Financial Frictions and the Re-distributive Effects of Exchange Rate Fluctuations*

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Abstract

In this paper, we study the re-distributive effects of exchange rate fluctuations in a model which captures three stylized facts about liability dollarization in emerging economies: i) it is generally non-financial firms and banks who borrow in foreign currency; ii) foreign currency debt is usually balanced by dollar bank deposits held by domestic households and iii) such deposits are generally mainly held by wealthier households. Our model is a small open economy HANK model in which households borrow and lend through a financial intermediary which is subject to a leverage constraint. Banks use their funds to finance purchases of domestic capital and local currency loans to households, while they raise deposits in both currencies, generating a currency mismatch in their balance sheet. When the local currency depreciates the net worth of financial intermediaries declines, causing banks to reduce the total supply of credit, while households increase their desired savings. As a result the economy falls into a recession. While richer households are relatively well insured given their foreign currency denominated asset holdings, borrowers suffer as the banks raise the interest rate spread on their debt, irrespectively of its currency composition.

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1 Introduction

In many emerging economies, the presence of debt denominated in foreign currency, a phenomenon also known as "debt dollarization", represents a serious financial stability risk. Such debt is usually issued by firms and banks.[1] When the exchange rate depreciates, the currency mismatch between assets and liabilities generates a decline in the net worth of financial and non-financial firms and, when financial frictions constrain the borrowing capacity of these firms, can generate a recession (see for example, Aoki et al. (2016), or Akinci and Queralto (2018)). However, recent papers, like Dalgic (2018) and Bocola and Lorenzoni (2019), have documented how country experiencing high levels of debt dollarization among banks and firms are also characterized by a large amount of dollar-denominated deposits held by households. As a result, these papers conjecture that high demand for dollar deposits by households, driven by hedging motives, might negatively affect the supply of local currency debt for financial and non-financial firms, hence forcing banks and firms to be exposed to currency risk. Furthermore the share of assets and liabilities denominated in foreign currency is very heterogeneous across households. In particular, as shown by Drenik, Pereira, and Perez (2018), across several emerging economies wealthier households hold a larger portfolio share of dollar deposits, whereas poor households, with zero or negative wealth, have essentially no direct exposure to foreign currency. Using micro data from Uruguay we see the same relation ship for the relation between net-liquid wealth and the share of it held in dollars (see Figure 1).

In this paper, we study the re-distributive effects of exchange rate fluctuations in a model which captures the three stylized facts about dollarization mentioned above: i) it is generally firms and banks who borrow in foreign currency; ii) foreign currency debt is usually balanced by dollar deposits held by households and iii) such deposits are mainly held by wealthier households. In particular, we introduce two non-standard elements into a small open economy New Keynesian model. First, we assume that households are heterogeneous in terms of their labor income, wealth, and face incomplete markets, similar to Drenik (2015) and De Ferra, Mitman, and Romei (2020). In addition, we assume that households borrow and lend through a leveraged financial intermediary in the spirit of Gertler-Karadi(2011), which uses funds also to finance domestic capital. Households can save either in bank deposits, denominated in local or foreign currency, or they can save in foreign bonds denominated in foreign currency. However, we assume that households can borrow, potentially in both currencies, only from domestic banks. Up to a first order approximation,

[1] In some countries also households borrow in foreign currency, see for example, the case of Hungary analyzed by De Ferra, Mitman, and Romei (2020)

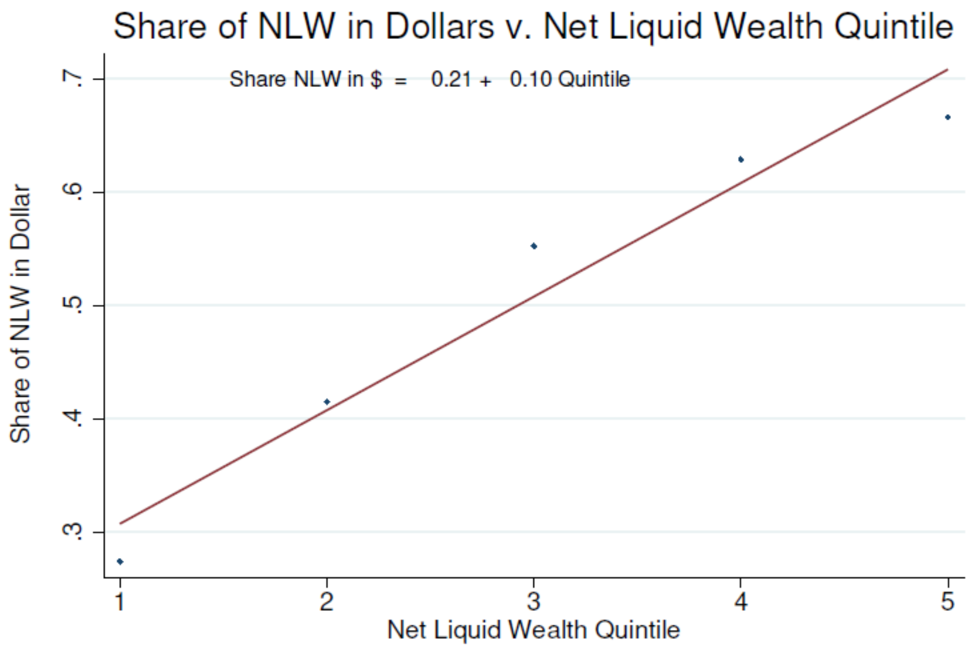

Figure 1: Dollar Deposits by Wealth Quintile in Uruguay

the households and banks currency portfolio choice, and consequently their exposure to currency mismatch is not determinate. In our current exercises, we assume that poor households borrow only in domestic currency, whereas the share of dollar savings increases with household wealth, in line with the evidence reported by Drenik, Pereira, and Perez (2018) for Uruguay. Through market clearing, our assumptions on households currency portfolio compositions determine also the bank's exposure to exchange rate fluctuations. As in Dalgic (2018) and Bocola and Lorenzoni (2019), a larger households demand for dollar deposits results in a more severe currency mismatch for financial intermediaries.

Our main experiment consists in studying the redistributive effects of an exchange rate depreciation brought about by an increase in the foreign interest rate. Since financial intermediaries lend to households and firms in local currency while they borrow both in local and foreign currencies, such depreciation results in a decline in banks net worth, which depresses domestic investments through a financial accelerator channel. As regards the heterogenous effects of the devaluation on households, wealthier households gain from their dollar savings, while, unlike De Ferra, Mitman, and Romei (2020), households in the left tail of the wealth distribution are not directly exposed to the depreciation, if not through the standard rebalancing effect in their consumption basket.

However, our model presents a novel mechanism through which a depreciation affects poor households. In fact, the banks agency problem generate an endogenous spread between bank lending and borrowing rates. Once bank net worth is eroded because of the currency mismatch of financial intermediaries, banks have to cut on their total lending, reducing loan supply to households and causing an increase in lending rates. As a result, borrowing households will have to cut spending, whereas savers are not exposed to such channel and instead increase consumption due to the higher value of their savings. However, since borrowers have a higher marginal propensity to consume (MPC), aggregate demand is depressed by the devaluation. Furthermore, the spread between the borrowing rate and the lending rate implies that, in steady state, there is a sizable mass of households with zero liquid wealth who behave like hand-to-mouth agents. These high-MPC households will reduce demand briskly in response to lower wages, amplifying the downturn. We illustrate this new mechanism with several quantitative exercises. In addition, we show that a calibration of the model with a larger share of dollar deposits results in a larger decline in consumption and investment, as banks have a more severe currency mismatch and, consequently, borrowing households are more negatively affected by a larger increase in spreads.

Our framework can potentially be used to study how the volatility of aggregate and idiosyncratic shocks affects the endogenous portfolio choice for financial firms and for households. In the appendix we show how to extend the method proposed in Devereux and Sutherland (2011) to a HANK model. This allows us to approximate optimal currency portfolio allocations at the household and bank level while keeping the flexibility of first order perturbation solutions.[2]

1.1 Related Literature

To be completed A recent wave of papers looks at various aspects of household heterogeneity in open emerging market economy models (see, for example, Zhou (2020), Hong (2020), Guntin, Ottonello, and Perez (2020), Guo, Ottonello, and Perez (2020), Villalvazo (2020)). Our paper differs from that literature by looking at the interaction of households with the financial sector, especially through the feedback effects induced by the currency composition of portfolios. This also separates our paper from Lee, Luetticke, and Ravn (2020), who look at the interaction between households and the bank in a closed economy, showing how the dynamics of endogenous household borrowing spreads influence idiosyncratic and aggregate consumption volatility.

[2] However, when implemented in the current setting we obtain essentially no dollar deposits in the financial sector as the optimal portfolio, dampening the effects of foreign interest rate shocks. While theoretically plausible this result does not match the data well.

2 Model

We introduce in a standard small open economy framework two novel elements: i) financially constrained intermediaries, as in Gertler and Karadi (2011) and ii) households facing uninsurable income risk as in an heterogeneous agents new keynesian (HANK) model, in the spirit of Bewley (1983), Huggett (1993), and Aiyagari (1994). In our model, financial intermediaries raise deposits, in either local or foreign currency, from domestic households (and potentially from abroad), and invest these funds in productive capital and in loans to households, which again can be in both currencies. Households can save either in bank deposits or in foreign bonds, but they can borrow only through bank loans, subject to a borrowing constraint. Banks face an agency problem, which implies an endogenous spread between the banks' lending rate and deposit rate. Such spread will fluctuate inversely with bank net worth. Another key element of our framework is that we will allow for different exposures to currency mismatch across households and banks, by allowing for different portfolio compositions. As a result, fluctuations in the exchange rate will have redistributive effects, which will have aggregate implications by affecting the net worth of banks and households, and by interacting with the heterogeneity in marginal propensities to consume across agents.

2.1 Households

Household derive utility from consumption of a bundle of home and foreign goods C_{it}, and derive disutility from labor $h_{i,t}$, according to

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[U\left(C_{it}\right) - \frac{h_{it}^{1+\varphi}}{1+\varphi} \right]$$

where C_{it} represents a CES aggregate of home and foreign good

$$C_{i,t} = \left[\chi \left(c_{i,t}^H\right)^{\frac{\rho-1}{\rho}} + (1-\chi)\left(c_{i,t}^F\right)^{\frac{\rho-1}{\rho}} \right]^{\frac{\rho}{\rho-1}} \tag{1}$$

Households can save and borrow in nominal domestic currency bonds, $\tilde{b}_{i,t}^H$ or in foreign bonds $\tilde{b}_{i,t}^F$, subject to the following borrowing constraint on the total amount borrowed

$$\tilde{b}_{i,t} = \tilde{b}_{i,t}^H + e_t \tilde{b}_{i,t}^F \geq -\bar{b} \tag{2}$$

Their budget constraint can be written as

$$P_t^H c_{i,t}^H + P_t^F c_{i,t}^F + \tilde{b}_{i,t}^H + e_t \tilde{b}_{i,t}^F = \omega_{i,t} W_t h_{i,t} + \tilde{R}_t^H \tilde{b}_{i,t-1}^H + e_t \tilde{R}_t^F \tilde{b}_{i,t-1}^F + \chi_{i,t} \Pi_t \tag{3}$$

where P_t^H is the price of home goods and P_t^F of foreign good, e_t is nominal exchange rate, W_t is the nominal wage expressed in local currency,Π_t represents dividends from banks and non-financial firms, and $\tilde{R}_t^H$ and $\tilde{R}_t^F$ are the one period nominal interest rate on domestic and foreign bonds. Households are subject to idiosyncratic labor productivity shocks ω_{it} and can transition between being a "worker" or a "capitalist". A worker has positive labor productivity but does not receive profits from banks and firms, that is $\omega_{i,t} > 0$ and $\chi_{i,t} = 0$. Capitalists do not work and their income comes only from dividends, that is $\omega_{i,t} > 0$ and $\chi_{i,t} = 1$. A key assumption in our model is that households can borrow, in either currency, only through an intermediary by asking for a loan. On the other hand, households can save either in bank deposits, which are issued in both currencies, or by purchasing foreing bonds. Implicitly we are assuming that only the financial intermediary has access to international markets to raise funds at the world interest rate, whereas households have to go through the bank in order to borrow.Financial intermediaries face an agency friction which limits their arbitraging capabilities. As a result, the bond interest rate will be different according to whether households are saving in bank deposits or borrowing through bank loans, that is

$$\tilde{R}_t^H = \begin{cases} \tilde{R}_{Dt}^H & \text{if } \tilde{b}_{i,t}^H \geq 0 \\ \tilde{R}_{Lt}^H & \text{if } \tilde{b}_{i,t}^H < 0 \end{cases} \tag{4}$$

$$\tilde{R}_t^F = \begin{cases} \tilde{R}_{Dt}^F & \text{if } \tilde{b}_{i,t}^F \geq 0 \\ \tilde{R}_{Lt}^F & \text{if } \tilde{b}_{i,t}^F < 0 \end{cases} \tag{5}$$

The interest rate on foreign bonds, $\tilde{R}_t^*$, is exogenously determined as is standard in small open economy models. An arbitrage condition implies that $\tilde{R}_{Dt+1}^F = \tilde{R}_{t+1}^*$, while the interest rate on domestic bank deposits is given by $\tilde{R}_{Dt}^H$. The interest rate on bank loans, in either local or foreign currency, are given by $\tilde{R}_{Lt}^H$ and $\tilde{R}_{Lt}^F$ respectively. As mentioned above, financial frictions in the banking sector will generate spreads between lending and borrowing rates, implying that $\tilde{R}_{Dt+1}^H < \tilde{R}_{Lt+1}^H$ and $\tilde{R}_{Dt+1}^F < \tilde{R}_{Lt+1}^F$.

If we define

$$x_{it} = \frac{e_t \tilde{b}_{i,t}^F}{\tilde{b}_{it}} \tag{6}$$

we can rewrite the budget constraint as

$$P_t^H c_{i,t}^H + P_t^F c_{i,t}^F + \tilde{b}_{it} = \omega_{i,t} W_t l_{i,t} + R_{bt} \tilde{b}_{it-1} + \gamma_{i,t} \Pi_t \tag{7}$$

where $\tilde{R}_{bt+1} = \left((1 - x_{it}) \tilde{R}_{t+1}^H + x_{it} \frac{e_{t+1}}{e_t} \tilde{R}_{t+1}^F \right)$ represents the nominal return on households bond portfolio.

As a result the FOCs are

$$\lambda_{i,t} = \frac{u_{C,i,t}^H}{P_t^H} \tag{8}$$

$$\lambda_{i,t} = \frac{u_{C,i,t}^F}{P_t^F} \tag{9}$$

$$\lambda_{i,t} = \beta E_t \lambda_{i,t+1} \tilde{R}_{t+1}^b + \gamma_{i,t} \tag{10}$$

$$E_t \lambda_{i,t+1} \tilde{R}_{t+1}^H = E_t \lambda_{i,t+1} \frac{e_{t+1}}{e_t} \tilde{R}_{t+1}^F \tag{11}$$

$$\lambda_t \omega_{i,t} W_t = v' \left(h_{i,t} \right) \tag{12}$$

where $u_{C,i,t}^H = U_{C,i,t} C'_{H,i,t}$ and $u_{C,i,,t}^F = U_{C,i,t} C'_{F,i,t}$ and $\gamma_{i,t}$ represents the multiplier on the borrowing constraint.

2.2 Banker

A representative banker manages a financial intermediary which raises deposits in both currencies from households, D_t^H and D_t^F, and invests these funds, together with the bank's net worth n_t, in capital K_t, and household loans in both currencies, L_t^H and L_t^F. Bankers are risk-neutral, and discount the future with a discount factor $\beta_{b,t} = 1/R_t$. In addition, bankers are subject to an agency problem as in Gertler and Kiyotaki (2015). In particular, after raising deposits, financial intermediarists can abscond a fraction θ_k of capital and a fraction θ_l of loans. In order to prevent bankers to run away with external funds, an incentive constraint is needed. Such constraint will imply a leverage constraint and consequently a limit to the bankers arbitrage capacity, resulting in a time varying spread on the return between banks assets and liabilities. As in Gertler and Kiyotaki

(2015), bankers die with a probability σ_b every period. Because of the spreads they make on their investments, bankers will find optimal to postpone consumption until they die. As a result, we can write the value function of the banker as

$$V_t^b = \max_{K_t, L_t^H, L_t^F, D_t^H, D_t^F} E_t \beta_{b,t} \left[(1-\sigma_b) n_{t+1} + \sigma_b V_{t+1} \right]$$

subject to

$$q_t^k K_t + L_t^H + e_t L_t^F \ \leq n_t + D_t^H + e_t D_t^F \tag{13}$$

$$n_{t+1} = R_{t+1}^k q_t^k K_t + \left(R_{Lt+1}^H L_t^H + e_{t+1} R_{Lt+1}^F L_t^F \right) - \left(R_{Dt+1}^H D_t^H + e_{t+1} R_{Dt+1}^F D_t^F \right) \tag{14}$$

$$V_t \geq \theta_k q_t^k K_t + \theta_l \left(L_t^H + e_t L_t^F \ \right) \tag{15}$$

Equation (13) represents the bank's balance sheet. Equation (14) describes the evolution of bank net worth, given by the difference between the return on bank assets, given by physical capital and loans to househols, and the interest payment on bank liabilities, that is deposits. The real return on capital is given by $R_{t+1}^k = \left(r_{t+1}^k + (1-\delta) q_{t+1}^k \right)$ where r_t^k is the capital rental rate and q_t^k is the price of capital, both expressed in terms of the home good. The real return on loans and deposits are given by $R_{Lt+1}^j = \frac{\tilde{R}_{Lt+1}^j}{\pi_{t+1}^H}$ and $R_{Dt+1}^j = \frac{\tilde{R}_{Dt+1}^j}{\pi_{t+1}^H}$ for $j = H, F$, where π_t^H represents home inflation. Equation (15) is the bank incentive constraint, which guarantees that the value of operating a bank is larger than the value of the funds the banker can abscond.

It can be shown that the banker's value function is linear in net worth, according to $V_t^b = \varphi_t n_t$, where φ_t only depends on aggregate variables. As a result, the FOCs for capital loans (in both currencies) and deposits (in both currencies) are

$$\beta_{b,t} E_t \Omega_{t+1} R_{t+1}^k - \upsilon_t = \theta_k \mu_t \tag{16}$$

$$\beta_{b,t} E_t \Omega R_{Lt+1}^H - \upsilon_t = \theta_L \mu_t \tag{17}$$

$$\beta_{b,t} E_t \Omega_{t+1} \frac{e_{t+1}}{e_t} R_{Lt+1}^F - \upsilon_t = \theta_L \mu_t \tag{18}$$

$$v_t = \beta_{b,t} E_t \Omega_{t+1} R^H_{Dt+1} \tag{19}$$

$$v_t = \beta_{b,t} E_t \Omega_{t+1} \frac{e_{t+1}}{e_t} R^F_{Dt+1} \tag{20}$$

where μ_t is the multiplier on the incentive constraint, v_t is the multiplier on the bank's balance sheet and $\Omega_{t+1} = (1 - \sigma_b + \sigma_b \varphi_{t+1})$.

The FOCs deliver the following no-arbitrage relationships

$$E_t \Omega_{t+1} \left(R^k_{t+1} - R^H_{Dt+1} \right) = \frac{\theta_k}{\theta_l} \mu E_t \Omega_{t+1} \left(R^H_{Lt+1} - R^H_{Dt+1} \right) \tag{21}$$

$$E_t \Omega_{t+1} R^H_{Dt+1} = E_t \Omega_{t+1} \frac{e_{t+1}}{e_t} R^F_{Dt+1} \tag{22}$$

$$E_t \Omega_{t+1} R^H_{Lt+1} = E_t \Omega_{t+1} \frac{e_{t+1}}{e_t} R^F_{Lt+1} \tag{23}$$

Furthermore, from the incentive constraint we obtain the following borrowing constraint

$$\phi_t \leq \frac{\beta_{b,t} E_t \Omega_{t+1} R^H_{Dt+1}}{\theta_k - \beta_{b,t} E_t \Omega_{t+1} \left(R^k_{t+1} - R^H_{Dt+1} \right)} \tag{24}$$

where ϕ_t represents an "adjusted" leverage

$$\phi_t = \left(\frac{q^k_t K_t + \frac{\theta_L}{\theta_K} L_t}{n_t} \right) \tag{25}$$

The equations above show that banks' total borrowing depends on bank net worth and on bank leverage, which moves inversely with credit spreads. As a result, depending on banks currency mismatch, unexpected fluctuations in exchange rates can have implications for lending conditions by affecting bank net worth.

Finally, new entrant bankers replace every period the portion $(1 - \sigma_b)$ of exiting ones, and they are endowed with and endowment $(1 - \sigma_k)\, \xi_b$ of either home good. As a result, the evolution of aggregate net worth, $N_{\ t}$, is given by

$$N_{t+1} = \sigma_b \left\{ R^k_{t+1} q^k_t K_t + \left(R^H_{Lt+1} L^H_t + e_{t+1} R^F_{Lt+1} L^F_t \right) - \left(R^H_{Dt+1} D^H_t + e_{t+1} R^F_{Dt+1} D^F_t \right) \right\} + (1 - \sigma_k)\, \xi_b \tag{26}$$

Exiting bankers pay dividends $\bar{N}_t$ to the capitalist households.

$$\bar{N}_{t+1} = (1 - \sigma_b) \left\{ R^k_{t+1} q^k_t K_t + \left(R^H_{Lt+1} L^H_t + e_{t+1} R^F_{Lt+1} L^F_t \right) - \left(R^H_{Dt+1} D^H_t + e_{t+1} R^F_{Dt+1} D^F_t \right) \right\} + \Pi^p_t + \Pi^I_t \tag{27}$$

2.3 Production

Final home good Y^H_t is a CES composite of different intermediate varieties, given by

$$Y^H_t = \left[\int Y^H_t (i)^{\frac{\varepsilon - 1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon - 1}} \tag{28}$$

so that the demand for each variety will be given by

$$Y^H_t (i) = \left(\frac{P_{H,t}(i)}{P_{H,t}} \right)^{-\varepsilon} Y_t \tag{29}$$

where the aggregate price levels for the two types of goods are given by

$$P_{H,t} = \left[\int (P_{H,t}(i))^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}} \tag{30}$$

Intermediate good is produced with CobbDouglas technology by monopolistically competitive intermediate-goods firms

$$Y^H_t = A_t K^{\alpha}_{t-1} H^{(1-\alpha)}_t \tag{31}$$

where H_t is aggregate labor demand. If we define P^m_t as marginal costs, then wages and the rental rate on capital, in units of home good, are given by

$$w_t = \frac{P^m_t}{P^H_t} (1 - \alpha) \frac{Y^H_t}{H_t} \tag{32}$$

$$r_t = \frac{P^m_t}{P^H_t} \alpha \frac{Y^H_t}{K_{t-1}} \tag{33}$$

Intermediate-goods firms are owned by a risk-neutral manager discounting the future at rate $1/R_{ss}$. As is standard in New-Keynesian models, they can reset prices only occasionally, with probability $(1-\gamma^p)$, as in Calvo (1983). As a result, their problem will consist in choosing the price $\bar{P}_{H,t}(i)$, in order to solve

$$\max E_t \sum_{j=0}^{\infty} \lambda_p^j \left(\frac{1}{R_{ss}}\right)^j \Lambda_{t,t+1} \left[\frac{\bar{P}_{H,t}(i)}{P_{t+j}^H} - \frac{P_{t+j}^m(i)}{P_{t+j}^H}\right] Y_{t+j}^H(i)$$

Real profits, arising from monopolistic competition in the final good sector are going to be given by

$$\Pi_t^p = Y_t^H \left(1 - \frac{P_t^m}{P_t^H}\right) \tag{34}$$

2.4 Labor Unions

We assume that labor markets are controlled by labor unions, who set the wage for different types of labor services subject to adjustment costs following the approach in Hagedorn, Luo, Manovskii, and Mitman (2019) and De Ferra, Mitman, and Romei (2020). This allows us to introduce nominal wage rigidities into our model in way that is easy to compare to representative agent models. Households provide labor services to a continuum of identical labor unions who sell them to competitive labor packers. These aggregate the services and rent them out to intermediate good producers. Labor packer combines labor using CES technology with elasticity ϵ^w. Unions set wages to maximize profits subject to Rotemberg costs with a scale Γ^W. Skipping the details of the decision problems for now we arrive at the following wage Philipps curve:

$$\theta^w(\pi_t^w - \bar{\pi}^w)\pi_t^w = (1-\epsilon^w)W_t + \epsilon^w \frac{H_t^{\phi}}{C_t^{-\sigma}} + \Gamma^W \mathbb{E}_t \beta_k (\pi_{t+1}^w - \bar{\pi}^w)\pi_{t+1}^w.$$

Here, H_t is the labor supply for all workers working for the union ($h_{i,t} = H_t \forall i$ as a result). C_t is total consumption and π_t^w is wage inflation and $\bar{\pi}^w$ is average wage inflation in the economy.

2.5 Capital producers

Capital producers have a technology allowing them to produce new capital goods by using home goods and subject to convex adjustment costs $\Psi(I_t, K_t)$. As a result, their optimization problem is

$$\max_{I_t} q_t^k I_t - [I_t + \Psi(I_t, K_t)]$$

and the implied first order condition is

$$q_t^k = \left[1 + \frac{d\Psi(I_t, K_t)}{I_t}\right] \tag{35}$$

Real profits are given by

$$\Pi_t^I = q_t I_t - [I_t + \Psi(I_t, K_t)] \tag{36}$$

Finally, capital evolves according to

$$K_t = (1 - \delta) K_{t-1} + I_t \tag{37}$$

2.6 Foreign Economy

The law of one price implies

$$P_t^F = e_t P_t^{F*} \tag{38}$$

so that we can normalize $P_t^{F*} = 1$ and obtain $P_t^F = e_t$.

We assume that the rest of the world's demand of the home good is a decreasing function of the terms of trade/relative price $S_t = P_t^H / P_t^F$, according to

$$C_t^{H*} = \bar{\psi} S_t^{-\psi} \tag{39}$$

2.7 Monetary policy

The monetary authority sets the nominal rate on local bonds according to a standard Taylor rule responding to domestic inflation

$$i_t = (1 - \rho) R_{zs} + \rho i_{t-1} + \kappa_\pi \pi_t^H + \varepsilon_{it}$$

where no arbitrage implies $i_t = R_{Dt+1}^H$.

2.8 Market Clearing

Capitalists profits include bank dividends $\bar{N}_t$, profits from monopolistic competition, Π_t^p, and profits from capital adjustment costs, Π_t^I, that is

$$\Pi_t = \bar{N}_t + \Pi_t^p + \Pi_t^I \tag{40}$$

Define $\Gamma_t(i)$ as the distribution of agents over the relevant state variable ω_{it} and total bond holding b_{it}, that is $\Gamma_t(i) = \Gamma_t(\omega_{it}, b_{it})$. In addition, define total households' total savings in home bonds and foreign bonds as

$$B_t^{H+} = \int 1_{b_{i,t} \geq 0} b_{i,t} (1 - x_{i,t}) \, d\Gamma_t(i) \tag{41}$$

$$B_t^{F+} = \int 1_{b_{i,t} \geq 0} b_{i,t} x_{i,t} d\Gamma_t(i)$$

and households total borrowing in home and foreing bonds as

$$B_t^{H-} = \int 1_{b_{i,t} < 0} b_{i,t} (1 - x_{i,t}) \, d\Gamma_t(i) \tag{42}$$

$$B_t^{F-} = \int 1_{b_{i,t} < 0} b_{i,t} x_{i,t} d\Gamma_t(i)$$

where $x_{i,t}$ represents household's i bond portfolio share in foreign bonds.

Then the market clearing for local currency bonds implies

$$D_t^H = B_t^{H+} \tag{43}$$

$$L_t^H = -B_t^{H-} \tag{44}$$

meaning that local deposits and loans are supplied only by domestic banks. In addition, since we assume that alos foreign currency loans are available only through the intermediary, we have that

$$L_t^F = -B_t^{F-} \tag{45}$$

Aggregate households consumption in home and foreign are given by

$$C_t^H = \int c_{i,t}^H d\Gamma_t(i) \tag{46}$$

$$C_t^F = \int c_{i,t}^F d\Gamma_t(i) \tag{47}$$

and market clearing for home good is

$$C_t^H + C_{bt}^H + C_t^{H*} + I_t + \Psi(I_t, K_t) = Y_t^H \tag{48}$$

while market clearing in labor market requires

$$\int \omega_{i,t} h_{i,t} d\Gamma_t(i) = H_t \tag{49}$$

Finally, the balance of payment, implied by aggregate households and banks budget constraints, is given by

$$C_t^{H*} - \frac{1}{S_t}\left(C_t^F + C_{bt}^F\right) = \left(B_t^+ - D_t\right) - \frac{\tilde{R}_{Dt}^b}{\pi_t^H}\left(B_{t-1}^+ - D_{t-1}\right)$$

where $B_t^+ = B_t^{H+} + B_t^{F+}$ and $D_t = D_t^H + D_t^F$.

3 Calibration

In this section we briefly summarize our preliminary calibration. Most parameters are set based on steady state targets with a Latin American country such as Mexico or Uruguay, as a rough data counterpart in mind[3]. A few parameters, which affect only the dynamics around the steady state, will be set to common values in the literature.

3.1 Production and Trade

We set the depreciation rate of capital (δ) to 0.025 and the curvature with respect to capital (α_k) in the production function to 0.33. We set the elasticity between varieties of the home good to 11 in order to target a small profit share. We set the trade elasticities for exports and imports to 1.5, well within common values in the macro literature. We set the import to GDP ratio to 20 percent

[3] See for example Cugat et al. (2019) and references therein.

and calibrate the scale of export demand to equilibrate the flows from imports and foreign holdings in steady state. Finally, we assume no price indexation and a calvo parameter of 0.85. We set the Rotemberg parameter to 100 roughly in line with a duration of one year in labor contracts in the corresponding Calvo based Philipps curve, when we set the elasticity of substitution to 11, which we use for both ϵ and ϵ_w.

3.2 Bankers

The world interest rate is set to 4 percent annualized. We assume a spread between deposits and the return of capital of 2 percent annualized, and of 8 percent annualized for household loans (in line with the evidence for Uruguay over the 2000s). We set the discount factor of the banker to one over the deposit rate. The survival rate of 5 years, slighly worse than in U.S. based calibrations. We set the scale parameter of the investment adjustment costs to 5, a common value in the literature. The endowment of new bankers is set to 1.2 percent of domestic production to target a leverage ratio of 6, within the typical values in the literature for banks.

3.3 Households

We calibrate the risk aversion to 2 and the Frisch elasticity to 1. We approximate the productivity process of the workers using a three state markov chain with values for volatiltiy (0.017) and persistence (0.966) based on Floden and Lindé (2001).The household discount factor is chosen to match a wealth to annual GDP ratio of 2.5, where wealth is the sum of net savings of the household sector and networth of the bankers. The borrowing limit is set to match 30 percent borrowers, a value in line with Uruguayan micro data. We assume that 5 percent of households are entrepreneurs. The persistence of the entrepreneurial state is set to 0.98 the same as the one implied by the discretization for the diagonal of the other earning states. To complete the probability matrix we assume that the chance of becoming an entrepreneur is the same for all workers and that after loosing the entrepreneur state a household draws his productivity from the ergodic distribution of skills.

3.4 Policy

We assume that the central bank sets the nominal interest rate on deposits based on a Taylor-type rule responding to inflation in the price of the home good with a coefficient of 1.5.

3.5 Household Portfolio

Our calibration implies, roughly, that 10% of households are at the borrowing constraint, 20% are unconstrained borrowers, 20% are hand-to-mouth agents with zero liquid wealth and 50% are savers. As mentioned in the introduction, in many emerging economies most dollar savings are held by wealthier households, whereas borrowing households only use local currency. As a result, in our baseline, we assume that the share of dollar deposits is a linear function of wealth, that is $x_i = \bar{\gamma} + \gamma b$, where γ is calibrated to match the implied slope of dollar holdings based on data from Uruguay in 2013, whereas $\bar{\gamma}$ is calibrated to obtain that in steady state 40% of aggretate bank deposits are in foreign currency, in line with the recent experience in Uruguay. Here, we use the Financial Survey of Uruguayan Households.

4 Quantitative Exercises

We solve our model using a first order perturbation and simulate the response to a surprise positive 25 bps shock to the foreign interest rate with persistence 0.7.

4.1 Transmission

To better illustrate the different channels affecting transmission of the foreign interest rate shock in the model we solve a sequence of 'simpler' models working towards our main one. We begin with a 'bankless' RANK version of the model in which, a) there is no bank, b) a representative agent replaces the heterogenous households. We calibrate the model to the same targets, assuming a utility cost of holding capital to keep the level of capital the same in steady state. In addition, we add a small cost of holding foreign bonds to induce stationarity. Figure 2 shows the response of this economy to the foreign interest rate shock. As foreign interest rates rises, the household wants to save more abroad. As a result exports rise, while consumption and investment fall and output expands. The exchange rate depreciates. This is the intertemporal channel common to most open economy DSGE models. In addition, a standard expenditure switching channel leads to lower demand for imports as the household rebalances his portfolio towards the local good. Notice also that local real interest rates also rise (bottom right).

In Figure 3 we add the bank to the RANK model. This adds two new transmission channels to the model. First, the rise in real rates means that the funding cost of banks increases, tightening the balance sheet of the intermediary and leading to a drop in asset prices. This amplifies the

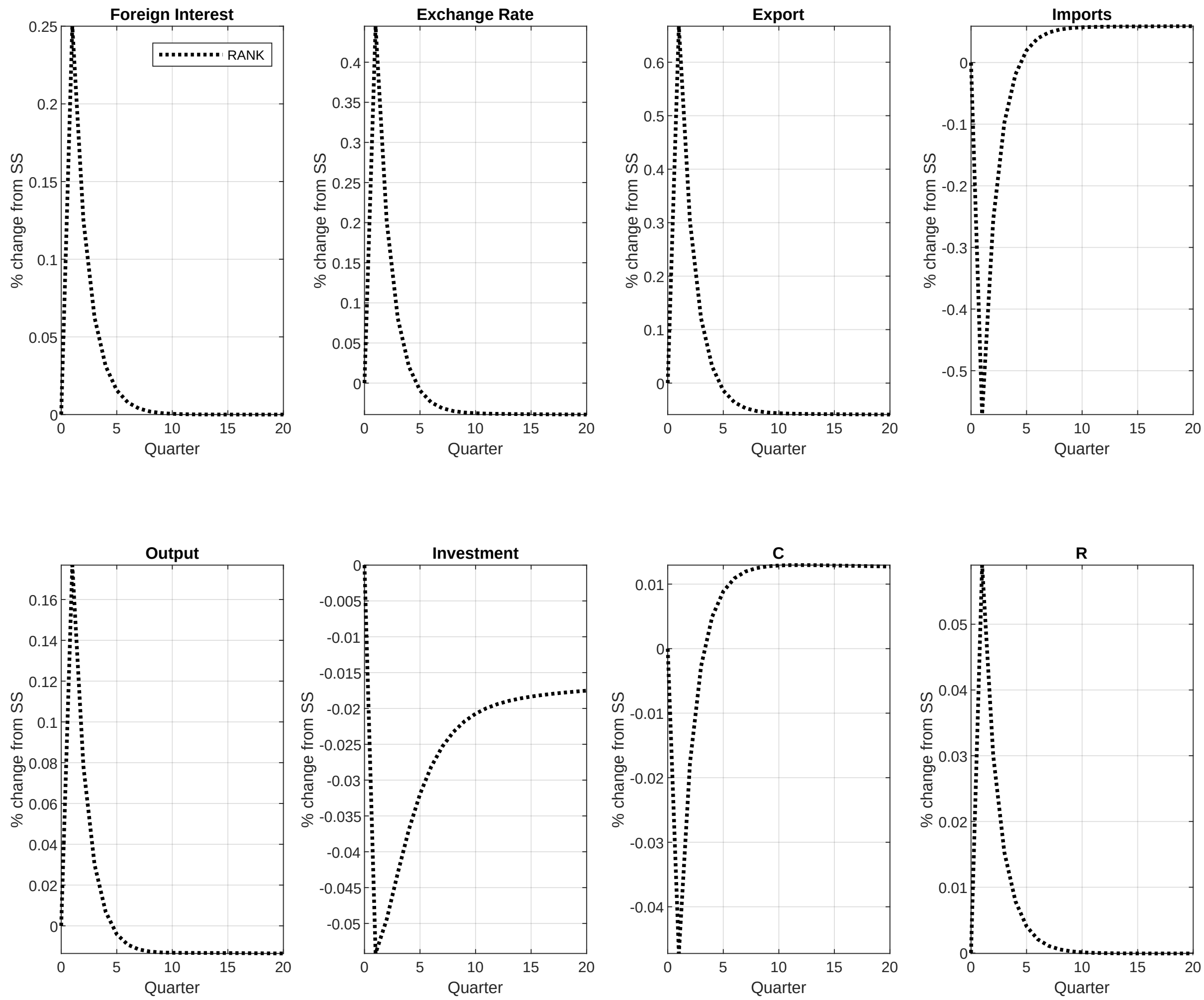

Figure 2: Devaluation: RANK

drop in investment and, therefore, supported by nominal rigidities, in output. Second, as a fraction of deposits is denominated in foreign currency the exchange rate depreciation further reduces the bank's networth after the shock leading to the same chain of events as the first channel. Looking at the figure the drop in investment is more than an order of magnitude larger and output is now falling.

In the next step we add the HANK model, but without bank, to the comparison. The results are shown in figure 4. The results are qualitatively similar to the 'bankless' RANK model. We see a few differences, especially looking at consumption. As output increases initially, constrained household spend the accompaning wage gains one for one on consumption goods. In addition, a

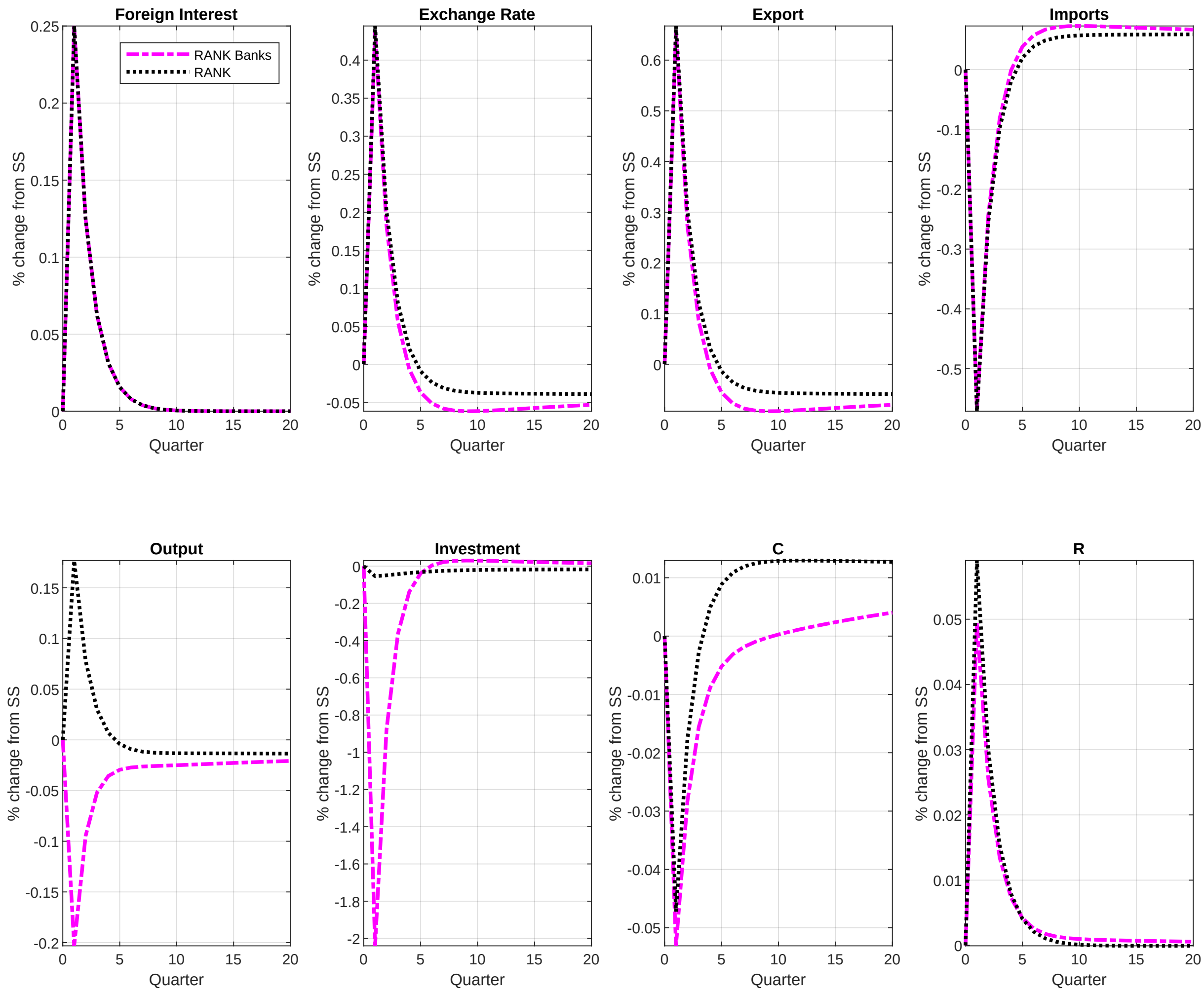

Figure 3: Devaluation: RANK vs RANK with Banks

mild increase in inflation initially gives a small windfall to borrowers. There is also a small boost from the depreciation to the wealth of rich households holding foreign assets, but these households have low MPCs, limiting the impact on consumption. The flipside of rising consumption is a stronger fall in investment.

Finally, Figure 5 brings all the ingredients together. The result is a stronger decline in economic activity, relative to all the previously discussed versions. First, as the addition of bank leads to a decline in output, hand to mouth households now cut back strongly on consumption amplifying the downturn. In addition, as borrowing spreads rise, borrowers also face higher costs of loans, which reduces their income in future periods and leads to a stronger intertemporal motive.

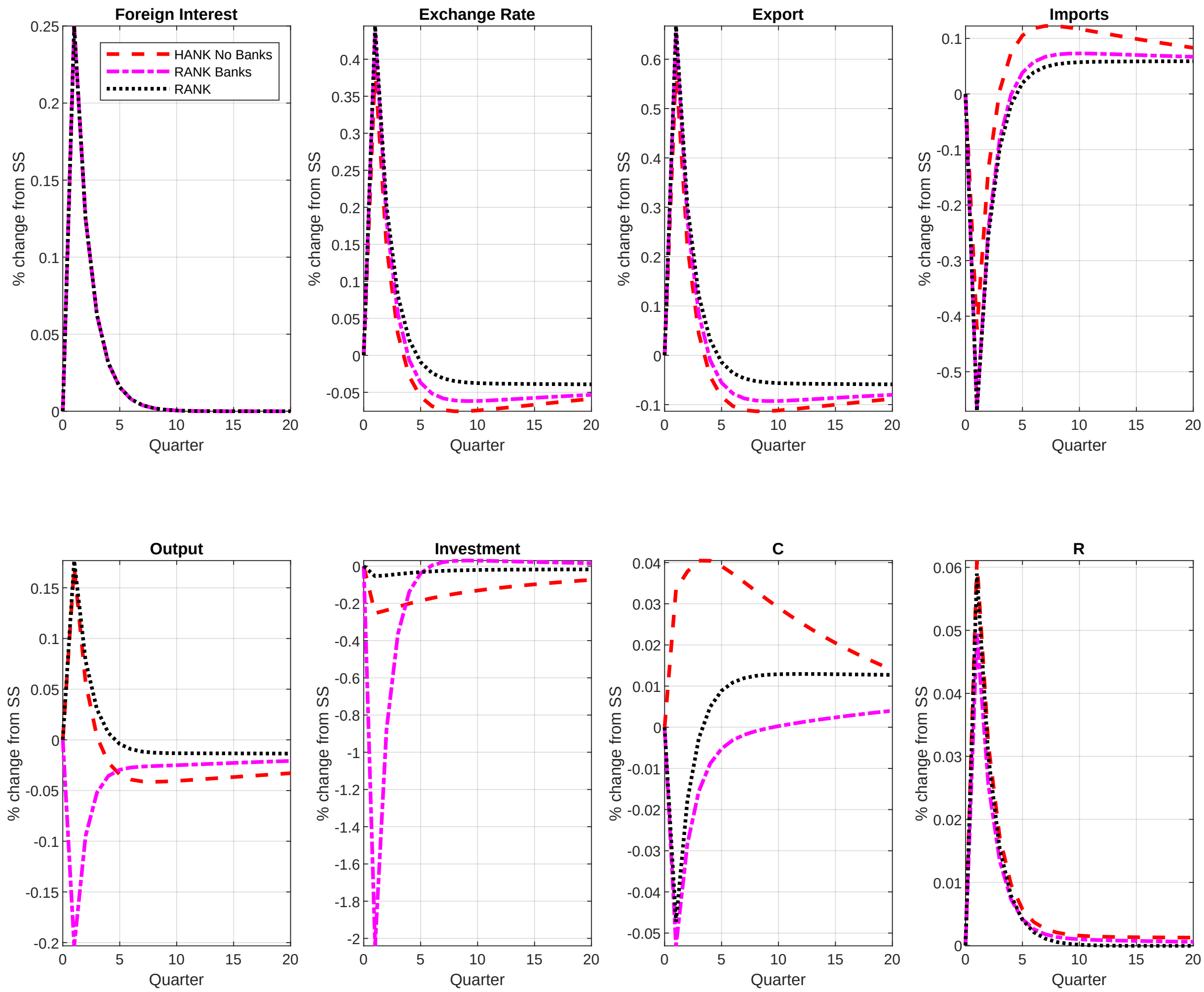

Figure 4: Devaluation: RANK vs RANK with Banks vs HANK

Figure 6 illustrates the transmission from the bank side a bit more in details, showing policy rates, inflation dynamics, both for the home good and the CPI, net worth and spreads.[4]

Figure 7 shows the consumption dynamics in the HANK models - together with the wage rate in units of the home good. Here we look at the total consumption response of various groups - we are not holding membership constant. While we already discussed the largest drivers of consumption of the wealth-poor, we can see that savers and, especially, entrepreneurs see mainly rising consumption. This is driven by a) a decline in the real rate below steady state after a few periods, b) rising income from firm holdings after a few periods, c) windfall gains from foreign

[4]The red line in the net worth plot can be ignored - it is generated by doing accounting in the bank-less model.

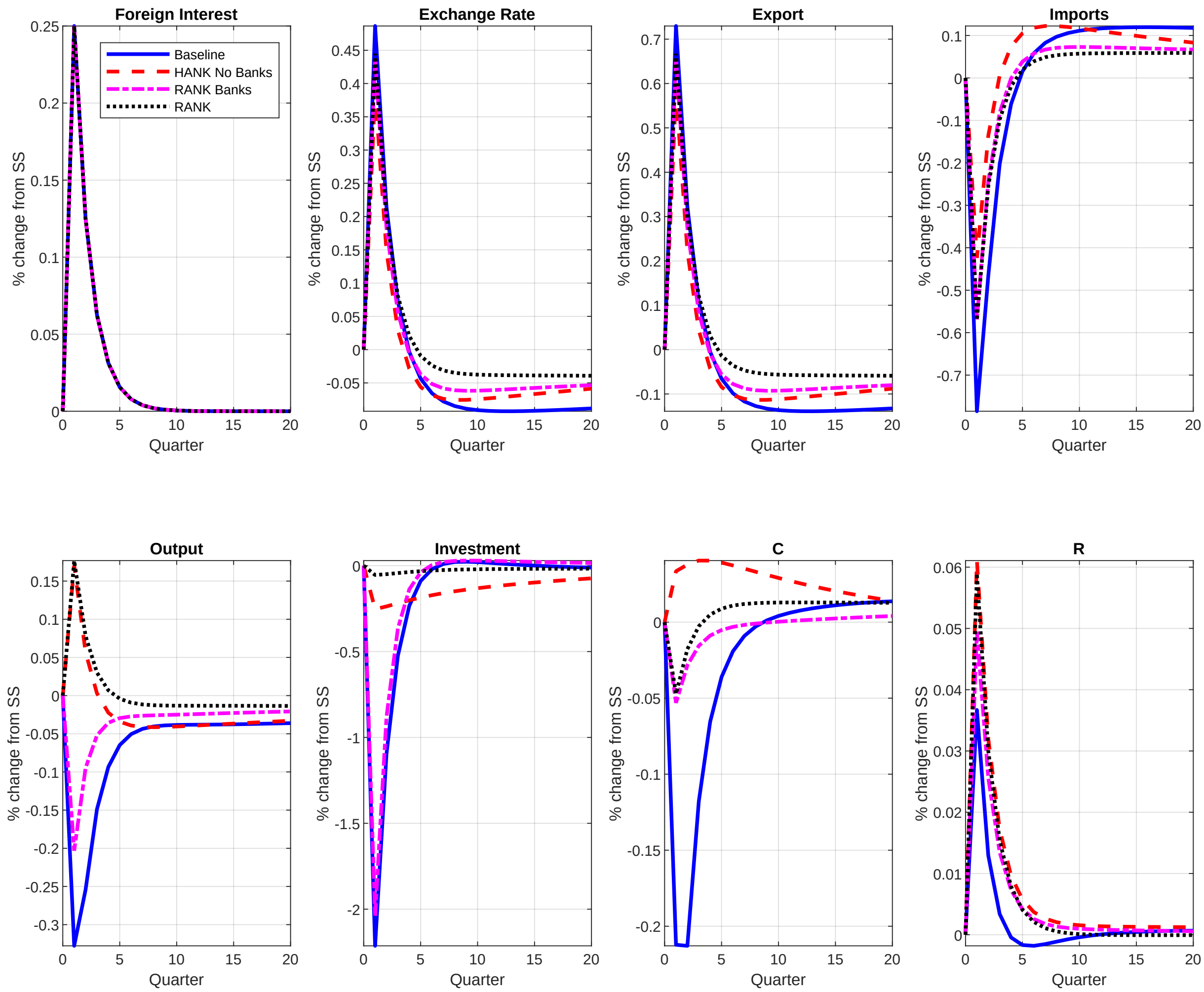

Figure 5: Devaluation: Baseline vs Other Models 1

currency denominated asset holdings in combination with a mild deflation on domestic prices.

4.2 Portfolios

Dollar deposits held by savers cause bank's currency mismatchh. As a result, the more insured the wealthier households are against interest rate shocks through dollar deposits, the worse the implications for poorer households and investment. To illustrate this we increase savers dollar deposit to double bank deposit dollarization to 80%. Figure 8 shows the results. The resulting recession is far worse compared to the baseline. What is particularly interesting here is a look at Figure 9, showing us that consumption of savers actually expands even more thanks, at least

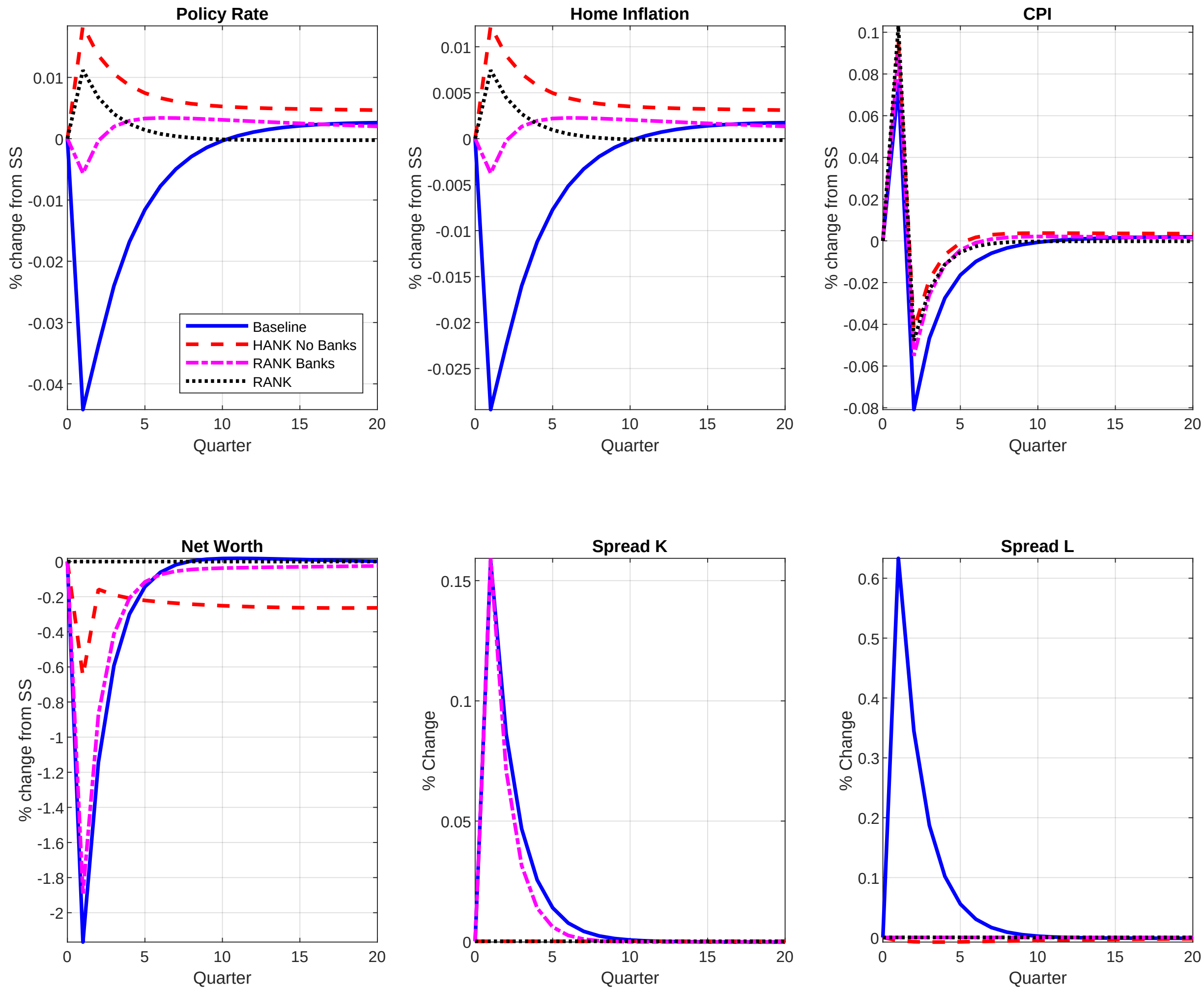

Figure 6: Devaluation: Baseline vs Other Models 2

partially, to windfall gains. It is the wealth-poor who drive the stronger consumption collapse. Both figures also show that having no dollar debt in the bank leads to a small boom after the interest rate shock with consumption gains for the poor. As such better insurance for some agents worsens the aggregate and distributional effects of the shock. [5]

[5] The magenta line is actually close to the response under the 'optimal' portfolio allocation. We can get a sense of why as consumption for most agents becomes more stable - and the same, in fact, applies to the balancesheet of the bank.

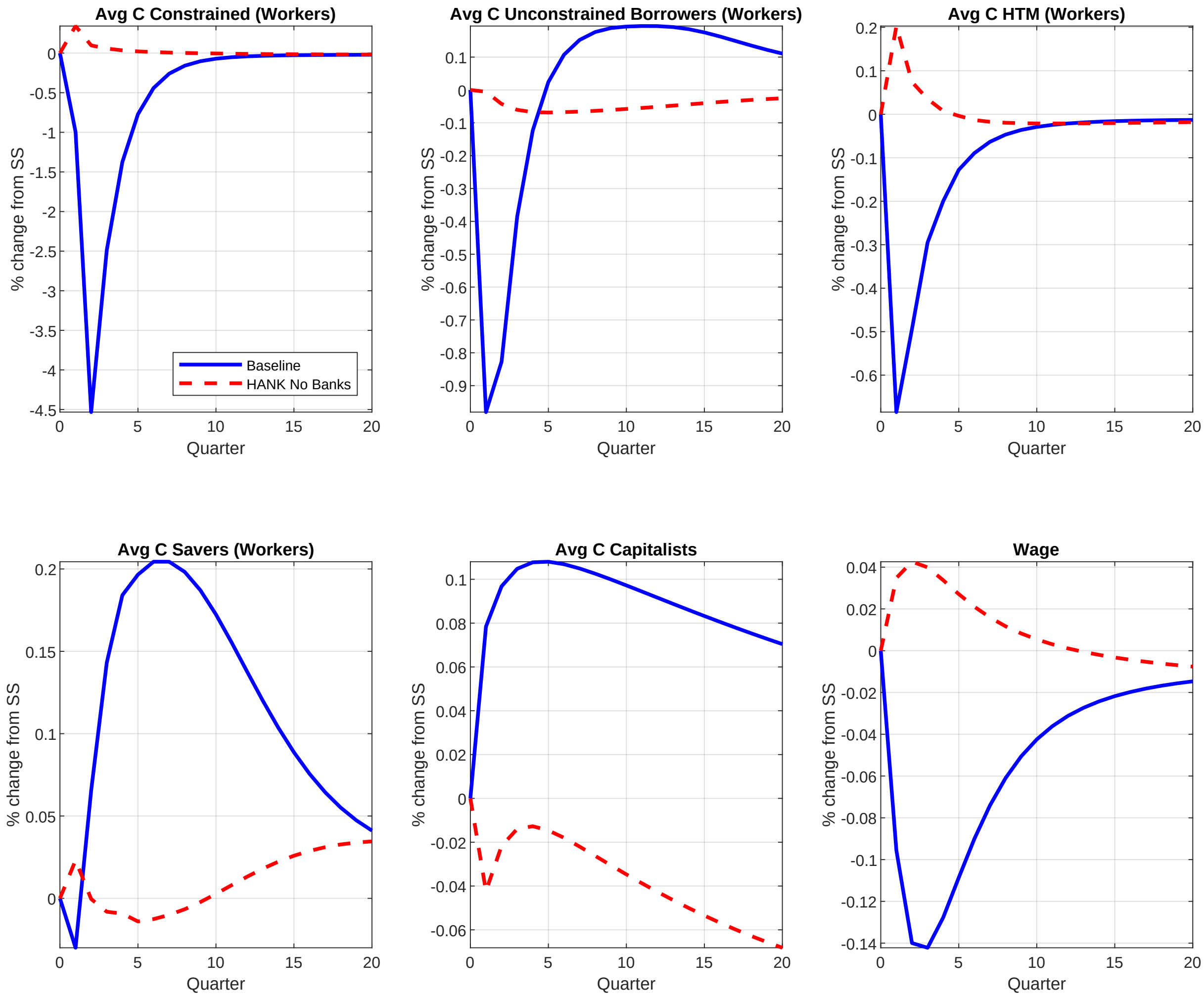

Figure 7: Devaluation: Consumption Dynamics

4.3 Monetary Policy

The central bank in our model can fight the depreciation by raising rates. This might have benefits here as it partially protects the networth of the bank. However, it might come at the cost of lower aggregate demand. To investigate this trade-off we assume now that monetary policy rule is given by

$$log(i_t) = log(R_{ss}) + \gamma_\pi log(\pi_t^H) + \gamma_e log\left(\frac{e_t}{e}\right).$$

This rule allows monetary policy to also respond to the exchange rate directly. In Figure 10 we look at the aggregate effects of the same shock but now with $\gamma_e = 1$ and $\gamma_e = 100$. Here,

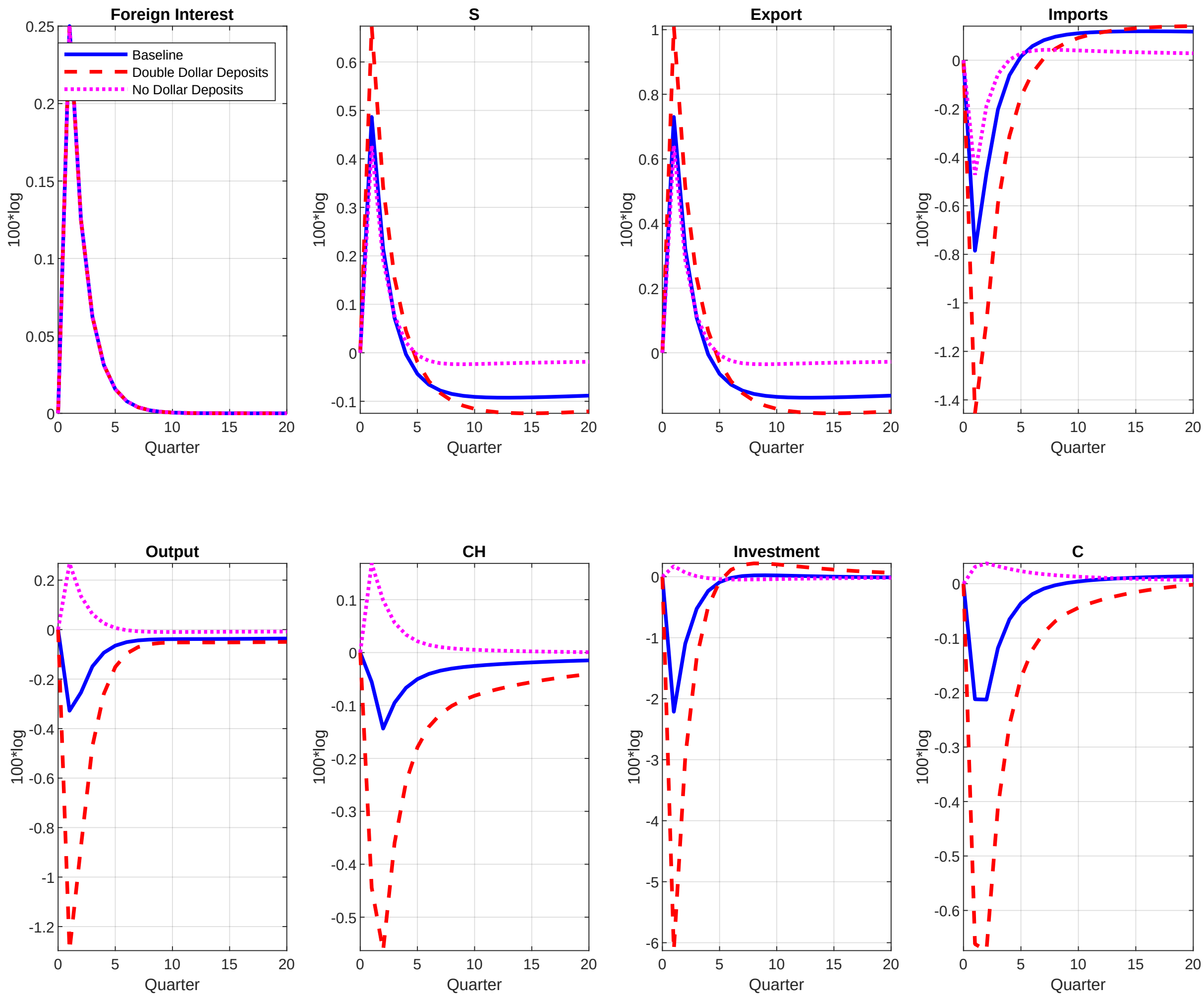

Figure 8: Devaluation: The Role of Household Portfolios 1

leaning against the depreciation turns out counterproductive as the rising domestic rates lead to an even deeper recession. However, as Figure 11 shows, with a higher dollarization of the bank's balancesheet some leaning against the depreciation might acutally be benefical as the resulting reduction in losses to the bank compensates for the rise in local rates.

Finally, figure 12 looks at the consumption equivalent gains, after a shock, for different household groups under different rules - which are in line with our consumption results.[6]

[6]We are working on a second order approximation of the value functions to look at ex-ante welfare effects of different policies.

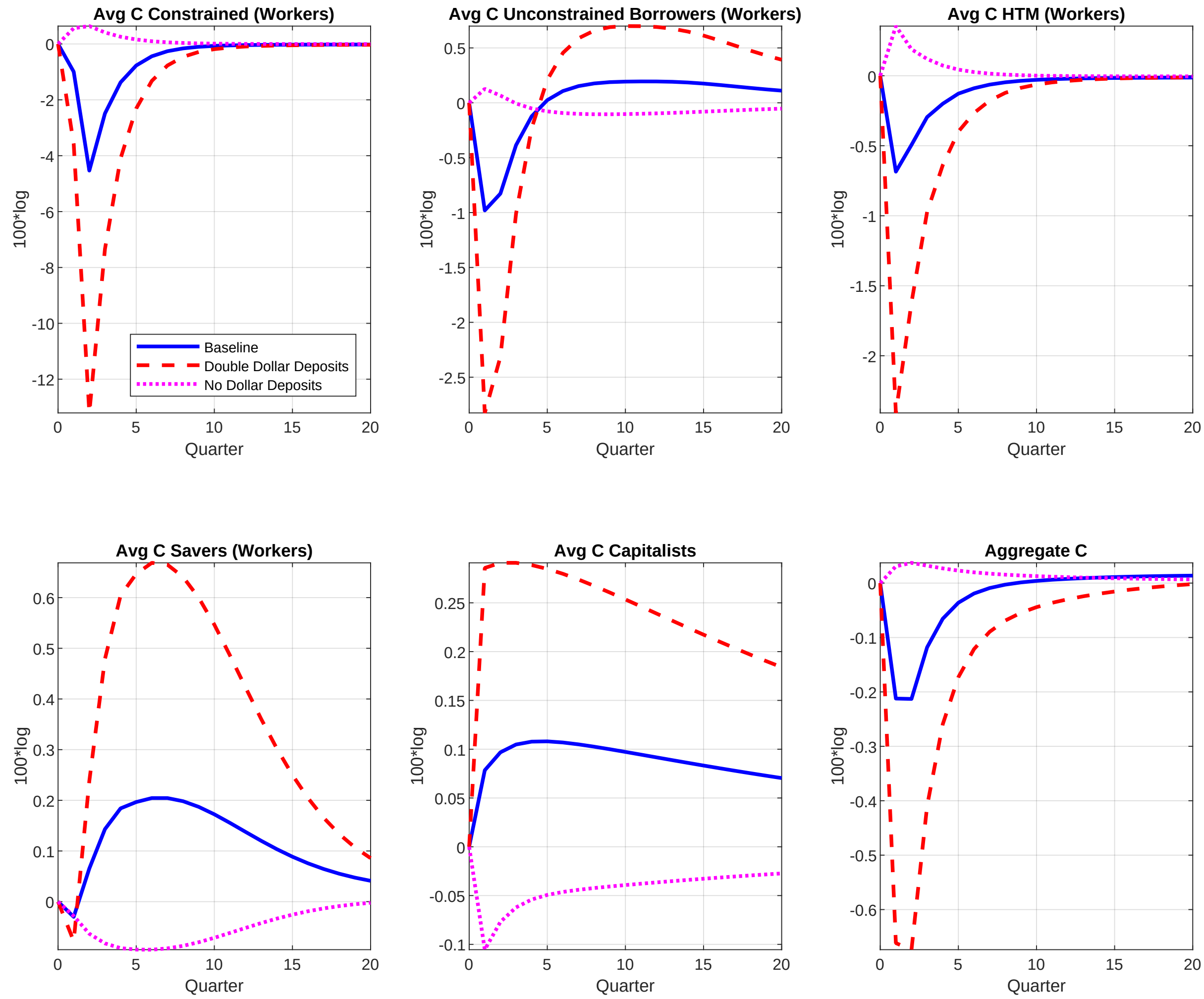

Figure 9: Devaluation: The Role of Household Portfolios 2

4.4 Robustness

We have experimented with imported intermediate goods and with dollar currency pricing of exports. These are commonly used ingredients in the emerging market literature that can help with the quantitative performance of small open economy models. While they matter for the exact response of the model to the interest rate shock, we found that our main results are robust to their addition.

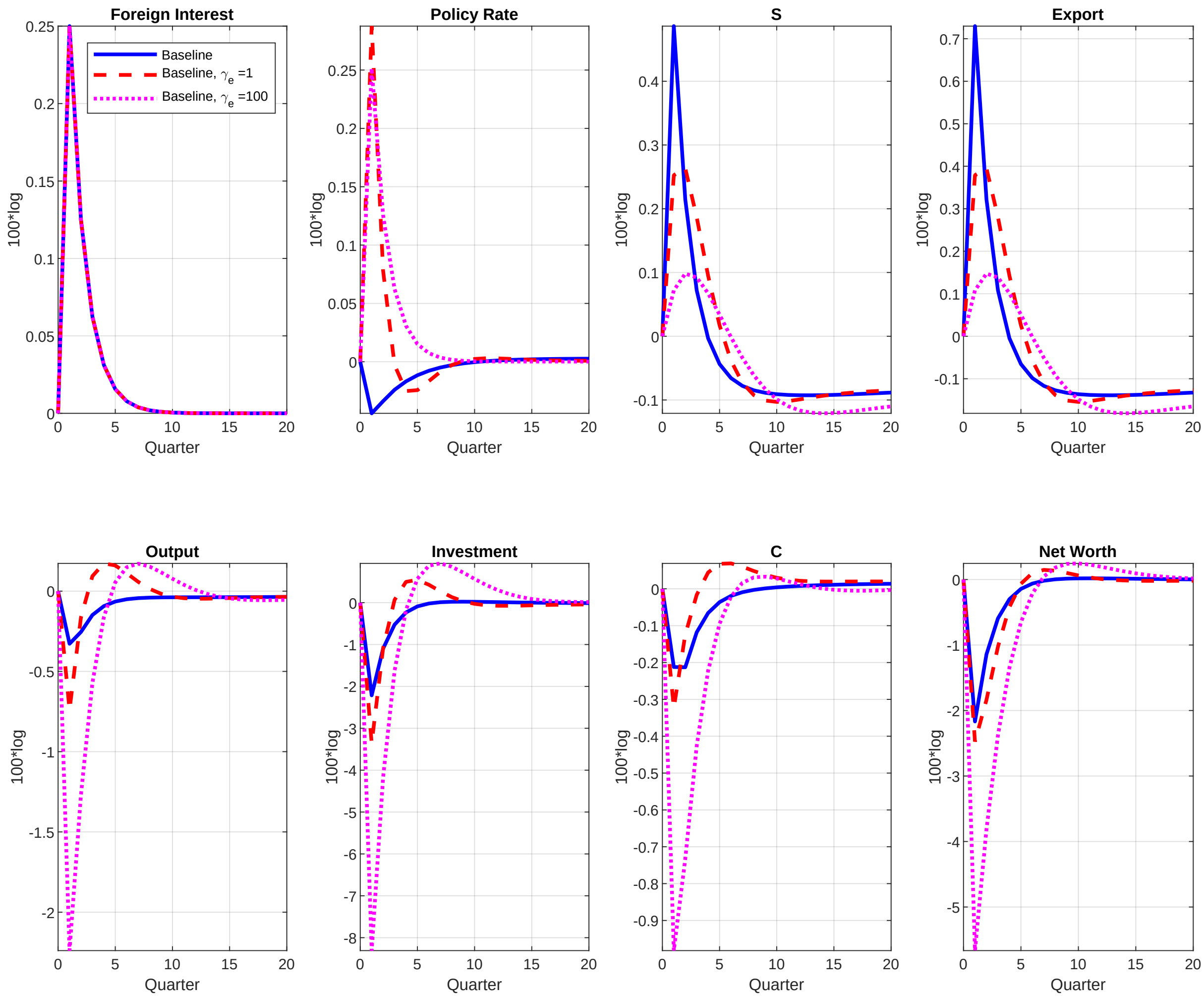

Figure 10: Exchange Rate Stabilization: Baseline

5 Conclusion

We construct a SOEM-HANK model with financial frictions and foreign currency deposits by households. We show that there are sizable interaction effects between the currency mismatch of savers, the bank, and the 'well-being' of borrowers. The central bank faces a trade-off between smoothing exchange rate fluctuations and exacerbating distributional effects of banks' financial frictions

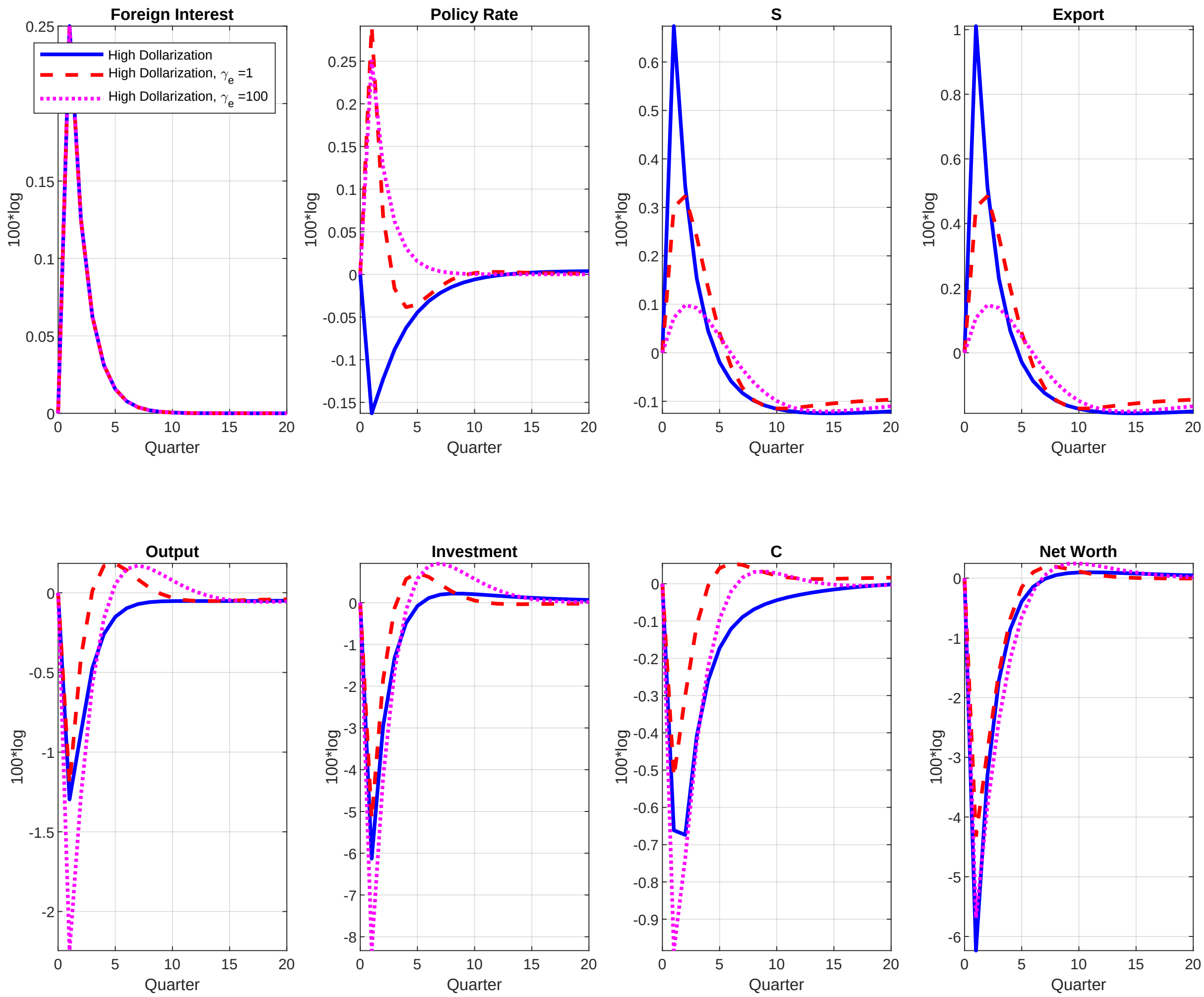

Figure 11: Exchange Rate Stabilization: High Dollarization (80%)

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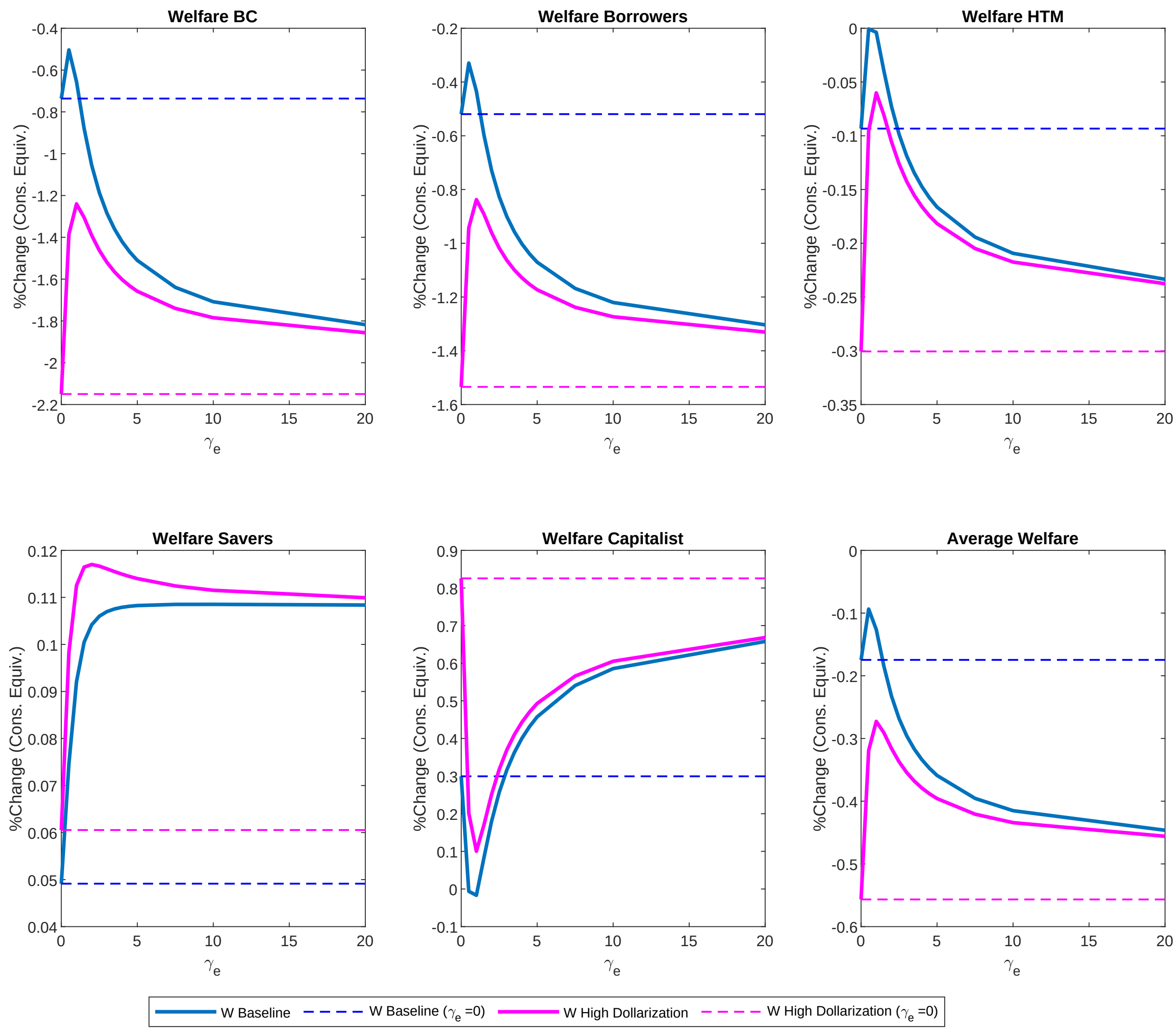

Figure 12: Exchange Rate Stabilization: Welfare

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A Optimal Portfolios

Household portfolio choice type j with assets a (returns here reflect movements in consumption price indices, when we do non-homothetic preferences we need to adjust the formular):

$$0 = \mathbb{E}_t \left[\sum_{i=1}^{s} \pi_{j,i} \left(c_{i,t+1} \left(a \right) \right)^{-\sigma} \left(R^h_{t+1} - R^f_{t+1} \right) \right]$$

Second order approximation (in logs) by terms (and surpressing the expectation operator):

- $\partial(c_{i,t+1}) : 0$
- $\partial(R^h_{t+1}) : \sum_{i=1}^{s} \pi_{j,i} \left(\bar{c}_i \left(a \right) \right)^{-\sigma} \bar{R}^h$

- $\partial(R^f_{t+1}) : -\sum_{i=1}^{s} \pi_{j,i} \left(\bar{c}_i\left(a\right)\right)^{-\sigma} \bar{R}^f$
- $\partial(c_{i,t+1})^2 : 0$
- $\partial(R^h_{t+1})^2 : \sum_{i=1}^{s} \pi_{j,i} \left(\bar{c}_i\left(a\right)\right)^{-\sigma} \bar{R}^h$
- $\partial(R^f_{t+1})^2 : -\sum_{i=1}^{s} \pi_{j,i} \left(\bar{c}_i\left(a\right)\right)^{-\sigma} \bar{R}^f$
- $\partial(c_{i,t+1}, R^h_{t+1}) : -\sigma\pi_{j,i} \left(\bar{c}_i\left(a\right)\right)^{-\sigma} \bar{R}^h$
- $\partial(c_{i,t+1}, R^f_{t+1}) : \sigma\pi_{j,i} \left(\bar{c}_i\left(a\right)\right)^{-\sigma} \bar{R}^f$
- $\partial(R^h_{t+1}, R^f_{t+1}) : 0$

As a result we get

$$0 = \mathbb{E}_t\left[\sum_{i=1}^{s} \pi_{j,i}\left(\bar{c}_i\left(a\right)\right)^{-\sigma}\left(\hat{R}^h_{t+1} - \hat{R}^f_{t+1} + 0.5\left((\hat{R}^h_{t+1})^2 - (\hat{R}^f_{t+1})^2\right) - \sigma\left(\hat{R}^h_{t+1} - \hat{R}^f_{t+1}\right)\hat{c}_{i,t+1}(a)\right)\right]$$

or

$$0 = \mathbb{E}_t\left[\hat{R}^h_{t+1} - \hat{R}^f_{t+1} + 0.5\left((\hat{R}^h_{t+1})^2 - (\hat{R}^f_{t+1})^2\right) - \frac{\sum_{i=1}^{s}\pi_{j,i}\left(\bar{c}_i\left(a\right)\right)^{-\sigma}}{\sum_{k=1}^{s}\pi_{j,k}\left(\bar{c}_k\left(a\right)\right)^{-\sigma}}\sigma\left(\hat{R}^h_{t+1} - \hat{R}^f_{t+1}\right)\hat{c}_{i,t+1}(a)\right]$$

This equation holds for all i and a

In addition, if we follow the same steps for the intermediary we get

$$0 = \mathbb{E}_t\left[\hat{R}^h_{t+1} - \hat{R}^f_{t+1} + 0.5\left((\hat{R}^h_{t+1})^2 - (\hat{R}^f_{t+1})^2\right) - \left(\hat{R}^h_{t+1} - \hat{R}^f_{t+1}\right)\hat{\Omega}_{t+1}\right]$$

Taking first differences we get

$$0 = \mathbb{E}_t\left[\left(\hat{R}^h_{t+1} - \hat{R}^f_{t+1}\right)\hat{\Omega}_{t+1} - \frac{\sum_{i=1}^{s}\pi_{j,i}\left(\bar{c}_i\left(a\right)\right)^{-\sigma}}{\sum_{k=1}^{s}\pi_{j,k}\left(\bar{c}_k\left(a\right)\right)^{-\sigma}}\sigma\left(\hat{R}^h_{t+1} - \hat{R}^f_{t+1}\right)\hat{c}_{i,t+1}(a)\right]$$

Now we take the HH budget constraint:

$$P_{c,t}c_t + a_t = w_t\omega_{i,t}h_{i,t} + \theta_{t-1}a_{t-1}R^h_t\frac{P_{c,t}}{P_{c,t-1}} + (1-\theta_{t-1})a_{t-1}R^f_t\frac{P_{c,t}}{P_{c,t-1}}$$

and re-write it as

$$P_{c,t}c_t + a_t = w_t\omega_{i,t}h_{i,t} + \theta_{t-1}a_{t-1}(R^h_t - R^f_t)\frac{P_{c,t}}{P_{c,t-1}} + a_{t-1}R^f_t\frac{P_{c,t}}{P_{c,t-1}}$$

Now, replace $\theta_{t-1}(R_t^h - R_t^f)$ with a shock $\xi_{i,t}$. Do the same for the entrepreneur, but use market clearing in the bond market. Also, to simplify notation define $r_{x,t} = (R_t^h - R_t^f)$.

Solve the model with the $\xi_{i,t+1}$ shocks. Then, impose that $\xi_{i,t+1} = \theta_i r_{x,t+1}$. From the model solution we get that

$$r_{x,t+1} = M^1 s_t + M^2 \epsilon_{t+1} + M^3 \xi_{t+1}.$$

Notice, that $M^1 = 0$ as, to first order, excess returns have no forecastable component. Plug in $\xi_{i,t+1} = \theta_i r_{x,t+1}$ to get

$$r_{x,t+1} = \frac{M^2 \epsilon_{t+1}}{1 - M^3\theta} =: \hat{M}\epsilon_{t+1}.$$

Next, notice that $\hat{\Omega}_{t+1}$ and $\hat{c}_{i,t+1}(a)$ are also linear functions. To economize on equations denote any of these elements as $\hat{x}_{t+1}$. Then,

$$\hat{x}_{t+1} = M_x^1 s_t + M_x^2 \epsilon_{t+1} + M_x^3 \xi_{t+1} = M_x^1 s_t + M_x^2 \epsilon_{t+1} + M_x^3 \theta \hat{M} \epsilon_{t+1}.$$

Plugging this into

$$0 = \mathbb{E}_t \left[\left(\hat{R}_{t+1}^h - \hat{R}_{t+1}^f \right) \hat{\Omega}_{t+1} - \frac{\sum_{i=1}^s \pi_{j,i} \left(\bar{c}_i \left(a \right) \right)^{-\sigma}}{\sum_{k=1}^s \pi_{j,k} \left(\bar{c}_k \left(a \right) \right)^{-\sigma}} \sigma \left(\hat{R}_{t+1}^h - \hat{R}_{t+1}^f \right) \hat{c}_{i,t+1}(a) \right]$$

and using that $\mathbb{E}_t \left[\epsilon_{t+1} s_t \right] = 0$

we arrive at

$$0 = \mathbb{E}_t \left[\left(\frac{M^2 \epsilon_{t+1}}{1 - M^3\theta} \right) \left((M_\Omega^2 + M_\Omega^3 \theta \hat{M}) \epsilon_{t+1} - \frac{\sum_{i=1}^s \pi_{j,i} \left(\bar{c}_i \left(a \right) \right)^{-\sigma}}{\sum_{k=1}^s \pi_{j,k} \left(\bar{c}_k \left(a \right) \right)^{-\sigma}} \sigma (M_{c_i(a)}^2 + M_{c_i(a)}^3 \theta \hat{M}) \epsilon_{t+1} \right)^T \right]$$

which becomes (denoting the variance-covariance matrix of shocks by Σ)

$$0 = \left(\frac{M^2}{1 - M^3\theta} \right) \Sigma \left(M_\Omega^2 + M_\Omega^3 \theta \frac{M^2}{1 - M^3\theta} - \frac{\sum_{i=1}^s \pi_{j,i} \left(\bar{c}_i \left(a \right) \right)^{-\sigma}}{\sum_{k=1}^s \pi_{j,k} \left(\bar{c}_k \left(a \right) \right)^{-\sigma}} \sigma \left(M_{c_i(a)}^2 + M_{c_i(a)}^3 \theta \frac{M^2}{1 - M^3\theta} \right) \right)^T$$

Multiply by $(1 - M^3\theta)$ twice (its a scalar)

$$0 = M^2 \Sigma \left(M_\Omega^2 (1 - M^3\theta) + M_\Omega^3 \theta M^2 - \frac{\sum_{i=1}^s \pi_{j,i} \left(\bar{c}_i \left(a \right) \right)^{-\sigma}}{\sum_{k=1}^s \pi_{j,k} \left(\bar{c}_k \left(a \right) \right)^{-\sigma}} \sigma \left(M_{c_i(a)}^2 (1 - M^3\theta) + M_{c_i(a)}^3 \theta M^2 \right) \right)^T$$

Transpose the whole system and sort by terms with θ

$$\left(M_{\Omega}^2 - \frac{\sum_{i=1}^{s} \pi_{j,i} \left(\bar{c}_i \left(a\right)\right)^{-\sigma}}{\sum_{k=1}^{s} \pi_{j,k} \left(\bar{c}_k \left(a\right)\right)^{-\sigma}} \sigma M_{c_i(a)}^2\right) \Sigma^T (M^2)^T$$

$$= \left(M_{\Omega}^2 M^3 \theta - M_{\Omega}^3 \theta M^2 + \frac{\sum_{i=1}^{s} \pi_{j,i} \left(\bar{c}_i \left(a\right)\right)^{-\sigma}}{\sum_{k=1}^{s} \pi_{j,k} \left(\bar{c}_k \left(a\right)\right)^{-\sigma}} \sigma \left(-M_{c_i(a)}^2 M^3 \theta + M_{c_i(a)}^3 \theta M^2\right)\right) \Sigma^T (M^2)^T$$

As $M_x^3 \theta$ are scalars we can move them out of the expressions and transpose to arrive at

$$\left(M_{\Omega}^2 - \frac{\sum_{i=1}^{s} \pi_{j,i} \left(\bar{c}_i \left(a\right)\right)^{-\sigma}}{\sum_{k=1}^{s} \pi_{j,k} \left(\bar{c}_k \left(a\right)\right)^{-\sigma}} \sigma M_{c_i(a)}^2\right) \Sigma^T (M^2)^T$$

$$= \theta^T \left((M^3)^T M_{\Omega}^2 - (M_{\Omega}^3)^T M^2 + \frac{\sum_{i=1}^{s} \pi_{j,i} \left(\bar{c}_i \left(a\right)\right)^{-\sigma}}{\sum_{k=1}^{s} \pi_{j,k} \left(\bar{c}_k \left(a\right)\right)^{-\sigma}} \sigma \left(-(M^3)^T M_{c_i(a)}^2 + (M_{c_i(a)}^3)^T M^2\right)\right) \Sigma^T (M^2)^T$$

Combining all matrices for all j, a cases we can solve for the portfolio shares by solving this linear system.