THE RIMINI CENTRE FOR ECONOMIC ANALYSIS

WP 07_11

Pierpaolo Pattitoni
Department of Management, University of Bologna, Italy

Marco Savioli
Department of Economics, University of Siena, Italy
Faculty of Economics-Rimini, University of Bologna, Italy
The Rimini Centre for Economic Research (RCEA), Italy

INVESTMENT CHOICES: INDIVISIBLE NON-MARKETABLE ASSETS AND BOUNDED RATIONALITY

The Rimini Centre for Economic Analysis
Legal address: Via Angherà, 22 – Head office: Via Patara, 3 - 47900 Rimini (RN) – Italy
www.rcfea.org - secretary@rcfea.org

Investment choices: Indivisible non-marketable assets and bounded rationality

Pierpaolo Pattitoni* Marco Savioli†

University of Bologna

January 19, 2011

Abstract

Several investment decisions deal with non-marketable assets. Non-marketable assets are available only to one investor and are often indivisible. This has relevant consequences on investor investment opportunities. Adhering to a mean variance representation of the investment space and considering a non-marketable asset (divisible or not), we derive some possible investment scenarios an investor may face. Furthermore, we show how bounded rationality affects investor portfolio choices. Our results define a set of conditions under which the non-marketable asset represents a good investment and show that, under certain assumptions, the efficient frontier exhibits non-linearities and intervals of discontinuity. That allows us to classify investors who can access a non-marketable investment as either entrepreneurs, who undertake it, or clerks, who invest their entire wealth on the market.

JEL classification: G11, D81.

Keywords: Portfolio Choice, Investment Decisions, Asset Indivisibility, Non-marketable Assets, Bounded Rationality.

Acknowledgements: We are grateful to Elettra Agliardi, Massimiliano Castellani, Vincenzo Denicolò (Department of Economics, University of Bologna), Roberto Dieci (Department of Mathematics for Economic and Social Sciences, University of Bologna), and Paolo Foschi (Department of Statistics, University of Bologna) for their helpful comments on an earlier version of this paper. We would also like to thank Carlo D'Adda, Alberto Dalmazzo, Ugo Pagano, Paolo Pin, Tiziano Razzolini, Roberto Renò, Paolo Vanin, Giulio Zanella and all the seminar participants at the Departments of Economics, University of Bologna and Siena, and all the participants to the RCEF conference 2010. The usual disclaimers apply.

*Email: pierpaolo.pattitoni@unibo.it. Department of Management, University of Bologna, Italy.

†Email: m.savioli@unibo.it. Department of Economics, University of Siena, and Faculty of Economics, Rimini, University of Bologna, Italy.

1 Introduction

One of the most relevant topics of economic and financial theory deals with investment decisions and optimal allocation of investor wealth. Theoretical models generally assume that investments are marketable, and investors are fully rational. However, these conditions rarely hold in reality.

In fact, non-marketable investment opportunities often occur. Non-marketable assets are available only to one particular investor and are often indivisible, since they require a fixed amount of investment to be undertaken. It is worth stressing that by saying "available only to one particular investor" we mean that only that investor can directly access it and that other agents can access it only through that investor. In this sense, the investment we consider is by definition out of the market and no market can exist for it.[1] Furthermore, investors often suffer from bounded rationality [Simon, 1957]: they either have a limited set of information, a limited capacity to process information, or the entire optimization process is too costly. The condition of bounded rationality drives investors toward satisfactory solutions rather than optimal ones. Both these characteristics have relevant consequences on optimal choices by investors, and must be taken into account.

Illegal or black market transactions, real investments of a family-managed business, and human capital investments of a particular individual are all examples of situations where the assumptions of the portfolio theory are likely to fail. From a personal perspective, these investments generally imply a "take it or leave it" choice for the individual. In all these cases, investors are generally required to commit a non-marginal amount of their wealth to the investment, and they suffer from an under-diversification condition.

Even if the situations presented are likely to occur in reality, to the best of our knowledge a contribution that deals with these problems in a general and exhaustive way is missing. Modern portfolio theory is introduced by Markowitz [1952], and Markowitz [1959]. These works establish mean-variance approach as the cornerstone of modern portfolio theory. A good review of portfolio theory can be found in Elton and Gruber [1997].

Following the modern portfolio theory framework, Tobin [1958] separation theorem leads to useful implications. According to the separation theorem, each investor can satisfy her investment needs combining the best portfolio of risky assets – identical for all individuals – with a risk-free asset. The optimal allocation between the efficient portfolio of risky assets and the risk-free asset depends on the investor's preferences. The separation theorem represents the starting point for the construction of the capital market line. In our work, however, the assumptions needed to validate this theorem are not always assured. In fact, we describe a situation where an investor has exclusive access to a non-marketable investment, which is not always infinitely divisible. In this case, the investor is likely to identify a new portfolio of risky assets that differs from that of all other individuals, and define a new personal efficient frontier which includes the

[1] Therefore, the literature on incomplete markets is not directly applicable to this context.

non-marketable investment.

Our model takes into account the strongest form of non-marketability: i.e., the investment is completely out of the market since it cannot directly be accessed by the market. Even if non-marketability is generally associated with illiquidity, our interpretation is different. In fact, an asset is illiquid if it is not "easy" or too costly to convert it in cash. Thus, the problem of illiquid assets deal with intertemporal choices, but we do not take this aspect into account. However, in what follows we present several studies that deal with illiquidity.

Mayers [1973] and Mayers [1976] presents the problem of non-marketable assets in relation to the determination of capital asset prices. His formalization considers a portfolio which includes a position in perfectly liquid marketable assets and a position in an illiquid non-marketable asset (e.g., human capital). He shows that the weights of the market portfolio vary across investors, and derives a single-period model of capital asset pricing. Differently from Mayers' works, we take into account the risk-free asset, we do not explicitly consider a particular utility function for the investor, and we deal with the non-marketable asset giving it an explicit weight. Furthermore, while Mayers mainly proposes a market equilibrium analysis aiming at the correct pricing of marketable and non-marketable assets, our analysis has a micro perspective focused exclusively on the correct evaluation of investment choices by individual investors.

Williams [1978] proposes a model where agents can choose a level of education, an example of non-marketable asset, and allocate their remaining wealth among risky and riskless assets. His results show that when the risk of human capital investment increases, risk-averse agents reduce their investments in human capital. Williams' findings stress the importance of the recognition of the uncertainty specific to the educational process, and show that the problems of choosing a level of education and the weights of a portfolio of financial assets are generally inseparable.

Several other contributions focus on the consequences of including non-marketable assets in portfolio choices. Svensson [1988] deals with the implicit pricing of non-traded assets. Longstaff [2009] considers a portfolio position in illiquid assets taking its temporal irreversibility into account. In particular, he finds that in the presence of illiquid assets agents choose not to hold the market portfolio but they tend to hold highly polarized portfolios. Our results confirm the departure from the market portfolio.

The problem of non-marketable investment evaluation has also been analyzed by empirical works. Kerins et al. [2004] use market data to estimate the cost of capital for well-diversified venture capital investors and under-diversified entrepreneurs. The authors support a direct relationship between the cost of capital and the entrepreneur's level of commitment to the non-marketable investment she is evaluating. Similar evidence is found in Müller [2008]. In particular, the author shows how idiosyncratic risk increases the cost of capital, since higher equity returns are required as compensation for under-diversification. Moskowitz and Vissing-Jørgensen [2002] analyze the problem of under-diversification for entrepreneurial households. Based on the data of the Survey of Consumer Finances (SCF), the authors find that in most cases households decide to invest on a single firm that they manage.

We contribute to the existing literature in more than a way. Considering a set of increasingly realistic assumptions about non-marketable asset divisibility and rationality, we analyze the problem of non-marketable investments in a systematic way. Furthermore, applying the original definition of bounded rationality, we show and measure how it affects investor portfolio choices, and, in particular, their portfolio risk. In addition, we stress that the presence of a non-marketable asset leads the investor to redefine the weights of the portfolio of only-risky assets. After that, we define a set of conditions under which the non-marketable asset represents a good investment. Finally, considering several scenarios an investor may face in reality, we depict the efficient frontier for the investor. In particular, we show that, under certain conditions, the efficient frontier exhibits non-linearities and intervals of discontinuity. This result permits us to divide investors among risk-tolerant entrepreneurs and risk-averse clerks.

Our analytical formalization is intentionally simple and limited to the single-period case (myopic portfolio choice). The single-period case is very important for both practitioners and academics: multi-period investment strategies involving hedging demands are prone to estimation errors that outweigh the gains of a more realistic dynamic specification. Furthermore, the single-period case is optimal when there is a single period horizon, investment opportunities are constant, investment opportunities are stochastic but unhedgeable, and agents utility is logarithmic [Brandt, 2009]. In all other cases, the single-period approach is a reasonable approximation that is useful to give a simplified representation of reality.

The remainder of the paper is structured as follows. Section 2 describes the theoretical framework we use in our analysis of investment choices. Varying the assumptions on indivisibility of the non-marketable investment and on rationality, we identify four possible cases an investor may face. Section 3 proposes a comparison among the solutions of these four cases. Afterwards, in section 4, we describe the efficient frontier for the investor considering plausible scenarios. The final section concludes the paper.

2 Four cases for a model of investment choices

In subsequent sections, we present four possible perspectives (cases) an investor faces when evaluating an investment opportunity. Case 1 refers to the traditional problem of a financial market with N risky assets and a risk-free asset. In this scenario, an investor allocates her wealth by choosing the weights for the risk-free asset and the N risky assets. The problem is well known in financial literature (see for example Elton et al. [2008]), but we report it to introduce the notation and to compare it with the results we obtain from the other cases.

In cases 2, 3 and 4, we assume that the investor has an exclusive access to a non-marketable investment besides the N assets and the risk-free asset. In that context, we consider a set of increasingly realistic assumptions. First, in case 2, we analyze the situation when the non-marketable investment is continuously divisible, i.e. its portfolio weight can assume any real value. We are aware that non-marketable investments are often indivisible, but we solve this case in order to obtain lower bound portfolios in terms of risk over all scenarios. Thus, they represent benchmark solutions obtained in ideal conditions.

When conditions are less than ideal, further analysis is necessary. Thus, in cases 3 and 4, we take a new set of more realistic assumptions into account. In case 3, we consider the instance when the non-marketable investment total cost is indivisible, and the investor commits its total amount to the investment. Finally, in case 4, we see how bounded rationality shifts the possible choices of the investor of case 3. In particular, we take into account the situation when she invests in the indivisible non-marketable asset and the limited set of information and resources forces her to choose only the weights for the risk-free asset and the risky assets portfolio as a whole. In this sense, the investor simplifies the optimization problem she faces, accepting a satisfactory solution rather than an optimal one.

2.1 N risky assets and a risk-free asset (case 1)

Consider a portfolio of N assets with weights $\boldsymbol{\omega}$. The return of this portfolio can be expressed by $\mu_P = \boldsymbol{\omega}'\boldsymbol{\mu}$, where $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)'$ is the vector of expected returns of the N risky assets. The variance of this portfolio is given by $\sigma_P^2 = \boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega}$, where $\boldsymbol{\Sigma}$ represents the positive-definite $N \times N$ variance-covariance matrix of the returns of the risky assets.

Now, suppose the market has a risk-free asset with return r_F besides N risky assets, and that lending and borrowing at the interest rate r_F are allowed. For a given value of portfolio expected return, $\mu_P = k$, investors prefer the portfolio with the lowest variance. They face the problem

$$\begin{cases} \min\limits_{\boldsymbol{\omega}} \frac{1}{2}\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} \\ \boldsymbol{\omega}'(\boldsymbol{\mu} - r_F\boldsymbol{\iota}) = k - r_F, \end{cases} \tag{1}$$

where $\boldsymbol{\iota} = (1, \ldots, 1)'$ denotes an N-dimensional vector in which all the elements are equal to 1. The constraints $\boldsymbol{\omega}'\boldsymbol{\mu} + \omega_F r_F = k$ and $\boldsymbol{\omega}'\boldsymbol{\iota} + \omega_F = 1$ are implicit in $\boldsymbol{\omega}'(\boldsymbol{\mu} - r_F\boldsymbol{\iota}) = k - r_F$.

Setting up the Lagrangian and solving the problem, the optimal portfolio weights and the portfolio variance are

$$\boldsymbol{\omega}_1 = \frac{k - r_F}{A}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_F\boldsymbol{\iota}), \tag{2}$$

$$\sigma_{P1}^2 = \boldsymbol{\omega}_1'\boldsymbol{\Sigma}\boldsymbol{\omega}_1 = \frac{(k - r_F)^2}{A}, \tag{3}$$

where $A \equiv (\boldsymbol{\mu} - r_F\boldsymbol{\iota})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_F\boldsymbol{\iota})$.

The variance of all portfolios obtained by varying k represents a parabola in the space $(\mu; \sigma^2)$, and consists of two straight lines with a common intercept at

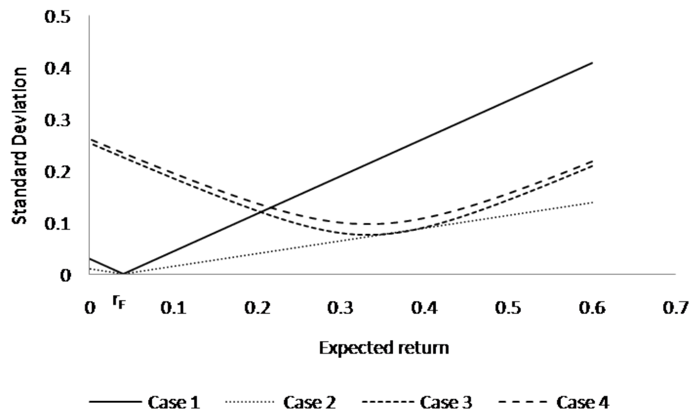

Figure 1: The frontiers of the four cases in the space $(\mu;\sigma)$

r_F in the space $(\mu;\sigma)$. However, as you can see in figure 1, only portfolios on the right line are relevant in practice ($k > r_F$), since they are Pareto-optimal portfolios.

A well known result in portfolio theory (see for example Elton et al. [2008]) refers to a particular portfolio, M, that is located on the identified efficient frontier, and is composed only by risky assets ($\omega_F = 0$). The weight vector of this portfolio is

$$\boldsymbol{\omega}_M = \frac{\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_F\boldsymbol{\iota}\right)}{\boldsymbol{\iota}'\boldsymbol{\Sigma}^{-1}\left(\boldsymbol{\mu} - r_F\boldsymbol{\iota}\right)}. \tag{4}$$

Let denote the expected return and the variance of M respectively by $\mu_M = \boldsymbol{\omega}'_M\boldsymbol{\mu}$, and $\sigma^2_M = \boldsymbol{\omega}'_M\boldsymbol{\Sigma}\boldsymbol{\omega}_M$.

If we assume that the N assets are all the risky assets in the universe of investments, and that all investors have the same beliefs about $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, the locus of combinations of expected returns and variance for efficient portfolios is the same for all investors and M is the market portfolio. It is worth noting that, by the separation theorem [Tobin, 1958], all the efficient portfolios can be obtained by a linear combination of the risk-free asset and M. In addition, it can be shown that

$$A = \frac{(\mu_M - r_F)^2}{\sigma^2_M} \equiv S^2, \tag{5}$$

where S corresponds to the Sharpe ratio. The Sharpe ratio, or the market price of risk, measures the excess return per unit of risk. By equations (3) and (5) we obtain the well known Capital Market Line $\mu_P = r_F + \dfrac{\sigma_P}{\sigma_M}(\mu_M - r_F)$.

2.2 N risky assets, a risk-free asset, and a non-marketable investment

In this section, we assume that an investor has exclusive access to a non-marketable investment besides the N assets and the risk-free asset. Let μ_I, σ_I^2, and ω_I denote respectively the expected return, the variance, and the portfolio weight on this new investment. Also let $\boldsymbol{\sigma} = (\sigma_{I,1}, \ldots, \sigma_{I,N})'$ be the vector of the N covariances among the risky assets and the non-marketable investment returns. We need to clarify that, coherently to our definition of non-marketable asset, μ_I, σ_I^2, and $\boldsymbol{\sigma}$ are free, in principle, to take any possible value. In particular, they are not influenced by the market pressures that drive $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

When the approach presented in case 1 is applied to investment choices that include a non-marketable investment available only to one investor, some differences must be taken into account. In particular, we need to specify a new set of constraints for the optimization problem. In cases 2, 3, and 4, we present the solution to the optimization problem considering several scenarios. In case 2, the non-marketable investment is continuously divisible. In case 3, we take a more realistic scenario into account considering the non-marketable investment as indivisible: we keep the amount of wealth spent in the non-marketable investment fixed. In case 4, other things being equal to case 3, we relax the assumption of complete rationality of the investor, so that, not being able to pursue the general minimization problem, she is limited to a sub-optimal choice. In particular, in the latter scenario, the investor can only choose the weights on the market portfolio M as a whole and the risk-free asset.

2.2.1 Continuously divisible non-marketable investment (case 2)

When the non-marketable investment is continuously divisible, the problem is

$$\begin{cases} \min\limits_{\boldsymbol{\omega},\omega_I} \dfrac{1}{2}\left(\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} + \omega_I^2\sigma_I^2 + 2\omega_I\boldsymbol{\omega}'\boldsymbol{\sigma}\right) \\ \boldsymbol{\omega}'(\boldsymbol{\mu} - r_F\boldsymbol{\iota}) = k - r_F - \omega_I(\mu_I - r_F), \end{cases} \tag{6}$$

where $\boldsymbol{\omega}$ and ω_I are determined simultaneously. Note that the constraints $\boldsymbol{\omega}'\boldsymbol{\mu} + \omega_F r_F + \omega_I\mu_I = k$ and $\boldsymbol{\omega}'\boldsymbol{\iota} + \omega_F + \omega_I = 1$ are implicit in the form $\boldsymbol{\omega}'(\boldsymbol{\mu} - r_F\boldsymbol{\iota}) = k - r_F - \omega_I(\mu_I - r_F)$. From an analytical standpoint this problem is simply the problem faced in case 1 with an additional investment and is part of the more general discussion on added assets (for a recent contribution on this topic see Wei-Guo et al. [2010].) However, it is worth presenting it explicitly to measure the effects of the non-marketable asset on the investor's personal portfolio.

Solving this problem, we get the weights and the variance

$$\begin{aligned}\boldsymbol{\omega}_2 &= \frac{k - r_F}{A + \dfrac{(\mu_I - r_F - B)^2}{\sigma_I^2 - C}} \boldsymbol{\Sigma}^{-1} \left(\boldsymbol{\mu} - r_F \boldsymbol{\iota} - \frac{\mu_I - r_F - B}{\sigma_I^2 - C} \boldsymbol{\sigma} \right), \\ \omega_{I2} &= \frac{k - r_F}{A + \dfrac{(\mu_I - r_F - B)^2}{\sigma_I^2 - C}} \left(\frac{\mu_I - r_F - B}{\sigma_I^2 - C} \right),\end{aligned} \tag{7}$$

$$\sigma_{P2}^2 = \frac{(k - r_F)^2}{A + \dfrac{(\mu_I - r_F - B)^2}{\sigma_I^2 - C}}, \tag{8}$$

where $B \equiv (\boldsymbol{\mu} - r_F \boldsymbol{\iota})' \boldsymbol{\Sigma}^{-1} \boldsymbol{\sigma}$ and $C \equiv \boldsymbol{\sigma}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\sigma}$.

As in case 1, the variance describes a parabola in the space $(\mu; \sigma^2)$, and consists of two straight lines with a common intercept in r_F in the space $(\mu; \sigma)$. It is worth noting that, in this latter space, the relevant part of the frontier coincide with the line on the right. (See figure 1.)

Let σ_{IM} be the covariance between the returns of the non-marketable investment and the market portfolio. σ_{IM} is obtained analytically by $\boldsymbol{x}_M' \boldsymbol{\Omega} \boldsymbol{x}_I$, where $\boldsymbol{x}_M = (\boldsymbol{\omega}_M', 0)'$, $\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{\sigma} \\ \boldsymbol{\sigma}' & \sigma_I^2 \end{bmatrix}$, and $\boldsymbol{x}_I = (0, \dots, 0, 1)'$. Developing the expression, we get $\sigma_{IM} = \boldsymbol{\omega}_M' \boldsymbol{\sigma}$. By the last expression, it is easy to recognize that

$$B = (\mu_M - r_F) \frac{\sigma_{IM}}{\sigma_M^2}. \tag{9}$$

Using equation (5), we are able to see easily that

$$\sigma_{P2}^2 = \frac{(k - r_F)^2}{S^2 + \dfrac{\alpha^2}{\sigma_I^2 - C}}, \tag{10}$$

where α is the Jensen's alpha, i.e. $\alpha = \mu_I - r_F - (\mu_M - r_F) \dfrac{\sigma_{I,M}}{\sigma_M^2}$. Jensen's alpha measures the excess return of the investment over its theoretical expected return determined by the CAPM. Note that, all things being equal, the portfolio total risk is inversely related to the square of the Sharpe ratio and the square of Jensen's alpha. Thus, whenever $\alpha \neq 0$, the portfolio risk in case 2 is smaller than that in case 1. In particular, when $\alpha < 0$, ω_{I2} will be negative.

2.2.2 Indivisible non-marketable investment (case 3)

In this section, we consider again an individual facing the portfolio problem, when she is the only one who can access the non-marketable investment. However, we make an additional assumption: the investment total cost I is indivisible and it has to be exactly covered. Therefore $\omega_I \equiv \dfrac{I}{W}$ is the fraction of the total wealth used for the non-marketable investment. This is a more realistic assumption than the one presented in the previous case, since non-marketable investments are often indivisible.

Therefore the problem is

$$\begin{cases} \min\limits_{\boldsymbol{\omega}} \dfrac{1}{2}\left(\boldsymbol{\omega}'\boldsymbol{\Sigma}\boldsymbol{\omega} + \dfrac{I^2}{W^2}\sigma_I^2 + 2\dfrac{I}{W}\boldsymbol{\omega}'\boldsymbol{\sigma}\right) \\ \boldsymbol{\omega}'(\boldsymbol{\mu} - r_F\boldsymbol{\iota}) = k - r_F - \dfrac{I}{W}(\mu_I - r_F). \end{cases} \tag{11}$$

Note that the value of ω_I is known, since it is fixed at $\omega_I \equiv \dfrac{I}{W}$.

Solving this problem, the weights and the variance are

$$\boldsymbol{\omega}_3 = \boldsymbol{\omega}_1 + \frac{I}{AW}(B - \mu_I + r_F)\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_F\boldsymbol{\iota}) - \frac{I}{W}\boldsymbol{\Sigma}^{-1}\boldsymbol{\sigma}, \tag{12}$$

$$\sigma_{P3}^2 = \sigma_{P1}^2 - 2\frac{k - r_F}{A}\frac{I}{W}(\mu_I - r_F - B) + \frac{I^2}{W^2}\left[\frac{(\mu_I - r_F - B)^2}{A} + \sigma_I^2 - C\right]. \tag{13}$$

σ_{P3}^2 describes a parabola in the space $(\mu;\sigma^2)$, and an hyperbola in the space $(\mu;\sigma)$. It is worth noting that, as it will be clearly stated in a subsequent observation, because of the constraint $\omega_I \equiv \dfrac{I}{W}$, neither the parabola nor the hyperbola intersect the μ-axis. As you can see in figure 1, efficient portfolios are those on the right side of the hyperbola.

After substituting A and B, we get the following expression:

$$\sigma_{P3}^2 = \sigma_{P1}^2 - 2\frac{k - r_F}{S^2}\frac{I}{W}\alpha + \frac{I^2}{W^2}\left[\frac{\alpha^2}{S^2} + \sigma_I^2 - C\right]. \tag{14}$$

Note that the total risk described in case 3 corresponds to the total risk in case 1 plus two terms, the latter always positive and the former negative if $I, \alpha > 0$.

2.2.3 Indivisible non-marketable investment and bounded rationality (case 4)

In this section, we consider the case when bounded rationality forces the investor to sub-optimal choices. Solving the problem described in the previous section requires an elaborate information processing strategy. However, a single investor may face some limitations on her ability to obtain and process the information required to monitor $N + 1$ assets.

To take this consideration into account, we assume that the investor simply holds a three-asset portfolio, consisting of a fixed position in the non-marketable investment, a position (ω_M) in the market portfolio (described in case 1)[2], and a position in the risk-free asset. In this sense, the investor exploits the knowledge

[2]Since we are dealing with bounded rationality, in this subsection we can consider the market portfolio as the average solution of all the agents on the market. If these agents have rational expectations, the average solution will tend to be the optimal one.

of all investors in the market, since their solution to case 1 determines the weights of the market portfolio, μ_M and σ^2_M.

Note that, assuming that case 1 is implicitly solved, the only additional information our investor needs to know is σ_{IM}, σ_I, and μ_I. Furthermore, bounded rationality avoids the optimization problem to the investor. In fact, given $\omega_I \equiv \frac{I}{W}$, the solution for ω_M comes out by the simple combination of the two usual constraints: $\omega_M \mu_M + \omega_F r_F + \omega_I \mu_I = k$ and $\omega_M + \omega_F + \omega_I = 1$.

$$\omega_M = \frac{1}{\mu_M - r_F}\left[(k - r_F) - \frac{I}{W}(\mu_I - r_F)\right]. \tag{15}$$

As we can see, in this case the share of wealth invested in the market is equal to the pursued excess return over the risk-free asset return minus $\frac{I}{W}$ times the non-marketable investment excess return over the risk-free asset return, all expressed as a percentage of the market excess return over the risk-free asset return.

The variance of this portfolio is

$$\begin{aligned}\sigma^2_{P4} = \quad & \frac{{\sigma_M}^2\left[(k - r_F) - (\mu_I - r_F)\frac{I}{W}\right]^2}{(\mu_M - r_F)^2} + \\ & +2\frac{I}{W}\frac{\sigma_{I,M}\left[(k - r_F) - (\mu_I - r_F)\frac{I}{W}\right]}{(\mu_M - r_F)} + \frac{I^2}{W^2}{\sigma_I}^2,\end{aligned} \tag{16}$$

which, as in case 3, describes a parabola in the space $(\mu; \sigma^2)$, and an hyperbola in the space $(\mu; \sigma)$. Because of the bounded rationality, the hyperbola in case 4 is located above the hyperbola in case 3. As you can see in figure 1, efficient portfolios are those on the right side of the hyperbola.

3 Solutions to the four cases: a comparison

In this section, we propose a comparison among the solutions to the four cases. This comparison allows us to understand when the exclusive access to the non-marketable investment under consideration may improve the risk-return trade-off for the investor, i.e. the investor faces a smaller risk to pursue the same expected return.

In each case we can express the standard deviation as function of the portfolio expected return. In this way, we are able to evaluate the relative convenience of the four solutions in terms of risk. That is the approach we follow in section 3.1. As we will see, there is no clear-cut order of dominance along all the range of variation of k between case 1 and 3, and between case 1 and 4. We focus on the comparison among these cases in section 3.2.

3.1 Standard deviations

- Case 1

By equation (3), we are able to get the standard deviation for case 1

$$\sigma_{P1} = \frac{k - r_F}{S}. \tag{17}$$

- Case 2

Similarly to case 1, by equation (10), we can obtain the standard deviation for case 2.

$$\sigma_{P2} = \frac{k - r_F}{\sqrt{S^2 + \dfrac{\alpha^2}{\sigma_I^2 - C}}}. \tag{18}$$

By comparing equations (17) and (18) the following observation is straightforward.

Observation 1. *When $\alpha = 0$ the frontier in case 2 corresponds exactly to that of case 1. When $\alpha \neq 0$, the frontier of case 2 has a slope smaller than that of case 1. In any case, the two frontiers start at the same point on the μ-axis, i.e. $k = r_F$.*

- Case 3

Case 3 differs from the previous ones, since the equation for σ_{P3} describes an hyperbola in the space $(\mu; \sigma)$. That is due to the extra constraint we impose on the fraction of wealth committed by the investor to the non-marketable investment. In appendix A, we obtain the standard form

$$\frac{\left(k - r_F - \dfrac{I}{W}\alpha\right)^2}{S^2 \dfrac{I^2}{W^2}(\sigma_I^2 - C)} - \frac{\sigma_{P3}^2}{\dfrac{I^2}{W^2}(\sigma_I^2 - C)} = -1, \tag{19}$$

which is an hyperbola in the variables (k, σ_{P3}). This hyperbola is centered at a point on the μ-axis with coordinates $(T, 0)$, where

$$T = r_F + \frac{I}{W}\alpha. \tag{20}$$

When $k = T$, the portfolio standard deviation, σ_{P3} reaches its minimum. Thus, it follows the subsequent observation.

Observation 2. *If $I, \alpha > 0$, σ_{P3} is at its minimum for a value of $k > r_F$.*

The upward sloping asymptote[3] of this hyperbola is

$$\sigma_P^{asym} = \frac{k - T}{S} = \frac{k - r_F}{S} - \frac{I}{W}\frac{\alpha}{S}. \tag{21}$$

Note that this asymptote is simply the frontier of case 1 translated by $-\frac{I}{W}\frac{\alpha}{S}$.[4] Thus, it is easy to prove the following observation.

Observation 3. *Whenever the conditions of observation 2 hold, that is $I, \alpha > 0$, an intersection point between the frontier of case 1 and case 3, k_{13}, exists. The frontier of case 3 dominates that of case 1 to the right of k_{13}, while the opposite applies to the left of k_{13}. If $\alpha = 0$, the intersection point does not exist, and the frontier of case 1 dominates along all the interval of variation of k.*

Observation 3 states that committing a fixed amount of one's wealth to a non-marketable investment can be convenient (there is an interval where the frontier of case 3 dominates that of case 1) if $\alpha > 0$ – i.e. whenever the non-marketable investment has a better risk-return trade-off compared to market assets with the same level of systematic risk.

The vertex of this branch of the hyperbola is located at $(T; \sigma_{P3}^{min})$. Using equation (19) we find

$$\sigma_{P3}^{min} = \left|\frac{I}{W}\right| \sqrt{\sigma_I^2 - C}. \tag{22}$$

Then it follows

Observation 4. *In case 3, the minimum standard deviation is greater than zero whenever $\sigma_I^2 > C$.*

Thus, even if the non-marketable investment has an advantageous risk-return trade-off, the investor can never eliminate risk if $\sigma_I^2 > C$.

As it is clear from figure 1, there is a tangency point between the frontier of case 2 and that of case 3. This point is located on the right of the vertex of the hyperbola, and corresponds to the point where the ω_I we get from the optimization procedure of case 2 coincides with the $\frac{I}{W}$ we assumed fixed in case 3.

- Case 4

[3] An hyperbola of the form $\frac{(x-T)^2}{b^2} - \frac{y^2}{a^2} = -1$ has an asymptote equal to $y = \frac{a}{b}(x - T)$.

[4] It is worth noting that, in the space $(\sigma; \mu)$, the asymptote becomes $\mu = r_f + \frac{(\mu_M - r_F)}{\sigma_M}\sigma_P^{asym} + \frac{I}{W}\alpha$. Henceforth, this asymptote is simply a translation of the Capital Market Line along the μ-axis of the amount $\frac{I}{W}\alpha$. The investor's individual frontier lies between the Capital Market Line and this asymptote.

Analogously to case 3, σ_{P4} describes an hyperbola in the space $(\mu;\sigma)$. In appendix B, we obtain the standard form

$$\frac{\left(k - r_F - \frac{I}{W}\alpha\right)^2}{S^2 \frac{I^2}{W^2}\left(\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2}\right)} - \frac{\sigma_{P4}^2}{\frac{I^2}{W^2}\left(\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2}\right)} = -1, \tag{23}$$

which is an hyperbola in the variables $(k,\, \sigma_{P4})$. Also this hyperbola, as the one in case 3, is centered at the point on the μ-axis with coordinates $(T, 0)$ and has the asymptote σ_P^{asym}.

When $k = T$, the portfolio standard deviation, σ_{P4}, reaches its minimum. Therefore, it can be easily proved the next observation.

Observation 5. *If $I, \alpha > 0$, σ_{P4} is at its minimum for a value of $k > r_F$.*

We recall by case 3 that the asymptote σ_P^{asym} is simply the frontier of case 1 translated by $-\frac{I}{W}\frac{\alpha}{S}$. Therefore, it follows the next observation.

Observation 6. *Whenever the conditions of observation 5 hold, that is $I, \alpha > 0$, an intersection point between the frontier of case 1 and case 4, k_{14}, exists. The frontier of case 4 dominates that of case 1 to the right of k_{14}, while the opposite applies to the left of k_{14}. If $\alpha = 0$, the intersection point does not exist, and the frontier of case 1 dominates along all the interval of variation of k.*

Even in the case of bounded rationality, we find that committing a fixed amount of one's wealth to a non-marketable investment can be convenient if $\alpha > 0$.

The vertex of this branch of the hyperbola is located at $(T; \sigma_{P4}^{min})$. Using equation (23) we find

$$\sigma_{P4}^{min} = \left|\frac{I}{W}\right| \sqrt{\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2}}. \tag{24}$$

Then it follows

Observation 7. *In case 4, the minimum standard deviation is greater than zero whenever $\sigma_I^2 > \frac{\sigma_{I,M}^2}{\sigma_M^2}$.*

Thus, even if the non-marketable investment has an advantageous risk-return trade-off, the investor can never eliminate risk if $\sigma_I^2 > \frac{\sigma_{I,M}^2}{\sigma_M^2}$. The condition $\sigma_I^2 > \frac{\sigma_{I,M}^2}{\sigma_M^2}$ has a significant financial meaning. Let us define the non-marketable return as $r_I = \gamma + \beta r_M + u$, where $\beta = \frac{\sigma_{I,M}}{\sigma_M^2}$. Suppose that the market return r_M has expected value $\mathbf{E}(r_M) \equiv \mu_M$ and variance $\mathbf{Var}(r_M) \equiv \sigma_M^2$;

the idiosyncratic component of the return, u, has expected value $\mathbf{E}(u) \equiv 0$ and variance $\mathbf{Var}(u) \equiv \varpi^2$; and the covariance between r_M and u is $\mathbf{Cov}(r_M, u) = 0$. It follows that the expected value of r_I is $\mu_I = \gamma + \beta\mu_M$, while its variance is $\sigma_I^2 = \left(\frac{\sigma_{I,M}}{\sigma_M^2}\right)^2 \sigma_M^2 + \varpi^2 = \frac{\sigma_{I,M}^2}{\sigma_M^2} + \varpi^2$. Therefore, $\frac{\sigma_{I,M}^2}{\sigma_M^2}$ measures the systematic risk, while ϖ^2 the idiosyncratic risk. In other words, by observation 7 we can say that the σ_{P4}^{min} is greater than zero whenever the non-marketable investment has a component of idiosyncratic risk. Since a single investment is likely to have an idiosyncratic risk component, the σ_{P4}^{min} will be virtually always greater than zero.

Since the only difference between case 4 and case 3 is the bounded rationality of the investor, we are able to measure its effect in terms of risk for any given k. The difference between σ_{P3} and σ_{P4} is at its maximum in correspondence of the two vertexes, and it declines the more we move to the right along the μ-axis, as the two hyperbolas converge to the same asymptote. Therefore, defining $BR(k)$ as the bounded rationality distortion, we get

$$BR(k) \leq \sigma_{P4}^{min} - \sigma_{P3}^{min} = \left|\frac{I}{W}\right| \left(\sqrt{\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2}} - \sqrt{\sigma_I^2 - C}\right).^5 \qquad (25)$$

3.2 Intersection points

Observations 3 and 6 show that one can determine where the solution to case 1 dominates the solutions to case 3 and 4 (or vice versa), only after one ascertains the existence of an intersection point between the frontiers. Thus, in this section, we look for the intersection points between the frontier of case 1 and the frontiers of cases 3 and 4. The intersection point is defined as the value of k that equates the standard deviations of two cases.

On one side, for cases 1 and 3, we have by equation (3)

$$A\sigma_P^2 = (k - r_F)^2. \qquad (26)$$

On the other side, by equation (13), we have

$$\begin{aligned} A\sigma_P^2 = \quad & (k - r_F)^2 - 2(k - r_F)\frac{I}{W}(\mu_I - r_F - B) + \\ & + \frac{I^2}{W^2}\left[(\mu_I - r_F - B)^2 - AC + A\sigma_I^2\right]. \end{aligned} \qquad (27)$$

[5] It is evident here as bounded rationality depends on the values C and $\frac{\sigma_{I,M}^2}{\sigma_M^2}$. In order to compare better the values, we can develop the expression

$$\frac{\sigma_{I,M}^2}{\sigma_M^2} = \boldsymbol{\sigma}' \frac{\boldsymbol{\omega}_M \boldsymbol{\omega}_M'}{\sigma_M^2} \boldsymbol{\sigma} = \boldsymbol{\sigma}' \frac{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_F \boldsymbol{\iota})(\boldsymbol{\mu} - r_F \boldsymbol{\iota})' \boldsymbol{\Sigma}^{-1}}{(\boldsymbol{\mu} - r_F \boldsymbol{\iota})' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_F \boldsymbol{\iota})} \boldsymbol{\sigma} = \frac{B^2}{A}.$$

Therefore, the greater the difference between C and $\frac{B^2}{A}$, the greater the distortion led by the bounded rationality.

Therefore

$$k_{13} = r_F + \frac{I}{2W}\frac{S^2(\sigma_I^2 - C) + \alpha^2}{\alpha}. \tag{28}$$

It is straightforward by equation (28) to prove the following observation.

Observation 8. *The intersection point between the frontier of case 1 and the frontier of case 3, k_{13}, is inversely related to the value of α and goes to infinity when $\alpha \to 0$.*

For cases 1 and 4, on one side, we have equation (3); on the other side, by equation (16), we have

$$\begin{aligned}\sigma_P^2 A = & \left[k - r_F - (\mu_I - r_F)\frac{I}{W}\right]^2 + \\ & +2\frac{I}{W}\frac{\sigma_{I,M}}{\sigma_M^2}\left[k - r_F - (\mu_I - r_F)\frac{I}{W}\right](\mu_M - r_F)+ \\ & +\frac{{\sigma_I}^2}{\sigma_M^2}(\mu_M - r_F)^2\frac{I^2}{W^2}.\end{aligned} \tag{29}$$

After some algebraic manipulation, we derive the expression

$$k_{14} = r_F + \frac{I}{2W}\frac{S^2\left(\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2}\right) + \alpha^2}{\alpha}. \tag{30}$$

It is then straightforward the following observation.

Observation 9. *The intersection point between the frontier of case 1 and the frontier of case 4, k_{14}, is inversely related to the value of α and goes to infinity when $\alpha \to 0$.*

4 The efficient frontier for different real scenarios and the choice of the investor

In the previous sections, we described the frontier for each of the four theoretical cases. In this section, we relate those theoretical cases to several scenarios an investor may face in reality. First, in reality, not all individuals have access to non marketable investments. Second, even if an investor identifies a viable non-marketable investment, she may choose not to undertake it accordingly to her risk-return preferences. Finally, investor's rationality may be bounded. Thus, the objective of this section is to depict the effective frontier that emerges from each possible scenario. Table 1 summarizes all scenarios we consider.

4.1 Four possible scenarios

- The investor does not identify the non-marketable investment

	Full rationality	Bounded rationality
Investment not available	Frontier of case 1	-
Continuously divisible investment	Frontier of case 2	-
Indivisible Investment	Frontier of case 1 or 3	Frontier of case 1 or 4

Table 1: Efficient frontiers and scenarios

This scenario coincides with case 1. The relevant frontier is the standard capital market line, which describes the locus of efficient portfolios when all investors have access to the same set of assets. Figure 1 (case 1) presents the frontier in the space $(\mu; \sigma)$. It is worth noting that, when $I = 0$, this frontier is the limit result for all the other scenarios.

- The investor identifies the non-marketable investment and can choose to commit any amount to it

In this scenario, which corresponds to case 2, the investor has exclusive access to a continuously divisible non-marketable investment. In this sense, the investment is available only to her. Being ω_I free to vary, this scenario is the best situation the investor may face among all the scenarios described, i.e. the solutions to case 2 are always at least as good as the solutions to all the other cases. To see this, note that cases 1, 3 and 4 may be interpreted as restricted versions of case 2, where additional constraints are imposed. In case 1, $\omega_I = 0$; in case 3, $\omega_I \equiv \frac{I}{W}$; and in case 4, $\omega_I \equiv \frac{I}{W}$ and $\omega \equiv \omega_M$. As previously mentioned, there is a tangency point between the frontier of case 2 and that of case 3, where $\omega_I = \frac{I}{W}$ (see figure 1).

- The investor identifies the non-marketable investment, but it is just a "take it or leave it" choice

In this scenario the investor has exclusive access to an indivisible non-marketable investment, therefore the amount she can commit to the investment is fixed and equal to I. It is important to note that the investor is not compelled to take on the non-marketable investment. Thus, for any given k, the frontier will be that of case 1 or that of case 3 according to the level of the standard deviation. In particular, solutions to case 1 may be Pareto-preferred for some intervals of k, but solutions to case 3 may be superior for other intervals. Depending on the value assumed by α, which measures the excess return of the non-marketable investment over the theoretical expected return at that level of risk, three frontier shapes emerge.

The first frontier (see figure 2) corresponds to the case when $\alpha = 0$. In this situation, the investor would disregard the opportunity to undertake the non-marketable investment, since the solutions to case 1 dominate those of case 3 for all possible values of k. In practice, since the non-marketable investment has an expected return in line with that of other market assets at the same level of risk, it is not worth taking on the non-marketable investment.

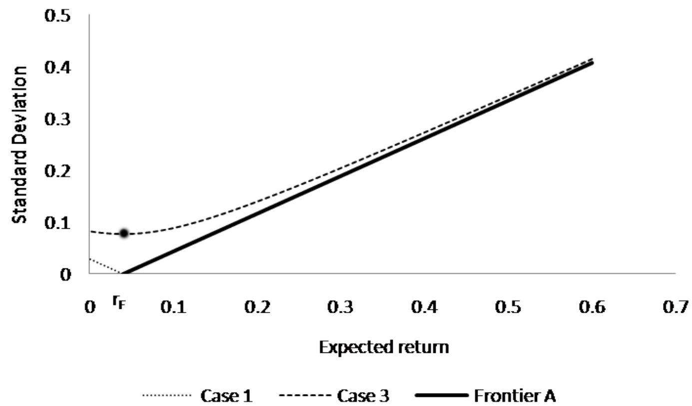

Figure 2: Relevant frontier when $\alpha = 0$

The second frontier occurs when α assumes a value between zero (excluded) and the value corresponding to the situation when the frontier of case 1 intersects the hyperbola of case 3 exactly on its vertex. Analytically, we get this value by substituting T for k and σ_{P3}^{min} for σ_{P1} in equation (17). The interval we obtain is $0 < \alpha \leq S\sqrt{\sigma_I^2 - C}$. The resulting frontier is shown in figure 3. The frontier is that of case 1 for low values of k, and that of case 3 for higher values of k. Thus, the investor's choice will ultimately depend on her risk-return preferences.

Finally, in figure 4, we can see the shape of the relevant frontier when $\alpha > S\sqrt{\sigma_I^2 - C}$. It is worth noting that between $k = S\sigma_{P3}^{min} + r_F$ and $k = T$ the frontier exhibits an interval of discontinuity. It would not be rational for the investor to choose k so that $S\sigma_{P3}^{min} + r_F < k < T$, since she can get $k = T$ for a lower level of risk. Besides the interval of discontinuity, the frontier is that of case 1 for low values of k, and that of case 3 for high values of k.

- The investor identifies the "take it or leave it" non-marketable investment, but her rationality is bounded

This last scenario is similar to the preceding one. The only difference is due to the bounded rationality of the investor. Now the relevant frontier is obtained combining the frontiers of case 1 and 4. Also in this scenario, there are three different shapes for the frontier that are very similar to those illustrated in figures 2, 3, and 4. The intervals of α which generate the three shapes are $\alpha = 0$, $0 < \alpha \leq S\sqrt{\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2}}$, and $\alpha > S\sqrt{\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2}}$.

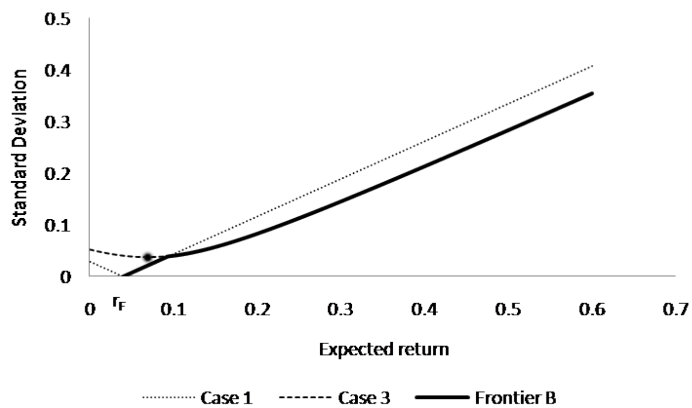

Figure 3: Relevant frontier when $0 < \alpha \leq S\sqrt{\sigma_I^2 - C}$

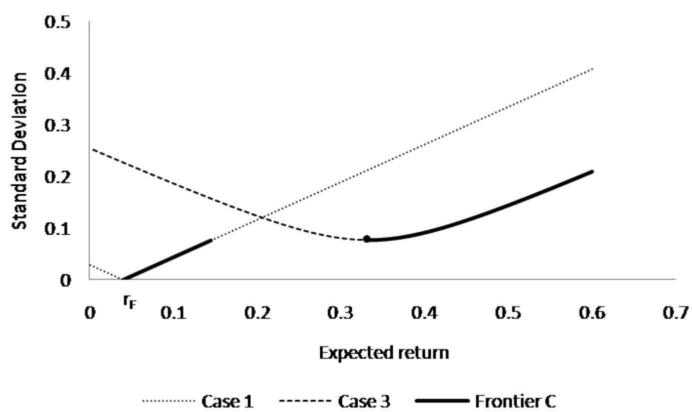

Figure 4: Relevant frontier when $\alpha > S\sqrt{\sigma_I^2 - C}$

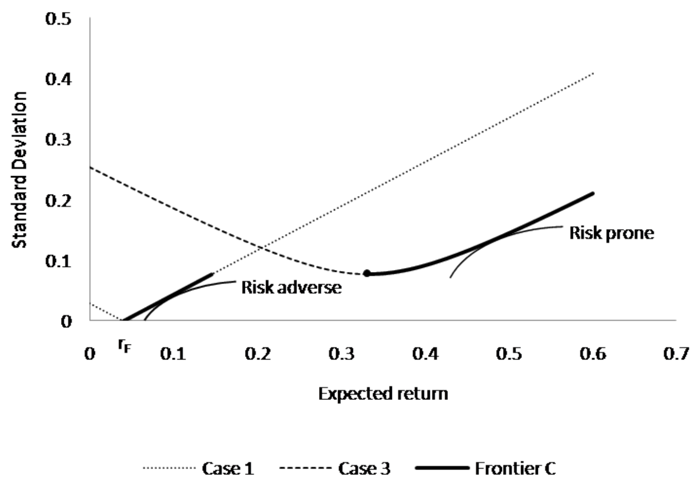

Figure 5: Indifference curves and investment choices

4.2 The choice of the investor: entrepreneurs vs. clerks

In reality individuals are heterogeneous in their risk-return preferences. Each frontier presented in the previous subsection leads to a possible optimum choice for the investor. The optimum choice is generally identified by the tangency point between her indifference curve and the frontier. This tangency point ultimately relates to the investor's risk-return profile: higher values of k are chosen by risk prone individuals, lower values of k are chosen by risk-averse individuals.

In the first place, the investor may or may not identify a non-marketable alternative. However, even when she identifies the opportunity of a non-marketable investment, this does not guarantee that she will undertake it. Her choice will depend on her attitude towards risk. In particular, scenarios 3 and 4 allow us to describe the entrepreneurial attitude. If she is a risk-averse individual, a clerk, she will not take on the investment. Conversely, if she is a risk-prone individual, an entrepreneur, she will take on the investment. As it is evident from figure 5, a clerk's indifference curve is tangent to the first part of the frontier (solution to case 1), an entrepreneur's indifference curve is tangent to the second part of the frontier (solution to case 3 or case 4).

5 Conclusions

Portfolio theory deals with investment decisions and optimal allocation of investor wealth. In the traditional framework, it is generally assumed that investments are marketable, continuously divisible, and investors are fully rational. However, reality often departs in a significant way from these assumptions. In order to give a theoretical basis to investment choices under all circumstances, in this paper we define what happens when we deviate from these ideal-school assumptions. To the best of our knowledge, there is yet to be a contribution that deals in a general and exhaustive way with investments that are by definition out of the market.[6]

We construct four cases describing different situations an investor may face, varying the assumptions on indivisibility of the non-marketable investment and on rationality of the individual. By the subsequent comparison of the results, we highlight several interesting implications as the value of the parameters α, S, $\frac{I}{W}$, σ_I^2, C and $\frac{\sigma_{I,M}^2}{\sigma_M^2}$ vary. First, we determine the shape of the efficient frontiers for each case we consider and then we identify the intersection points among these efficient frontiers. We also measure the effects of bounded rationality in terms of additional risk of the investor portfolio. We show that an important role is played by Jensen's alpha and Sharpe ratio. In particular, when $\alpha = 0$, there is no convenience for the investor to include the non-marketable investment in her portfolio. Furthermore, we highlight that, when the investment needs a fixed amount to be undertaken, the portfolio risk is always greater than zero, i.e. it cannot be eliminated.

In last sections, we compare the solutions to the four cases by taking into account several scenarios that may occur in reality. This allows us to identify new shapes of the efficient frontier, some of which are non-linear and exhibit intervals of discontinuity. In particular, when the investor identifies the non-marketable investment, but it is just a "take it or leave it" choice, there are some values of α that allow us to split investors between clerks and entrepreneurs according to their attitude towards risk. This may suggest why some individuals engage in personal enterprises and others do not.

Most of the shortcomings of our work are consistent with those of traditional portfolio theory. For example, the mean-variance representation of the space of investments disregards higher moments of asset return distributions. Furthermore, we consider only one period in our analysis. Considering higher moments or a multi-period analysis could shed more light on the dynamic effects of non-marketability and bounded rationality (at the price of a greater analytical complexity). We intend bounded rationality as a limiting factor of the investor's processing ability in her optimization problem, or as a lack in her availability of information on the risky assets market. We know that other more complex development of bounded rationality may be taken into account. However, we intentionally use a simple definition of bounded rationality, which is enough for our purpose. Lastly, we consider only one non-marketable asset. Considering more than one non-marketable asset could enrich our understanding of the interaction among marketable and non-marketable investments. Another direction for further research may address in depth how different kinds of utility functions interact with the frontiers we identify.

[6] We recall that our definition of non-marketability differs from illiquidity.

References

M. W. Brandt. Portfolio choice problems. *Handbook of financial econometrics: Tools and techniques*, pages 269–336, 2009.

E. J. Elton and M. J. Gruber. Modern portfolio theory, 1950 to date. *Journal of Banking & Finance*, 21:1743–1759, 1997.

E. J. Elton, M. J. Gruber, S. J. Brown, and W. N. S. Goetzmann. *Modern portfolio theory and investment analysis.* Wiley India Pvt. Ltd., 2008.

F. Kerins, J. K. Smith, and R. Smith. Opportunity cost of capital for venture capital investors and entrepreneurs. *Journal of Financial and Quantitative Analysis*, 39(2):385–405, June 2004.

F. A. Longstaff. Portfolio claustrophobia: asset pricing in markets with illiquid assets. *The American Economic Review*, 99(4):1119–1144, 2009.

H. Markowitz. Portfolio selection. *Journal of Finance*, 7:77–91, 1952.

H. Markowitz. *Portfolio selection: efficient diversification of investments.* Wyley, New York, 1959.

D. Mayers. Nonmarketable assets and the determination of capital asset prices in the absence of a riskless asset. *The Journal of Business*, 46(2):258–267, April 1973.

D. Mayers. Nonmarketable assets, market segmentation, and the level of asset prices. *Journal of Financial and Quantitative Analysis*, 11(1):1–12, March 1976.

T. J. Moskowitz and A. Vissing-Jørgensen. The returns to entrepreneurial investment: a private equity premium puzzle? *The American Economic Review*, 92(4):745–778, 2002.

E. Müller. How does owners' exposure to idiosyncratic risk influence the capital structure of private companies? *Journal of Empirical Finance*, 15(2):185–198, September 2008.

H. Simon. A behavioral model of rational choice. In *Models of Man, Social and Rational: Mathematical Essays on Rational Human Behavior in a Social Setting*. Wiley New York, 1957.

L. E. O. Svensson. Portfolio choice and asset pricing with nontraded assets. *NBER Working Paper Series*, (2774), November 1988.

J. Tobin. Liquidity preference as behavior towards risk. *The Review of Economic Studies*, 25(2):65–86, February 1958.

Z. Wei-Guo, X. Wei-Lin, and X. Wei-Jun. A possibilistic portfolio adjusting model with new added assets. *Economic Modelling*, 27(1):208–213, 2010.

J. T. Williams. Risk, human capital, and the investor's portfolio. *The Journal of Business*, 51(1):65–89, January 1978.

Appendix A

By equation (13)

$$\sigma^2_{P3} = \frac{(k-r_F)^2}{A} - 2\frac{(k-r_F)}{A}\frac{I}{W}(\mu_I - r_F - B) + \frac{I^2}{AW^2}(\mu_I - r_F - B)^2 + \frac{I^2}{W^2}(\sigma^2_I - C)$$

$$\sigma^2_{P3} = \frac{1}{A}\left[k - r_F - \frac{I}{W}(\mu_I - r_F - B)\right]^2 + \frac{I^2}{W^2}(\sigma^2_I - C)$$

$$\sigma^2_{P3} - \frac{1}{A}\left[k - r_F - \frac{I}{W}(\mu_I - r_F - B)\right]^2 = \frac{I^2}{W^2}(\sigma^2_I - C)$$

$$\frac{\sigma^2_{P3}}{\frac{I^2}{W^2}(\sigma^2_I - C)} - \frac{\left[k - r_F - \frac{I}{W}(\mu_I - r_F - B)\right]^2}{A\frac{I^2}{W^2}(\sigma^2_I - C)} = 1$$

Appendix B

By equation (16)

$$\sigma^2_{P4}\frac{(\mu_M - r_F)^2}{\sigma^2_M} = \left[k - r_F - (\mu_I - r_F)\frac{I}{W}\right]^2 +$$

$$+2\frac{I}{W}\frac{\sigma_{I,M}}{\sigma^2_M}\left[k - r_F - (\mu_I - r_F)\frac{I}{W}\right](\mu_M - r_F) + \frac{{\sigma_I}^2}{\sigma^2_M}(\mu_M - r_F)^2\frac{I^2}{W^2}$$

$$\sigma^2_{P4}\frac{(\mu_M - r_F)^2}{\sigma^2_M} = \left[k - r_F - (\mu_I - r_F)\frac{I}{W} + \frac{\sigma_{I,M}}{\sigma^2_M}(\mu_M - r_F)\frac{I}{W}\right]^2 -$$

$$-\frac{\sigma^2_{I,M}}{\sigma^4_M}(\mu_M - r_F)^2\frac{I^2}{W^2} + \frac{\sigma^2_I}{\sigma^2_M}(\mu_M - r_F)^2\frac{I^2}{W^2}$$

$$\sigma^2_{P4}\frac{(\mu_M - r_F)^2}{\sigma^2_M} = \left\{k - r_F - \frac{I}{W}\left[\mu_I - r_F - (\mu_M - r_F)\frac{\sigma_{I,M}}{\sigma^2_M}\right]\right\}^2 +$$

$$+\frac{(\mu_M - r_F)^2}{\sigma^2_M}\frac{I^2}{W^2}\left(\sigma^2_I - \frac{\sigma^2_{I,M}}{\sigma^2_M}\right)$$

$$\frac{\sigma^2_{P4}}{\frac{\sigma^2_M}{(\mu_M - r_F)^2}} - \left\{k - r_F - \frac{I}{W}\left[\mu_I - r_F - (\mu_M - r_F)\frac{\sigma_{I,M}}{\sigma^2_M}\right]\right\}^2 =$$

$$= \frac{(\mu_M - r_F)^2}{\sigma_M^2} \frac{I^2}{W^2} \left(\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2} \right)$$

$$\frac{\sigma_{P4}^2}{\frac{I^2}{W^2} \left(\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2} \right)} - \frac{\left\{ k - r_F - \frac{I}{W} \left[\mu_I - r_F - (\mu_M - r_F) \frac{\sigma_{I,M}}{\sigma_M^2} \right] \right\}^2}{\frac{(\mu_M - r_F)^2}{\sigma_M^2} \frac{I^2}{W^2} \left(\sigma_I^2 - \frac{\sigma_{I,M}^2}{\sigma_M^2} \right)} = 1$$