error, show that an overinvestment is required and derive the best volatility used by program traders in calculating their trading strategy. Sircar and Papanicolaou (1998) study the interaction between reference traders and program traders and the feedback effects of program traders on the underlying asset, deriving a feedback adjusted option price and the optimal hedging strategy. Frey (1998), in a continuous time version of Jarrow (1992, 1994), study the replication of derivative securities from the viewpoint of large traders, whose trades have a non-negligible effect on the asset price. In a similar vein, Schönbucher and Wilmott (2000) discuss the pricing, hedging and replication of options if a larger trader, for whom the market is illiquid, interacts with

[1] Brennan and Schwartz (1989), on the contrary, study the effects of portfolio insurance on financial markets abstracting from possible liquidity problems. They assume that agents are only concerned with long-term prospects of the assets and do not change their expectations in reaction to changes in current prices. As a result, markets are very liquid and the feedback effects of hedging on volatility become very small.

small traders. Among the contributions related to the role of portfolio insurance strategies in the market, some relevant papers have demonstrated the possibility of multiple equilibrium prices in illiquid markets. Gennotte and Leland (1990) show that information differences among market participants can cause markets to be relatively illiquid and discontinuities (or "crashes") can occur even with relatively little hedging. Like Gennotte and Leland (1990), Donaldson and Uhlig (1993) show that the existence of atomistic porfolio insurers increase the variance of possible equilibrium prices and can lead to situations with many potential equilibrium prices for a single set of fundamentals, unless large porfolio insurers act in a centralized way.

All the above-mentioned papers assume that program traders can buy and sell assets without incurring transaction costs. But, as a matter of fact, transaction costs are non-negligible in asset markets. Our paper, which is most closely aligned with Frey and Stremme (1997), extends the literature on market equilibrium models with feedback effects caused by dynamic hedging to the case of transaction costs.

If we introduce transaction costs, then it is no longer optimal to adjust the portfolio continuously. There are two main approaches in the literature taking the effects of transaction costs into account: the first considers discrete adjustments of the portfolio, where the time step of portfolio rebalancing is exogenously given, while the second considers traders as continuously monitoring the price of the underlying asset, although adjusting their portfolio only if the gain from adjustment is greater than the cost of adjustment. This latter approach can be subdivided into two further approaches: the first is called local in time (Leland (1985), Hoggard, Whalley and Wilmott (1994)), while the second is called global in time (Davis, Panas and Zariphopoulou (1993), Whalley and Wilmott (1997), Constantinides and Zariphopoulou (1999), (2001)). The former is a non-optimizing approach, where re-hedging is based on minimizing the variance of the hedged portfolio, while the latter is an optimizing one and is based on utility maximization and stochastic optimal control. The option value is obtained by a comparison of the maximum utilities of trading with and without the obligation of fulfilling the option contract at expiry (see, for example, Wilmott (2000) for a review). In this framework, Davis, Panas and Zariphopoulou (1993) consider European option pricing with proportional transaction costs charged on sales and purchases of stock. Whalley and Wilmott (1997) provide an asymptotic analysis of Davis, Panas and Zariphopoulou (1993) in the limit of small transaction costs. Constantinides and Zariphopoulou (1999, 2001) derive in closed form bounds to the reservation price of a call option for a large class of utility functions.

All these papers are concerned with the pricing of derivatives in the presence of transaction costs, but do not deal with the hedging effects in a market model.

Our paper considers a "finitely liquid" market model and, following a local in time approach, assumes that hedging takes place at flexible stochastic trading periods, instead of fixed interval times, and that transaction costs are fixed costs, instead of proportional costs[2].

Once transaction costs are taken into account, an appropriate hedging strategy has to be found since it is no longer optimal to rehedge immediately the portfolio as the price of the asset changes. In his seminal paper Leland (1985) first introduced proportional transaction costs and developed a pricing model with a modified option replicating strategy depending on the level of transaction costs and the revision interval. Such strategy replicates the option inclusive of transaction costs, with an error that is uncorrelated with the market and is claimed to approach zero as the revision period becomes shorter. Inclusive of transaction costs, the bid-ask spread of the underlying asset becomes larger and the accentuation of up and down movements of the asset price is modelled as if the volatility of the actual asset price is higher. Kabanov and Safarin (1997) calculate the limiting error in Leland's hedging strategy for the approximate pricing of the European call. They partially correct a result in Leland (1985), showing that such limiting error equals zero only when the level of transaction costs decreases to zero as the revision interval tends to zero. Bensaid, Lesne, Pages and Scheinkman (1992) deal with a discrete time model with proportional transaction costs and derive sequentially optimal portfolios, finding dominating strategies of the (S,s)-type and a range within which the derivative price should lie, defining its bid-ask spread.

Most above-mentioned papers with transaction costs assume a given fixed revision interval. In this paper instead we consider a stochastic revision interval. For this purpose we introduce adjustment hazard functions for each program trader, in a way which is new in this literature. Following Whalley and Wilmott (1993) and Henrotte (1993) we define a confidence level for the deviation of the risky asset position from the perfect hedge such that for each agent inaction is optimal as long as the hedging unbalance level is below a tolerance level $\tilde{H}_0$, while the position should be rebalanced once the hedging unbalance level is above $\tilde{H}_0$. The parameter $\tilde{H}_0$ gives a measure of the maximum expected risk in the portfolio.

We will assume that $\tilde{H}_0$ is partly deterministic and partly stochastic: the former captures the influence of transaction costs, while the latter captures stochastic contingencies. Given these assumptions, we can define an adjustment hazard function for each program trader. Following Caballero and Engel (1993, 1999) we study the aggregate dynamics of the adjustment hazard rates. Then, we study the resulting price dynamics of the risky asset. We show that the average size of the adjustment depends on the adjustment size function, which depends on transaction costs as well. Our main results are in keeping with the literature on increased market volatility from hedging strategies; however, we specify in which way markets are finitely liquid when transaction costs are introduced and the role of transaction costs in determining the size and the frequency of adjustment.

The paper is organised as follows. In Section 2 the adjustment hazard func-

[2] The model could be extended trivially in order to consider also proportional transaction costs.

tion is formally introduced and the model is presented. Section 3 contains the main results. Finally, Section 4 concludes.

2 The basic model

Suppose that there are two types of traders operating in a market, where there is a risky asset and a riskless one (a pure discount bond): program traders and reference traders. Program traders use a dynamic hedging strategy, while reference traders are small price takers, which include market makers and market timers, providing liquidity for market transactions. In what follows, we do not model the reference traders' investment problem explicitly, but rather model the aggregate behaviour of them. We assume that there is a continuum of reference traders, such that the effects of transaction costs on the aggregate demand function of reference traders are negligible. The reference traders are unaware of the program traders' presence and of their trading strategy (so that we can avoid strategic trading). Furthermore, we suppose that reference traders have perfect information about the fundamentals of the risky asset. The aggregate demand function of reference traders is denoted by $D(t, F_t, S_t)$, where S is the price of the risky asset, t is time and F can be interpreted in different ways. For example, Frey and Stremme (1997) and Sircar and Papanicolaou (1998) assume F to be the aggregate income of reference traders, while Platen and Schweizer (1994) assume F to be an unspecified liquidity demand, and others assume F to be the fundamental value of the firm. We follow this latter approach, and make the following assumptions about the aggregate demand of reference traders:

Assumption 1.

a) $D(t, F_t, S_t)$ *is a smooth function*

b) *there exists* $d > 0$ *such that* $\frac{\partial D}{\partial S} \leqslant -d$

c) $\frac{\partial D}{\partial F} > 0$

Assumption 1.b) indicates that as the price of the asset increases, demand decreases, so that, everything else being equal, the reference traders would like to hold more assets if the price is low and fewer if it is high. Assumption 1.c) means that as the fundamental value of the asset increases, demand increases as well. In Assumption 1.b) the derivative of demand with respect to price is negative. Economically, ruling out the derivative to be zero means that demand does react to price changes; otherwise it would not be possible to find an equilibrium by adjusting the price and the market would be illiquid. Actually, Assumption 1.b) implies that the derivative of demand with respect to price is bounded below. This latter requirement can be relaxed, following a compactness argument, where Assumption 1.b) is satisfied by a suitable sequence of $D^n(.)$, $n = 1, 2, ...$converging uniformly to D.

Let us normalize the total supply of the risky asset to one. Thus, in the absence of program traders, equilibrium is guaranteed by the following market clearing condition

$$D\left(t, F_t, S_t^*\right) = 1 \tag{1}$$

For every pair (t, F_t) the equation (1) has exactly one solution in S_t denoted by $\varphi(t, F_t)$. Thus, the equilibrium price of the asset, i.e. $S_t^* = \varphi(t, F_t)$, is a function of its fundamental value. We will call S_t^* the normal price of the asset.

We are looking for a diffusion process for the asset price of the form

$$dS_t = \mu_S\left(S_t, t\right) dt + \sigma_S\left(S_t, t\right) dW_t \tag{2}$$

where W_t denotes a Wiener process. Once we have specified the demand function of the reference traders and the stochastic process for the fundamental value we can determine the dynamics of the asset price S_t^*.

We will assume that the dynamics of the fundamental value of the risky asset follow a diffusion process of the form

$$dF_t = \mu_F\left(F_t, t\right) dt + \sigma_F\left(F_t, t\right) dW_t. \tag{3}$$

Using the equilibrium condition (1), the dynamics of the fundamental value (3) and the fact that we are looking for a diffusion process for the risky asset of the type (2), we have that, in equilibrium, the following condition has to be satisfied

$$\begin{array}{c} 0 = \left(D_S \sigma_S\left(S_t^*, t\right) + D_F \sigma_F\left(F_t, t\right)\right) dW_t + \\ + \left(D_S \mu_S\left(S_t^*, t\right) + D_F \mu_F\left(F_t, t\right) + \right. \\ + \frac{1}{2} D_{SS}\left(\sigma_S\left(S_t^*, t\right)\right)^2 + \frac{1}{2} D_{FF}\left(\sigma_F\left(F_t, t\right)\right)^2 + \\ \left. + D_{SF} \sigma_S\left(S_t^*, t\right) \sigma_F\left(F_t, t\right) + D_t\right) dt \end{array} \tag{4}$$

In order to satisfy condition (4) we need the stochastic as well as the deterministic term in equation (4) equal to zero; therefore, we obtain the following moments for the risky asset price dynamics:

$$\sigma_S\left(S_t^*, t\right) = -\sigma_F\left(F_t, t\right) \frac{D_F}{D_S}$$

$$\begin{aligned} \mu_S\left(S_t^*, t\right) \quad = \quad & -\frac{1}{D_S}\left[D_t + D_F \mu_F\left(F_t, t\right) + \frac{1}{2} D_{SS}\left(\sigma_F\left(F_t, t\right) \frac{D_F}{D_S}\right)^2 + \right. \\ & \left. + \frac{1}{2} D_{FF}\left(\sigma_F\left(F_t, t\right)\right)^2 - D_{SF}\left(\sigma_F\left(F_t, t\right)\right)^2 \frac{D_F}{D_S}\right] \end{aligned} \tag{5}$$

Thus, the price dynamics of the risky asset S_t^* follow a diffusion process (4), where $\sigma_S(S_t^*, t)$ and $\mu_S(S_t^*, t)$ are given by expressions (5).

In the next Section we are going to study the aggregate demand of program traders. Then, we plug the aggregate demand of program traders into expression (1) and study its implications for the price dynamics of the risky asset.

2.1 Aggregate demand of program traders

In this Section we specify the aggregate demand of program traders, which follow a dynamic hedging strategy. For simplicity, all program traders act collectively, so that the hedging strategy for a portfolio of payoffs is just the portfolio of the hedging strategies for the individual payoffs, and we can consider just a representative program trader. As we argued in Section 1, in the case of no transaction costs, continuous adjustment of the portfolio is optimal. But once we introduce transaction costs, and in particular fixed costs of adjustment, continuous adjustment of the portfolio is no longer optimal.

Let us define a confidence level $\tilde{H}_0$ and suppose it is a function $\tilde{H}_0(c, H_0)$, where $c \geqslant 0$ is the deterministic component, while H_0 is the stochastic component. The deterministic component c captures the influence of the size of the transaction costs on the confidence level. An increase in the transaction costs increases c. The stochastic component H_0 captures the influence of stochastic contingencies on the confidence level.

Denote by G the portfolio of the representative program trader and by $V(S, \sigma, \tau, K)$ the option value, where S is the current underlying asset price, K is the strike price, τ is the time to maturity and σ is volatility.

We define a probability of adjusting the portfolio in the following way

$$\Pr\left(\tilde{H}_0(c, H_0) \leqslant |\eta_{t+dt}| \;\middle|\; |\eta_t| < \tilde{H}_0(c, H_0)\right) = h(\eta, c)\, dt$$

where $\eta = \eta(S, \sigma, \tau, K) = \Delta(S, \sigma, \tau, K) - G$, and $\Delta(S, \sigma, \tau, K) = \frac{\partial V(S,\sigma,\tau,K)}{\partial S}$.

We assume that the distribution of H_0 is the same for each program trader, and thus, the probability of portfolio rehedging depends just on the hedging unbalance level η and on the level of the deterministic component c. We make the following assumptions about the adjustment hazard function $h(\eta, c)$:

Assumption 2.

a) $h(\cdot)$ *is a smooth function*

b) $h_\eta(\eta, c) > 0$ *and* $h_c(\eta, c) < 0$

c) $\lim_{c\to 0} h(\eta, c) = 1$ *and* $\lim_{c\to\infty} h(\eta, c) = 0$.

Assumption 2.b) states that the probability of adjustment increases as the hedging unbalance level increases and/or as the size of the transaction costs decreases. Assumption 2.c) implies that, if the size of the transaction costs is vanishing small, then the probability of portfolio adjustment converges towards one, that is, we have continuous adjustment or dynamic delta hedging, while, if the transaction costs are infinitely large, then the probability of portfolio adjustment becomes vanishing small.

The hedging unbalance level η has a common element for each program trader, which is the price of the underlying S, while there are also idiosyncratic components such as the strike price K and time to maturity τ. Thus, as the price of the underlying asset changes, the hedging unbalance level for each program trader changes as well, while the way it changes depends on the distribution of strike prices and of the time to maturity.

By aggregating program traders over their hedging unbalance levels we have that the average demand of the risky asset over a time interval dt becomes

$$g(S,\sigma,c,t) = \int_{\Re} \eta h(\eta,c) f(\eta)\, d\eta$$

where $f(\eta)$ is the distribution of the hedging unbalance level over the program traders. This latter distribution is not time-invariant since, as the time goes on τ changes, and thus, according to the distribution of τ, the distribution of η changes. We are going to assume a continuous and random influx and outflux of program traders from the asset market, and heterogeneity in the distribution of strikes. Thus, we have that the average demand over a small time period dt of program trader is given by

$$\Psi(S,\sigma,c) = \int\limits_{\Re_+^2 \times R} \eta(S,\sigma,\tau,K)\, h(\eta(S,\sigma,\tau,K),c)\, f(\eta)\, \upsilon(dK \oplus d\tau \oplus d\eta) \quad (6)$$

where υ has a smooth density function with respect to a Lebesgue-measure. Thus, expression (6) represents the demand for the underlying asset of the program traders over a small time period dt. We are interested to see how this demand changes as the price of the underlying changes.

Consider the change in the price of the risky asset of size dS. Using expression (6) we have:

$$\begin{aligned}\Psi(S+dS,\sigma,c) = \int\limits_{\Re_+^2 \times R} & [\Delta(S+dS,\sigma,\tau,K) - \Delta(S,\sigma,\tau,K) + \eta] \times \\ & \times h(\Delta(S+dS,\sigma,\tau,K) - \Delta(S,\sigma,\tau,K) + \eta, c) \times \\ & \times f(\eta)\, \upsilon(dK \oplus d\tau \oplus d\eta)\end{aligned}$$

Taking Taylor expansion of $\Delta(S+dS,.)$ and $h(\Delta(S+dS,.) - \Delta(S,.) + \eta)$ around S and η respectively, we have

$$\Delta(S+dS,.) - \Delta(S,.) = \Delta_S dS + \frac{1}{2}\Delta_{SS}(dS)^2$$

and

$$\begin{aligned}h(\Delta(S+dS,.) - \Delta(S,.) + \eta, c) = {} & h(\eta,c) + h_\eta(\eta,c)\left(\Delta_S dS + \tfrac{1}{2}\Delta_{SS}(dS)^2\right) + \\ & + \tfrac{1}{2} h_{\eta\eta}(\eta,c)\left(\Delta_S dS + \tfrac{1}{2}\Delta_{SS}(dS)^2\right)^2\end{aligned}$$

Now we can calculate $d\Psi(S,\sigma,c) = \Psi(S+dS,\sigma,c) - \Psi(S,\sigma,c)$ as follows

$$d\Psi(S,\sigma,c) = \int\limits_{\Re_+^2\times R} \left\{ \left[\Delta_S dS + \tfrac{1}{2}\Delta_{SS}(dS)^2 \right] \left[h(\eta,c) + h_\eta(\eta,c)\left(\Delta_S dS + \right.\right.\right.$$
$$\left.+\tfrac{1}{2}\Delta_{SS}(dS)^2\right) + \tfrac{1}{2}h_{\eta\eta}(\eta,c)\left(\Delta_S dS + \tfrac{1}{2}\Delta_{SS}(dS)^2\right)^2 \Big] +$$
$$+\eta h_\eta(\eta,c)\left(\Delta_S dS + \tfrac{1}{2}\Delta_{SS}(dS)^2\right) +$$
$$\left.+\tfrac{1}{2}\eta h_{\eta\eta}(\eta,c)\left(\Delta_S dS + \tfrac{1}{2}\Delta_{SS}(dS)^2\right)^2 \right\} f(\eta)\,\upsilon(dK\oplus d\tau\oplus d\eta)$$

Since $(dS)^\theta = 0$ for $\theta > 2$ we obtain, after rearranging terms

$$d\Psi(S,\sigma,c) = dS \int\limits_{\Re_+^2\times R} \left[h(\eta,c) + \eta h_\eta(\eta,c) \right] \Delta_S f(\eta)\,\upsilon(dK\oplus d\tau\oplus d\eta) +$$
$$+\tfrac{1}{2}(dS)^2 \int\limits_{\Re_+^2\times R} \left\{ \left[h(\eta,c) + \eta h_\eta(\eta,c) \right] \Delta_{SS} + \right.$$
$$\left. + \left[2h_\eta(\eta,c) + \eta h_{\eta\eta}(\eta,c) \right] (\Delta_S)^2 \right\} f(\eta)\,\upsilon(dK\oplus d\tau\oplus d\eta)$$

This latter expression can be rewritten as follows

$$d\Psi(S,\sigma,c) = H_1(S,\sigma,c)\,dS + \frac{1}{2}H_2(S,\sigma)(dS)^2 \tag{7}$$

where

$$H_1(S,\sigma,c) = \int\limits_{\Re_+^2\times R} \tilde{h}(\eta,c)\,\Delta_S(S,\sigma,K,\tau)\,f(\eta)\,\upsilon(dK\oplus d\tau\oplus d\eta) \tag{8}$$

$$H_2(S,\sigma,c) = \int\limits_{\Re_+^2\times R} \left\{ \tilde{h}(\eta,c)\,\Delta_{SS} + \tilde{h}_\eta(\eta,c)\,(\Delta_S)^2 \right\} f(\eta)\,\upsilon(dK\oplus d\tau\oplus d\eta)$$

and where $\tilde{h}(\eta,c) = \frac{\partial}{\partial\eta}\eta h(\eta,c) = h(\eta,c) + \eta h_\eta(\eta,c)$ and $\tilde{h}_\eta = \frac{\partial}{\partial\eta}\tilde{h}(\eta,c) = 2h_\eta(\eta,c) + \eta h_{\eta\eta}(\eta,c)$.

Let us introduce the following technical assumption:

Assumption 3. *There exist functions υ_1 and υ_2 such that $\upsilon(dK\oplus d\tau\oplus d\eta) = \upsilon_1(d\eta)\,\upsilon_2(k,\tau)dKd\tau$ where υ_1 and υ_2 have a smooth density function with respect to a Lebesgue measure; υ_2 has a compact support in $R_+\times[0,\infty)$.*

Assumption 3 implies that the distribution of strike prices/times to maturity and hedging unbalance levels is relatively heterogeneous. It will become clear that under Assumption 3 we can control the feedback effects on market volatility (see Proposition 2).

With Assumption 3, we can rewrite expression (8) as follows

$$H_1(S,\sigma,c) = \tilde{H}(c)\,\tilde{\Gamma}(S,\sigma) \tag{9}$$

where

$$\tilde{H}(c) = \int_{\Re} \tilde{h}(\eta, c) f(\eta) \upsilon_1(d\eta) \tag{10}$$

$$\tilde{\Gamma}(S, \sigma) = \int_{\Re_+^2} \Delta_S(S, \sigma, K, \tau) \upsilon_2(k, \tau) dK d\tau \tag{11}$$

$\tilde{H}(c)$ indicates the stationary average size of the adjustment, given a change in the hedge unbalance level. $\Delta_S(S, \sigma, K, \tau)$ is known in the option pricing literature as the parameter gamma, and it indicates, in the absence of transaction costs, how often a position must be rehedged in order to maintain a delta-neutral position. Thus, $\tilde{\Gamma}(S, \sigma)$ is the stationary average value of the gamma, which indicates, in the absence of transaction costs, how often in the stationary state, a position must be adjusted on average in order to keep delta-neutral positions.

$H_1(S, \sigma, c)$ indicates the stationary average adjustment, given a change in the price of the risky asset. $H_1(S, \sigma, c)$ depends: a) on the average size of adjustment, and thus on the properties of the adjustment hazard function and on the stationary state distribution of unbalance levels, and b) on the frequency of adjustment, which depends on the stationary average sensibility of the delta with respect to the price of the underlying. Notice that Assumption 2 implies that $\frac{\partial}{\partial c} H_1(S, \sigma, c) < 0$ and $\lim_{c \to 0} H_1(S, \sigma, c) = \tilde{\Gamma}(S, \sigma)$, while $\lim_{c \to \infty} H_1(S, \sigma, c) = 0$. In other words, $H_1(S, \sigma, c)$ is a decreasing function of c.

Furthermore, if the size of the transaction costs is vanishing small, then $H_1(S, \sigma, c)$ converges towards the stationary average value of the gamma and so we are back to the case of dynamic hedging strategies, while if the transaction costs are very large, then no portfolio adjustment occurs at all. Finally, by Assumption 2 we have $H_1(S, \sigma, c) \geqslant 0$.

3 Positive feedback effects from hedging

As we pointed out before, reference traders have perfect information about the fundamental value of the risky asset. Thus, a reduction in the fundamental value leads to a decrease in the price of the risky asset. Given this decrease, program traders will sell the risky assets in order to adjust their portfolio. This latter leads to a further price reduction of the risky asset, which now will be lower than its normal level, i.e. $S_t < S_t^*$. Thus, the action of program traders leads to potential gains for liquidity providers, such as market makers and market timers (see Grossman, 1988). These latter could buy the assets since their actual price is now lower than their normal price. In this way, liquidity providers have a stabilising function. Such ability to exploit gains from excess volatility of price dynamics depends on some parameters, for example, the cost of capital, transaction costs (c) and also the information about how many agents are using a dynamic hedging strategy. If liquidity providers commit insufficient capital, then their stabilising function will be reduced.

Let us indicate by $\rho(c, \Im) \in [0,1]$ a parameter measuring market liquidity, which is related to the action of the liquidity providers (market timers and market makers) in response to the program traders' demand, which may affect the price dynamics. $\rho(c, \Im)$ is a function of c, the transaction costs, and $\Im \in \Re_+$, which captures the effects of other variables on market liquidity, such as lack of information on hedging activity and cost of capital. We take $\Im$ and c as exogenous variables. We make the following assumptions about the behaviour of $\rho(c, \Im)$:

Assumption 4.

a) $\frac{\partial \rho}{\partial c} > 0$, $\frac{\partial \rho}{\partial \Im} > 0$

b) $\lim_{c \to \infty} \rho(c, \Im) = 1$

c) $\lim_{(c,\Im) \to (0,0)} \rho(c, \Im) = 0$

d) $\lim_{c \to 0} \rho(c, \Im) \geqslant 0$ *and* $\lim_{c \to 0} \rho(c, \Im) > 0$ *as long as* $\Im > 0$.

Assumption 4.a) implies that an increase in the transaction costs and in the exogenous parameter $\Im$ reduces the liquidity of the market; 4.b) implies that as the transaction costs diverge towards infinity, the market is completely illiquid; 4.c) implies that the market is perfectly liquid if the supply of capital is perfectly elastic, information is perfect and if there are no transaction costs; 4.d) implies that if transaction costs are vanishing small, then the market will still be illiquid, where the size of illiquidity depends on the size of the exogenous variable $\Im$.

Let us add the demand of program traders to the demand of reference traders. It can be done since reference traders are supposed to be unaware of the presence of program traders, otherwise they would condition $D(t, F_t, S_t)$ on the program traders' strategy itself. Using the market clearing condition, we have that the equilibrium price has to satisfy the following condition

$$D(t, F_t, S_t) + \rho(c, \Im)\,\Psi(S_t, \sigma, c) = 1 \qquad (12)$$

According to Assumption 4, for $(c, \Im) \to (0,0)$ the action of the program traders has a negligible effect on the price dynamics of the risky asset. Thus, each deviation of prices from their normal level will be eliminated through the action of market timers. On the other side, if the supply of capital is not perfectly elastic or market timers do not have perfect information, then market timers cannot completely eliminate the effect of the action of the program traders (see Grossman, 1988). Therefore, $\rho(c, \Im) = 0$ denotes a perfectly liquid market, that is, liquidity providers are able to neutralize program traders' demand and thus there is no deviation of the asset price from its fundamental value; as long as $\rho(c, \Im) > 0$ the market for the underlying asset is only finitely liquid, and for $\rho(c, \Im) = 1$ it becomes completely illiquid. Notice that an increase in the transaction costs increases the weight of the portfolio insurance in the aggregate demand.

Now we can prove the main result.

Proposition 1 *The diffusion process governing the dynamics of the asset price is of the form (2) with parameters:*

$$\sigma_S(S_t, t; \sigma, c) = -\frac{D_F}{D_S + \rho(c, \Im)\tilde{H}(c)\tilde{\Gamma}(S_t, \sigma)}\sigma_F(F_t, t) \tag{13}$$

$$\begin{gathered}\mu_S(S_t, t; \sigma, c) = -\Theta[D_t + D_F\mu_F(F_t, t) + \\ -D_{SF}D_F\ \Theta\ (\sigma_F(F_t, t))^2 + \\ +\tfrac{1}{2}((D_{SS} + \rho(c, \Im) H_2(S_t, \sigma, c))(D_F\ \Theta\sigma_F(F_t, t))^2 + D_{FF}(\sigma_F(F_t, t))^2)]\end{gathered}$$

where $\Theta = \left[D_S + \rho(c, \Im)\tilde{H}(c)\tilde{\Gamma}(S_t, \sigma)\right]^{-1}$.

Proof. Taking total differential of (12) and using (7) we have that

$$\begin{gathered}0 = D_t + D_S dS + D_F dF + \tfrac{1}{2}D_{SS}(dS)^2 + D_{SF}dSdF + \\ +\tfrac{1}{2}D_{FF}(dF)^2 + \rho(c, \Im) H_1(S_t, \sigma, c)\, dS + \tfrac{1}{2}\rho(c, \Im) H_2(S_t, \sigma, c)(dS)^2\end{gathered}$$

Using the stochastic processes (4) and (3) we can rewrite this condition as follows

$$\begin{gathered}0 = [D_t + D_S\mu_S(S_t, t) + D_F\mu_F(F_t, t) + \tfrac{1}{2}D_{SS}(\sigma_S(S_t, t))^2 + \\ +D_{SF}\sigma_S(S_t, t)\,\sigma_F(F_t, t) + \tfrac{1}{2}D_{FF}(\sigma_F(F_t, t))^2) + \\ +\rho(c, \Im) H_1(S_t, \sigma, c)\,\mu_S(S_t, t) + \tfrac{1}{2}\rho(c, \Im) H_2(S_t, \sigma, c)(\sigma_S(S_t, t))^2]dt + \\ +[D_F\sigma_F(F_t, t) + D_S\sigma_S(S_t, t)) + \rho(c, \Im) H_1(S_t, \sigma, c)\,\sigma_S(S_t, t)]\, dW_t\end{gathered} \tag{14}$$

Since (14) has to be true, we need the deterministic as well as the stochastic term in (14) to be equal to zero. Thus, we obtain the values in expression (13) for $\sigma_S(S_t, t)$ and $\mu_S(S_t, t)$. Obviously, we need $\sigma_S(S_t, t) \geqslant 0$, and this is true for appropriate values of d in Assumption 1. Thus the price of the underlying follows a non-linear diffusion process (2). ■

From expression (13) in Proposition 1 we observe that the larger the average gamma, the larger the volatility of the asset price of the underlying asset. Since gamma indicates how often a position must be rehedged on average in order to maintain a delta-neutral position, the higher is its average value, the more frequently an adjustment occurs. At the same time, $\tilde{H}(c)$ indicates the average size of the adjustment, given a change in the hedge unbalance level. This latter depends on the properties of the adjustment hazard function and on the size of the deterministic component of the confidence level c. This latter component depends directly on the size of the transaction costs. Thus, the larger the transaction costs, the lower $\tilde{H}(c)$. On the other side, the larger the transaction costs, the less liquid the market, and therefore the weight of the demand of the portfolio insurance in the aggregate demand of the risky asset increases. Thus, the effect of a change in the transaction costs on the volatility of the underlying asset is a priori ambiguous. Given Assumptions 2 and 4 we have that if the transaction costs are infinitely high, then $\lim_{c\to\infty} \rho(c, \Im) \tilde{H}(c) = 0$ and so on average no portfolio adjustment occurs and there will be no feedback effect, i.e. $S_t = S_t^*$. On the other side, if transaction costs are vanishing small, then, since $\lim_{c\to 0} \tilde{H}(c) = 1$, we are back to dynamic hedging strategies where the liquidity of the market $\rho(0, \Im)$ depends just on the size of $\Im$, and thus we are back to a situation like the one studied by Frey and Stremme (1997).

It will become clear in view of Proposition 2 that an appropriate choice of parameter values prevents $\sigma_S(S_t, t; \sigma, c)$ from becoming

negative.

We can summarize our results as follows. The volatility of the underlying asset is larger, the larger is the frequency of adjustment and/or the larger the size of adjustment ($\tilde{H}(c)$). The volatility depends also on $\rho(c, \Im)$. The more illiquid is the market, the higher is the influence of the hedging activity of the program traders on the price dynamics of the underlying asset, that is, the higher is the feedback effect.

Notice that as long as $\rho(c, \Im) > 0$ and $c < \infty$ the price dynamics of the risky asset S_t are different from the dynamics of the normal price S_t^*. In particular, comparing (13) with (5) we observe that there exists an excess volatility due to the hedging activity of program traders. The size of the excess volatility depends on the liquidity of the market ($\rho(c, \Im)$) and on the aggregate characteristics of program traders ($\tilde{H}(c)\tilde{\Gamma}(S_t, \sigma)$). But since expression (13) for the volatility $\sigma_S(S_t, t; \sigma, c)$ still depends on the "input volatility" σ, consistency requires that the input volatility σ be equal to the actual observed volatility. In other words, we have to solve a fixed point problem.

In solving the fixed point problem, we will make use of the following Lemma.

Lemma *(i) $\tilde{\Gamma}$ is a bounded function of σ; (ii) for $\sigma \geqslant \sigma_0$, with $0 < \sigma_0 < \infty$, $\frac{\partial}{\partial\sigma}\tilde{\Gamma}$ is a bounded function of σ.*

Proof. (i) The following equalities hold, because of the definition of Δ and Assumption 3:

$$\tilde{\Gamma} = \int\int \frac{\partial\Delta}{\partial S} v_2(K, \tau)\, dK d\tau = -\int\int \frac{K}{S}\frac{\partial\Delta}{\partial K} v_2(K, \tau)\, dK d\tau$$

$$\int\int \Delta \frac{\partial}{\partial K}\left(\frac{K}{S} v_2\right) dK d\tau$$

since v_2 has a compact support. Since $0 \leqslant \Delta \leqslant 1$, we get:

$$\left|\tilde{\Gamma}\right| \leqslant \int\int \left|\frac{\partial}{\partial K}\left(\frac{K}{S} v_2\right)\right| dK d\tau$$

that is, $\tilde{\Gamma}$ is a bounded function of σ.

(ii) Recall that $\frac{\partial \Delta}{\partial S} = \frac{N'(d_1)}{\sigma S\sqrt{\tau}}$, where $d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$.

Then, $\frac{\partial}{\partial \sigma}\left(\frac{N'(d_1)}{\sigma S\sqrt{\tau}}\right) \leqslant N''(d_1)\frac{1}{\sigma S\sqrt{\tau}} - \frac{N'(d_1)}{\sigma^2 S\sqrt{\tau}} \leqslant \frac{1}{\sigma_0 S\sqrt{\tau}} + \frac{1}{\sigma_0^2 S\sqrt{\tau}}$, that is, $\frac{\partial}{\partial \sigma}\tilde{\Gamma}$ is a bounded function of σ for $\sigma > \sigma_0$. ■

We are now able to state and proof the following Proposition.

Proposition 2 *Under Assumptions 1-4, there exists a solution of the fixed point problem*

$$\sigma_S(S_t, t; \sigma, c) = -\frac{D_F}{D_S + \rho(c, \Im)\tilde{H}(c)\tilde{\Gamma}(S_t, \sigma)}\sigma_F(F_t, t) \tag{15}$$

provided that $\rho(c, \Im)$ is sufficiently small.

Proof. Let us put $M(\sigma) = \sigma_S(S, t; \sigma, c) = -\frac{D_F \sigma_F(F_t, t)}{D_S + \rho(c,\Im)\tilde{H}(c)\tilde{\Gamma}(S_t,\sigma)}$. We have to show that $\left|\frac{\partial M(\sigma)}{\partial \sigma}\right| \leqslant M < 1$, with $0 \leqslant M \leqslant 1$, in order to apply the contraction mapping theorem.

We have that:

$$\left|\frac{\partial M(\sigma)}{\partial \sigma}\right| = \left|\frac{D_F \sigma_F \rho\left(\frac{\partial \tilde{H}}{\partial \sigma}\tilde{\Gamma} + \tilde{H}\frac{\partial \tilde{\Gamma}}{\partial \sigma}\right)}{\left(D_S + \rho\tilde{H}\tilde{\Gamma}\right)^2}\right|$$

Let us first consider the denominator. If $|\rho| < \varepsilon$ we get, for some $\tilde{e}$:

(16) $\left|D_S + \rho\tilde{H}\tilde{\Gamma}\right| \geqslant |D_S| - |\rho|\left|\tilde{H}\tilde{\Gamma}\right| \geqslant d - \varepsilon\tilde{e} \geqslant \frac{d}{2}$

provided that $\varepsilon \leqslant \frac{d}{2\tilde{e}}$. (16) holds because of Assumption 1, the Lemma, and in view of the fact that $\tilde{H}$ is a bounded function of σ, since $0 \leqslant \Delta \leqslant 1$, and $h(\eta, c)$, $f(\eta)$, $v_1(d\eta)$ are bounded functions of σ.

Let us consider the numerator. We get, with suitable constants $\tilde{\tilde{e}}$, J:

$$\left|-D_F \sigma_F \rho\left(\frac{\partial \tilde{H}}{\partial \sigma}\tilde{\Gamma} + \tilde{H}\frac{\partial \tilde{\Gamma}}{\partial \sigma}\right)\right| =$$
$$= |D_F \sigma_F|\,|\rho|\left|\frac{\partial \tilde{H}}{\partial \sigma}\tilde{\Gamma} + \tilde{H}\frac{\partial \tilde{\Gamma}}{\partial \sigma}\right| \leqslant D_F \sigma_F |\rho|\tilde{\tilde{e}} \leqslant J\varepsilon$$

It holds because of the Lemma and in view of the fact that $\tilde{H}$ and $\frac{\partial \tilde{H}}{\partial \sigma}$ are bounded functions of σ. Therefore, $\left|\frac{\partial M(\sigma)}{\partial \sigma}\right| \leqslant \frac{4J\varepsilon}{d^2} = M \leqslant 1$, which holds for $\varepsilon \leqslant \min\left\{\frac{d}{2\tilde{e}}, \frac{d^2}{4J}\right\}$.

Finally, we have to check that $\sigma_S(S_t, t; \sigma, c) = M(\sigma) \geqslant \sigma_0$, where $0 < \sigma_0 < \infty$, as required by the Lemma. Since $\frac{-D_F \sigma_F}{D_S + \rho\tilde{H}\tilde{\Gamma}} = \frac{D_F \sigma_F}{-D_S - \rho\tilde{H}\tilde{\Gamma}} \geqslant \frac{D_F \sigma_F}{-D_S} > 0$, we can put $\sigma_0 = \frac{D_F \sigma_F}{-D_S}$, which completes the proof.. ■

Proposition 2, which ensures existence and consistency of the equilibrium, puts a restriction on the market weight $\rho(c, \Im)$ of program traders and thus makes the notion of "finitely liquid" market more precise.

Moreover, Proposition 2 specifies the extent to which it is appropriate to use Black-Scholes strategies for hedging purposes. In practice, most traders base their strategies on the classical Black-Scholes formula, which assumes constant volatility σ, while we recognized that the correct value is $\sigma_S(S, t; \sigma, c)$, incorporating the feedback effects due to the interaction between the degree of market liquidity, the size and the frequency of adjustment and transaction costs. In the presence of feedback effects, Black-Scholes strategies based on the assumption of a constant volatility produce a tracking error that is almost surely non zero. El Karoui, Jeanblanc-Picqué and Shreve (1998) show how to derive a formula for the tracking error, which measures the difference between the actual and the theoretical value of a self-financing hedge portfolio for a European call calculated from the Black-Scholes formula with constant volatility. Proposition 2 gives us an insight about the behaviour of the tracking error: clearly, for $\rho(c, \Im)$ sufficiently small, as required in Proposition 2, the tracking error vanishes.

4 Conclusion

We extend the analysis of feedback effects of dynamic hedging strategies on the underlying asset in the case of finitely liquid markets to the case of fixed costs of transactions. Our results are in keeping with the literature on increased market volatility from dynamic hedging strategies. However, in this paper we specify in which way markets are finitely liquid when transaction costs are introduced and the role of transaction costs in determining the size and the frequency of adjustment. Our results can be of interest for applications, since we provide a quantitative estimate of the increased volatility with transaction costs, which establishes a precise interaction between the degree of market liquidity, the size and frequency of adjustment, and transaction costs. We show that the action of program traders leads to an excess volatility of the asset price, the size of which depends on the average size of adjustment and on the average gamma. The former depends on the properties of the adjustment hazard function and on the size of the transaction costs, while the latter indicates how often, on average, a position must be rehedged in order to maintain a delta neutral position. Finally, we show how the fixed point problem for the volatility of the asset price can be solved, to conclude that from a practical viewpoint it may be reasonable to use Black-Scholes strategies based on $\sigma_S(S_t, t; \sigma, c)$ for hedging purposes provided that the difference between $\sigma_S(S_t, t; \sigma, c)$ and σ is sufficiently small.

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