Government Debt Maturity Structure, Fiscal Policy, and Default[*]

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June 18, 2019

Abstract

I develop a tractable model to study the optimal debt maturity structure and fiscal policy in an environment with incomplete markets, lack of commitment, and opportunity to default by the government. The default on public debt is endogenous and the real interest rate reflects the default risk and the marginal rate of substitution between present and future consumption. The maturity is used to resolve the time-consistency problem: The present government can incentivize future governments to stick to an ex ante optimal sequence of fiscal policies and interest rates. I show that if both risk-free interest rates and risk premiums can be manipulated, the optimal maturity structure tends to have a decaying profile: The government issues debt at all maturity dates, but the distribution of payments over time is skewed toward the short-term end. Debt maturity data across countries are consistent with model predictions.

[*]I would like to thank Mark Aguiar, Mike Golosov, Oleg Itskhoki, Nobu Kiyotaki, Dmitriy Mukhin, Michael Dobrew, Michael Burda, Iqbal Zaidi, Yuriy Gorodnichenko and the participants of the seminars at Princeton University, University of Melbourne, AQR Capital Management, Kyiv School of Economics, National Bank of Ukraine, US Naval Academy, Bank of Poland, and Humboldt University as well as the participants of Fall Midwest Macroeconomic Meetings 2018 and Spanish Economic Symposium 2018 for their comments and suggestions. The views in this paper are mine as well as all typos and mistakes.

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1 Introduction

Debt maturity structure is an important element of optimal fiscal policy, especially in light of recent sovereign debt crises. The consensus is that debt maturity is used to minimize the costs of lack of commitment. In their seminal paper, Lucas and Stokey (1983) derive the classic result that in an environment with endogenous risk-free interest rates and no default the government should issue consol bonds, i.e., the optimal maturity is spread out or flat. In contrast, Aguiar et al. (2016) study an open economy with default but exogenous risk-free interest rates. The authors demonstrate that the time-consistency problem can be resolved if the government issues only short-term debt and abstains from any active issuance or repurchase of long-term liabilities.

In this paper, I combine both sources of time inconsistency - manipulation of risk-free interest rates and debt dilution due to option to default - within a unified framework. I develop a tractable model to study the optimal fiscal policy and optimal debt maturity structure in an environment with incomplete markets, lack of commitment to fiscal policies, and endogenous default on public debt, and show that, if a government can alter both risk-free rates and risk premiums, the optimal maturity structure exhibits a decaying profile, i.e., total payments due at a later maturity date are lower. This prediction is in line with empirical data as observed term structures of most countries are neither flat nor short but skewed toward the short end.

The model features a benevolent government and a continuum of atomistic households with strictly concave utility functions over private consumption. Households are the only lenders to the government. The government cannot commit to either future fiscal policies or to repay its debt, and sets fiscal policies, restructures its debt portfolio, and decides whether or not to default sequentially. The markets are incomplete, and the set of financial instruments is limited to bonds with various maturities. Interest rates reflect both the probability of default and marginal rate of substitution between present and future consumption.

Default is modeled as a stochastic outside option that can be exercised at the beginning of every period. Whenever the value of the outside option exceeds the value of repaying debt, the default option is triggered. Default is costly to sustain a positive amount of debt in equilibrium. The value of the outside option is the only shock in the model. In addition, the value of default is continuously distributed to allow smoothness in default probability.

I analyze the Markov perfect competitive equilibrium in which all decisions are made sequentially and are functions of payoff-relevant state variables: the outstanding debt at various maturities and the value of the outside option. I characterize the optimal allocation by considering the modified commitment problem as the benchmark: A contract that allows the government to commit to predetermined fiscal policies but not to abstain from default. In other words, the planner simultaneously makes the fiscal decisions for all future periods and can promise to pursue the plan: however, it cannot promise to repay debt if the outside option is preferred. The optimal allocation of the modified commitment problem defines the fiscal plan: the sequence of budget surpluses needed to repay debt contingent on no prior default.

In this model, the Markov perfect competitive equilibrium is efficient in the sense that the sequential policy maker follows the ex ante optimal fiscal plan and sticks to ex ante optimal risk-free interest rates and default probabilities. Even though the government cannot commit to future policies, it can set the term structure of its liabilities so that it has no incentive to deviate from the plan in the future. Why might the government be willing to distort the ex ante optimal allocation in the future? At every date, the value of outstanding debt must be financed by future budget surpluses. Therefore, a deviation from the plan can be ex post beneficial if the market value of outstanding debt is decreased. For example, if debt is mostly short term, then reallocation of budget surpluses leading to a decrease in the value of short-term debt at the expense of an increase in the value of long-term debt might be optimal ex post, and vice versa. However, the government can structure its debt maturity such that any such distortion strictly reduces the budget set of the government by increasing the value of outstanding debt and, hence, deviations are not optimal.

The main result is that in the presence of default risk, the government issues more short-term debt than long-term debt. Moreover, the optimal maturity structure has a decaying profile: The government issues debt at all maturity dates, but the distribution of payments over time is skewed toward the short-term end. The average maturity depends on the relative sensitivity of risk-free interest rates and risk premiums. The term structure is shorter if risk-free interest rates are less responsive to changes in government policies. On the other hand, if a deviation from a fiscal plan has a negligible effect on default risk, then the optimal maturity structure is approximately flat.

To gain intuition, suppose that the government can distort only risk-free rates and the default risk is absent. Then the optimal maturity is flat, meaning that the total payments due at different maturity dates are constant. Any deviation from the fiscal plan that increases the budget surplus in one period and decreases it in another period does not lead to a decrease in the value of outstanding debt; this is because changes in risk-free rates are proportional and offset each other. However, if debt is skewed toward the short- or long-term end, decreasing the price of a larger stock of debt at the expense of increasing the price of a lower stock of debt allows the government to reduce the value of total debt.

Now consider an environment in which risk-free rates are exogenous, but the default risk is positive and increasing in total debt issued. The next period, the government can affect default probabilities in future periods by increasing or decreasing the budget surplus. Thus, it can affect the value of debt that matures in subsequent periods. However, the government cannot manipulate the price of a one-period debt issued in the preceding period, because it cannot alter the default probability in the current period. The reason is that all government fiscal policies are conducted conditional on no prior default in that period. Therefore, the optimal debt policy prescribes issuance of one-period bonds only.

Finally, suppose the government can manipulate risk-free interest rates and default probabilities. In such an environment, the price of debt with longer maturity is more sensitive to potential future distortions compared to the price of debt with shorter maturity. Consider a deviation from ex ante optimal fiscal plan that implies reallocation of budget surpluses between two subsequent periods, keeping the market value of budget surpluses constant. This perturbation causes proportional changes in risk-free interest rates. The probability of default in the later period changes as a higher or lower budget surplus in that period corresponds to a higher or lower value of pursuing the fiscal plan and, hence, to a lower or higher default risk. However, this deviation does not affect the default probability in the earlier period, as the deviation described does not change government welfare in the earlier period. Therefore, change in the price of debt with longer maturity reflects distortions in both the risk-free interest rate and risk premium, while the price of debt with shorter maturity varies due only to changes in the risk-free interest rate. A deviation from the ex ante fiscal plan has an offsetting effect on that value of shorter- and longer-term debt if the stock of debt with shorter maturity is larger. Extending this result to a finite-period model leads to the conclusion that the optimal term structure must be decreasing in maturity date.

The benefit of using the modified commitment problem is that it allows me to characterize the optimal maturity structure in an infinite-period model with various maturities available to the government. First, I solve for the optimal path of fiscal policies. Then I solve for the maturity structure that renders the ex ante plan incentive-compatible for future government.

In quantitative exercises, I solve for the optimal maturity structure under different shocks to taste parameter for public spending, different initial debt-to-GDP ratios and various initial maturity structures.

My analysis implies that the data on the maturity structure of developed economies is broadly consistent with normative analysis of the optimal debt policy under lack of commitment and opportunity to default. According to this model, lengthening government debt maturity would cause an increase in long-term risk-free rates and default probabilities, as such term restructuring would incentivize future governments to over-borrow compared to the ex ante optimal policy.

Related Literature

As already mentioned, the paper bridges the gap between two literatures that study lack of commitment due to risk-free rates manipulation and risk premiums manipulation in isolation. I build on the work of Aguiar, Amador, Hopenhayn and Werning (2016) by introducing endogenous risk-free interest rates as in Lucas and Stokey (1983).

This paper also relates to the literature that investigates time consistency of fiscal *and* monetary policy. Alvarez et al. (2004) show that Ramsey policy can be made time consistent under the Friedman rule, i.e., zero nominal interest rate is optimal. Persson et al. (2006) argue that time consistency can be achieved by structuring government nominal and indexed debt in an environment where positive nominal interest rates are optimal. In my paper, the focus is on the option of outright government default which is missing from the discussed studies, however, nominal debt and, hence, government's ability to inflate away debt is absent in my paper. I find that the fiscal policy is time-consistent in a weaker sense, as discussed in Aguiar et al. (2016): A government follows an optimal sequence of fiscal policy decisions conditional on no prior default.

Maturity structure can be also used to hedge a government against fiscal shocks. Angeletos (2002) shows that in an environment with perfect commitment but incomplete markets, state-contingent debt can be replicated by maturity structure of non-contingent debt providing complete insurance to the government. According to quantitative exercises discussed in Buera and Nicolini (2004), such an insurance requires very large debt positions relative to GDP. However, Debortoli et al. (2017) show that such large debt positions are not sustainable in an environment with lack of commitment as a government has an incentive to distort risk-free interest rates to alter the value of outstanding debt. Moreover, the authors find that the optimal maturity structure is approximately flat because minimizing the costs associated with the lack of commitment is quantitatively much more important than minimizing the costs associated with the lack of insurance. The latter conclusion rationalizes the focus of the paper on the commitment problem and abstraction from hedging motive by setting deterministic fiscal shocks.

Maturity has been studied in international quantitative sovereign debt models. Aguiar and Gopinath (2006) were among the first to present a quantitative model with endogenous default decision in an environment with incomplete markets, as in the seminal paper by Eaton and Gersovitz (1981). Hatchondo and Martinez (2009) and Chatterjee and Eyigungor (2012) find that exogenously lengthening debt maturity by introducing a consol bond with a decaying coupon rate improves quantitative fit of such models. Arellano and Ramanarayanan (2012) extend their framework by allowing a sovereign to choose between consol bonds with different decaying rates, and the authors show that average maturity shortens in an event of a crisis. Short-term debt in these models minimizes an incentive to dilute the value of longer-term debt, while long-term debt serves as a hedging against income shocks. In these models, maturity structure of debt has a decaying profile by construction, while in my model I show that such debt structure is optimal. However, in contrast to the aforementioned studies, the role of long-term debt in this paper is to minimize risk-free interest rate distortions, while hedging motive is absent due to deterministic fiscal shocks and constant endowment.

Open economy and corporate finance literature often emphasizes the disciplining role of short-term debt. Jeanne (2009) demonstrates that short-term debt can incentivize a government to pursue a creditor-friendly policy as debt is rolled-over conditional on policy implementation. In Calomiris and Kahn (1991) and Diamond and Rajan (1991) short-term debt provides a creditor an option to liquidate project. In this model, lenders are atomistic and cannot directly affect government's decisions, instead, a time-inconsistent government uses debt maturity to discipline itself in the future.

The remainder of the paper is organized as follows. Section 2 describes the finite-period model. Section 3 considers the modified commitment problem and shows that the Markov perfect competitive equilibrium is equivalent to the allocation under commitment. In Section 4, I show that maturity is tilted toward the short end in a three-period example. In Section 5, I demonstrate that the maturity has a decaying profile in a multi-period model. Section 6 presents numerical results of three-period and six-period models, and Section 7 concludes.

2 Model

In this section, I describe a model that allows to study the optimal maturity structure of government debt in an environment with lack of commitment to fiscal policies and to pay debt.

The economy is closed and consists of a government and a unit mass of atomistic households. The time is discrete and indexed by $t = 0, 1, 2, ...$ I assume that the government starts with no debt to abstract away from potential effect of initial debt maturity structure on government debt policy.

Preferences and Endowment. A representative household values private consumption and government spending:

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t\left(u(c_t)+\theta_t\omega(g_t)\right) \tag{1}$$

where u and ω are continuously differentiable, strictly increasing, concave functions and $\beta \in (0, 1]$ is the discount factor. θ_t represents taste parameter for public spending. Larger θ implies higher marginal utility of government expenditures and, hence, households would prefer more resources to be spent on public goods. I assume that all θ_t are deterministic and known at date 0. The government is benevolent and shares the same preferences.

There is no capital in the economy. Each period a representative household is endowed with $1-\tau$ units of consumption and the government is endowed with τ units of consumption. Every period the resource constraint has to be satisfied

$$c_t + g_t = 1 \, \forall t \tag{2}$$

I make the following assumptions about u and ω:

Assumption 1.
(i) $\theta_0 > \theta_1 = \theta_2 = ... = 1$;
(ii) $\omega'(\tau) \geq u'(1-\tau)$.

Assumption (i) states that there is an incentive to have higher public spending in the very first period than in consecutive periods. Assumption (ii) states that if the budget surplus is non-negative then the marginal utility of public spending weekly exceeds marginal utility of private consumption. Together these two assumptions ensure that the government has an incentive to reallocate resources for public spending from future periods to the initial period by issuing some debt. Moreover, Assumption (ii) also implies that per period utility is strictly decreasing in the budget surplus if the latter is positive.

Bond Markets and Default. The government borrows from households. I assume that state-contingent bonds are not available, and the set of financial instruments is limited to discount bonds with different maturities. Define by b_t^{t+k} the government debt held by a household that is issued at date t and promises to pay one unit of consumption at $t+k$ and let q_t^{t+k} be the price of the bond. Without loss of generality the government rebalances its portfolio each period, i.e., it buys back all the outstanding debt and issues new debt at all maturities.

The government can default on its debt. Why default can be optimal for the government in this model? Assumption 1 guarantees that the government issues some positive amount of debt in the very first period. It also implies that starting from period 1, as the government has to pay its debt back, the marginal utility of private consumption is lower than marginal utility of government consumption. Therefore, reallocation of goods from private consumption to public spending is beneficial so that default (if costless) is always optimal.

In order to sustain some positive debt in equilibrium I assume that default isn't costless. More specifically, I follow Aguiar et al. (2018) and assume that every period the government has an outside option V_t^{def} that can be achieved upon default. V_t^{def} is drawn from continuous distribution F that has bounded support $[V_{min}; V_{max}]$. I make the following assumptions about the outside option:

Assumption 2. *Outside option:*
(i) $V_{max} \leq \sum_{j=t}^{\infty} \beta^j \left(u(1-\tau) + \theta_j \omega(\tau)\right)$ $\forall t = 0,\, 1,\, ...;$
(ii) $\exists g_{min} > 0$*:* $V_{min} > \sum_{j=t}^{\infty} \beta^j \left(u(1-g_{min}) + \theta_j \omega(g_{min})\right)$ $\forall t = 0,\, 1,\, ...;$
(iii) F *is strictly increasing on* (V_{min}, V_{max}) *and* $f(V_{max}) = 0$*;*
(iv) V_t^{def} *is independent across time and independent of debt portfolio.*

Restriction (i) ensures that the government will never choose an outside option if debt positions are zero. In addition, it guarantees that some positive level of debt can be sustained in equilibrium. Restriction (ii) implies that the government always defaults if the debt position is high enough and government spending is sufficiently low. Assumption (iii) allows to avoid kinks in the pricing functions which ensures that the equilibrium can be characterized by first-order necessary conditions. The assumption of independence in (iv) is made to abstract from using maturity structure for hedging motives.

Timing and Government Problem. At the beginning of every period, the government decides whether or not to default on its debt. If it defaults, it receives the outside option value V_t^{def}. Otherwise, the government sets government expenditures, buys back existing debt and issues new debt. The default decision precedes any fiscal decisions, and the government is not allowed to default until the beginning of the next period once new debt has been issued. This timing rules out the possibility of self-fulfilling debt crises, as discussed by Cole and Kehoe (2000).

I focus on a Markov perfect competitive equilibrium in which the government makes decisions sequentially as functions of payoff-relevant variables: the outstanding bond holdings and period t. Denote by $\boldsymbol{b}_t = (b_t^{t+1},\, b_t^{t+2},\, ...)$ the vector of bond holdings issued at period t and let $\boldsymbol{q}_t = (q_t^{t+1},\, q_t^{t+2},\, ...)$ be the vector of corresponding bond prices.

To simplify notation, it is useful to define the contingent budget surplus as the difference between endowment of and spending by the government if it does not default:

$$s_t = \tau - g_t$$

Consumption is then defined as $c_t = 1 - \tau + s_t$ and government spending is $g_t = \tau - s_t$. Therefore, setting contingent budget surpluses is equivalent to choosing government expenditures. Conditional on no default the budget constraint in every period satisfies:

$$s_t + \boldsymbol{q}_t(s_t, \boldsymbol{b}_t) \cdot \boldsymbol{b}_t \geq (1, \boldsymbol{q}_t(s_t, \boldsymbol{b}_t)) \cdot \boldsymbol{b}_{t-1} \tag{3}$$

The right-hand side of (3) is the market value of outstanding debt. The left-hand side is the sum of budget surplus and the market value of newly issued debt.

Let $V_t(\boldsymbol{b}_{t-1})$ be the value of the government if it does not prefer the outside option V_t^{def}:

$$V_t(\boldsymbol{b}_{t-1}) = \max_{s_t, \boldsymbol{b}_t} \left\{ u(1 - \tau + s_t) + \theta_t \omega(\tau - s_t) + \beta \cdot \mathbb{E} \max \left\{ V_{t+1}(\boldsymbol{b}_t);\, V_{t+1}^{def} \right\} \right\} \tag{4}$$

subject to $s_t \in (-(1-\tau),\, \tau)$ and the budget constraint (3)

and denote by $\rho_t(\boldsymbol{b}_{t-1}) = \{s_t^\star(\boldsymbol{b}_{t-1}),\, \boldsymbol{b}_t^\star(\boldsymbol{b}_{t-1})\}$ the optimal government fiscal and debt policies, conditional on no default at t.

Household Optimization and Bond Prices. In any competitive equilibrium, household optimality conditions must be satisfied. A representative household takes into account the future government policies that are reflected in risk-free interest rates and risk premiums. The price of a bond that matures in $k \geq 1$ periods can be defined recursively as

$$q_t^{t+k}(s_t, \boldsymbol{b}_t) = \beta \frac{u'(1 - \tau + s_{t+1}^\star(\boldsymbol{b}_t))}{u'(1 - \tau + s_t)} \cdot (1 - \pi_{t+1}(\boldsymbol{b}_t)) \cdot q_{t+1}^{t+k}(\rho_{t+1}(\boldsymbol{b}_t)) \tag{5}$$

where $\pi_{t+1}(\boldsymbol{b}_t)$ defines the probability of default in the next period:

$$\pi_{t+1}(\boldsymbol{b}_t) = \text{Prob}(V_{t+1}^{def} > V_{t+1}(\boldsymbol{b}_t)) = 1 - F(V_{t+1}(\boldsymbol{b}_t)) \tag{6}$$

Definition of Markov Perfect Competitive Equilibrium. The Markov Perfect Competitive Equilibrium of the economy consists of the value function $V_t(\boldsymbol{b}_{t-1})$, the fiscal policy function $\rho_t(\boldsymbol{b}_{t-1})$ and the pricing function $\boldsymbol{q}_t(s_t, \boldsymbol{b}_t)$ such that:

(i) the value function $V_t(\boldsymbol{b}_{t-1})$ solves the Bellman equation (4) given the fiscal policy function $\rho_t(\boldsymbol{b}_{t-1})$ and the pricing function $\boldsymbol{q}_t(s_t, \boldsymbol{b}_t)$;

(ii) the fiscal policy function $\rho_t(\boldsymbol{b}_{t-1})$ maximizes the right-hand side of (4) subject to the budget constraint (3), taking into account the pricing function $\boldsymbol{q}_t(s_t, \boldsymbol{b}_t)$;

(iii) the pricing function $\boldsymbol{q}_t(s_t, \boldsymbol{b}_t)$ satisfies the first-order condition of household utility maximization (5) given the fiscal policy function $\rho_t(\boldsymbol{b}_{t-1})$.

The key objective is to characterize the optimal maturity structure of debt. ((discuss why it is important))

3 Time Consistency of a Markov Perfect Competitive Equilibrium

In this section, I show that the time consistency of fiscal policy carries over in environments in which interest rates reflect both the default probability, as in Aguiar et al. (2018), and the endogenous marginal rate of substitution between present and future consumption, as in Lucas and Stokey (1983).

Lucas and Stokey (1983) argue that if debt commitments are binding, then in an environment with no capital the discretionary fiscal policy is time consistent. However, they show that time consistency does not carry over in a monetary economy, in which future governments can inflate away the value of outstanding debt. Analogous conclusion applies to the model studied in this paper: If default risk is positive, it is reflected in bond prices and, hence, the first-best allocation is generally not attainable for a Markov perfect competitive equilibrium. Nevertheless, the optimal fiscal policy can still be characterized by considering the modified commitment problem in which fiscal commitments are binding but debt commitments are not. Then the fiscal policy is "time consistent," in a sense that the sequential government sticks to the ex ante optimal fiscal plan and follows it as long as the government does not default.

3.1 The Modified Commitment Problem

To characterize the optimal maturity debt structure of a government that cannot commit (a Markov government), it is useful to consider the following planning problem. Consider a government (a planner) who can commit to fiscal policies conditional on sequential default decisions not being preferred. In other words, at date 0 a planner simultaneously makes fiscal decisions for all periods, and it can promise to follow the plan; however, the planner defaults whenever the value of the outside option is higher than the value of pursuing the fiscal plan. I call this the modified commitment problem due to the planner's inability to commit to pay its debt.

Initially, the planner sets fiscal policies for current and all future periods. Define by a fiscal plan $\boldsymbol{s}_t = (s_t,\, s_{t+1},\, ...)$ a sequence of contingent budget surpluses from period t onwards. Let $W_t(\boldsymbol{s}_t)$ be the value of fiscal plan $\boldsymbol{s}_t$, that is defined recursively as

$$W_t(\boldsymbol{s}_t) = u(1-\tau+s_t) + \theta_t \omega(\tau - s_t) + \beta \cdot \mathbb{E} \max \left\{ W_{t+1}(\boldsymbol{s}_{t+1}),\, V_{t+1}^{def} \right\} \tag{7}$$

Notice that any fiscal plan $\boldsymbol{s}_0$ uniquely determines bond prices. Iterating equation (5) forward we can derive the bond prices at the initial period:

$$q_0^t(\boldsymbol{s}_0) = \beta^t \frac{u'(1-\tau+s_t)}{u'(1-\tau+s_0)} \cdot \prod_{k=1}^{t} F(W_k(\boldsymbol{s}_k)) \tag{8}$$

where $\prod_{k=1}^{t} F(W_k(\boldsymbol{s}_k))$ defines the probability of repaying debt issued at date 0, which matures at t, i.e., the probability that the planner does not default at dates 1, 2, ..., t.

Any fiscal plan $\boldsymbol{s}_0$ must satisfy the dynamic budget constraint:

$$s_0 + \sum_{t=1}^{\infty} \beta^t \frac{u'(1-\tau+s_t)}{u'(1-\tau+s_0)} s_t \prod_{k=1}^{t} F(W_k(\boldsymbol{s}_k)) \geq b_{-1}^0 + \sum_{t=1}^{\infty} \beta^t \frac{u'(1-\tau+s_t)}{u'(1-\tau+s_0)} b_{-1}^t \prod_{k=1}^{t} F(W_k(\boldsymbol{s}_k))$$

or equivalently

$$\sum_{t=0}^{\infty} \beta^t u'(1-\tau+s_t) s_t \prod_{k=1}^{t} F(W_k(\boldsymbol{s}_k)) \geq \sum_{t=0}^{\infty} \beta^t u'(1-\tau+s_t) b_{-1}^t \prod_{k=1}^{t} F(W_k(\boldsymbol{s}_k)) \tag{9}$$

The left-hand side represents the present value of contingent budget surpluses, while the right-hand side is the market value of outstanding debt (both left-hand side and right-hand side are adjusted by $u'(1-\tau+s_0)$). Loosely speaking, any outstanding debt must be financed by future budget surpluses.

Let $\hat{W}_0(\boldsymbol{b}_{-1})$ be welfare of the planner at the initial period if the planner prefers not to default. Then the modified commitment problem is to design an optimal fiscal plan $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ to maximize the planner's welfare subject to the dynamic budget constraint and non-negativity constraints:

$$\hat{W}_0(\boldsymbol{b}_{-1}) = \max_{\{s_0\}} W_0(\boldsymbol{s}_0) \tag{10}$$

subject to $s_t \in (-(1-\tau),\, \tau)$, $\forall t$ and (9)

Importantly, the maturity structure is completely irrelevant for the planning problem. As long as the government can commit to the sequence of budget surpluses, default probabilities and risk-free interest rates remain constant. Therefore, bond prices do not change and there are infinitely many ways to implement the allocation, with multiple maturities available every period. This does not apply to a Markov government, as the inherited maturity structure affects government decisions in future periods.

First-Order Optimality Condition

To simplify notation, let $u_t = u(1-\tau+s_t)$, $\omega_t = \omega(\tau-s_t)$, $u'_t = u'(1-\tau+s_t)$, $\omega'_t = \omega'(\tau-s_t)$ and $u''_t = u''(1-\tau+s_t)$. In addition, let

$$Pr_t^{t+k} = \prod_{i=t+1}^{t+k} F(W_i(\boldsymbol{s}_i)) \tag{11}$$

denote the probability of no default from period $t+1$ to $t+k$. Note that this probability is a function of $\boldsymbol{s}_{t+1}$ but does not depend on s_k, $k \le t$.

Finally, let

$$S_t = \sum_{k=0}^{\infty} \beta^{k-t} Pr_t^{t+k} u'_{t+k} s_{t+k}, \tag{12}$$

$$B_{-1,\,t} = \sum_{k=0}^{\infty} \beta^{k-t} Pr_t^{t+k} u'_{t+k} b_{-1}^{t+k}. \tag{13}$$

Then S_0 corresponds the market value of contingent budget surpluses (the left-hand side of (9). Similarly, $B_{-1,0}$ is the market value of outstanding debt $\boldsymbol{b}_{-1}$ at date 0 if the planner pursues fiscal plan $\boldsymbol{s}_0$ (the right-hand side of (9). In general, S_t shows the value of the stream of contingent budget surpluses (s_t, s_{t+1}, ...) in period t. Analogously, $B_{-1,\,t}$ shows the value of debt which matures in period t or later (b_{-1}^t, b_{-1}^{t+1}, ...). In a stationary economy with $s_t = s_{t+1}$ $\forall t$ S_t is constant. In addition, if initial maturity structure is flat then $B_{-1,\,t}$ also remains unchanged over time.

Optimal fiscal plan $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ satisfies the first-order necessary conditions of the modified commitment problem (10):

$$\frac{\frac{\partial}{\partial s_{t+1}} W_0(\boldsymbol{s}_0)}{\frac{\partial}{\partial s_t} W_0(\boldsymbol{s}_0)} = \frac{\frac{\partial}{\partial s_{t+1}} \left(S_0 - B_{-1,0}\right)}{\frac{\partial}{\partial s_t} \left(S_0 - B_{-1,0}\right)} \tag{14}$$

The left-hand side of (14) is the marginal rate of substitution between contingent budget surpluses at period t and $t+1$. The marginal rate of substitution shows by how much the budget surplus at period t can be decreased if the planner increases the budget surplus at $t+1$ by one unit, keeping planner's welfare (7) constant. Note that the marginal rate of substitution depends only on fiscal plan $\boldsymbol{s}_0$ and does not depend on the initial debt composition $\boldsymbol{b}_{-1}$. The right-hand side of the equation (14) shows the rate at which the planner at period 0 can transfer budget surpluses from $t+1$ to t, keeping the budget constraint satisfied with equality. In equilibrium, any marginal deviation from the optimal fiscal plan which satisfies the budget constraint (9) does not lead to an increase in the planner's welfare.

Lemma 1. The Optimality Condition
The first-order necessary condition (14) can be simplified as follows:

$$\frac{\theta_{t+1}\omega'_{t+1}-u'_{t+1}}{\theta_t\omega'_t-u'_t}=\frac{\frac{\partial\left(u'_{t+1}\cdot(s_{t+1}-b^{t+1}_{-1})\right)}{\partial s_{t+1}}+\frac{\frac{\partial Pr^{t+1}_t}{\partial s_{t+1}}}{Pr^{t+1}_t}\left(S_{t+1}-B_{-1,\,t+1}\right)}{\frac{\partial\left(u'_t\cdot(s_t-b^t_{-1})\right)}{\partial s_t}} \tag{15}$$

To get intuition, consider the following perturbation: suppose the government increases s_t and decreases s_{t+1}[1] keeping $W_0(\boldsymbol{s}_0)$ constant. Note that $W_0(\boldsymbol{s}_0)$ remains constant only if $W_t(\boldsymbol{s}_t)$ does not change. An increase in s_t decreases $W_t(\boldsymbol{s}_t)$ by $(\theta_t\omega'_t-u'_t)\triangle s_t$. A reduction in s_{t+1} has several effects. First, it increases $W_{t+1}(\boldsymbol{s}_{t+1})$ by $\left(\theta_{t+1}\omega'_{t+1}-u'_{t+1}\right)(-\triangle s_{t+1})$. Second, as default is triggered if $V^{def}_{t+1}>W_{t+1}(\boldsymbol{s}_{t+1})$ and $W_{t+1}(\boldsymbol{s}_{t+1})$ goes up, the probability of receiving the outside option V^{def}_{t+1} decreases while the probability of repaying debt and receiving $W_{t+1}(\boldsymbol{s}_{t+1})$ increases. However, for a small $\triangle s_{t+1}$ the second effect disappears because an increase in $\mathbb{E}\max\left\{W_{t+1}(\boldsymbol{s}_{t+1}),\,V^{def}_{t+1}\right\}$ due to increase in probability of $W_{t+1}(\boldsymbol{s}_{t+1})$ is offset by a decrease in expected value of V^{def}_{t+1}[2]. Finally, ratio of $\triangle s_t$ and $-\triangle s_{t+1}$ is proportional to $\theta_t\omega'_t-u'_t$ and $\beta Pr^{t+1}_t\left(\theta_{t+1}\omega'_{t+1}-u'_{t+1}\right)$. The term βPr^{t+1}_t disappears because it cancels out with the right-hand side.

Now consider the right-hand side of (15). The optimality of fiscal plan requires that such perturbation does not allow the planner to relax the budget constraint (9). How the perturbation in s_t and s_{t+1} keeping $W_t(\boldsymbol{s}_t)$ constant affects the budget constraint? There are three channels. First, the present value of budget surpluses changes as a result of changes in s_t and s_{t+1}. The second channel is through risk-free interest rates as u'_t and u'_{t+1} change. Note that this channel alters the present value of budget surpluses in periods t and $t+1$ and bonds maturing in t and $t+1$ only. These two channels reflect the denominator and the first term of numerator of the right-hand side in (15). Third, the perturbation distorts default risks. However, as $W_k(\boldsymbol{s}_k)$ changes only for $k=t+1$ and remains constant for all other periods, the perturbation affects only Pr^{t+1}_t - the default risk in period $t+1$. The last

[1] or vice versa
[2] see Lemma A.1 {{check and link}} in Appendix for formal proof.

channel affects present value of budget surpluses in periods $t+1$ and onwards, as well as all outstanding debt maturing in $t+1$ and later.

3.2 Characterization of the Modified Commitment Problem

It is useful to define $\bar{s} > 0$ as the maximum stationary budget surplus for which the default risk remains zero every period:

$$\frac{u(1-\tau+\bar{s}) + \omega(\tau - \bar{s})}{1-\beta} = V_{max}.$$

Suppose that the initial debt is zero. Note that the optimality condition (15) is satisfied for all $s_1 = s_2 = ... = \bar{s}$ because $\theta_1 = \theta_2 = ... = 1$. Let $\bar{s}_0 < 0$ be the budget surplus such that the dynamic budget constraint (9) is satisfied. As the left-hand side of (15) is strictly decreasing in θ_t, we conclude that there exists a unique $\bar{\theta} > 1$ which satisfies the optimality condition (15) for $t = 0$, $s_0 = \bar{s}_0$ and $s_1 = s_2 = ... = \bar{s}$.

Suppose that a representative household has a constant relative risk aversion utility function:

Assumption 3. $u(c) = \frac{c^{1-\gamma_C}}{1-\gamma_C}$, $\gamma_C \geq 0$.

Then we can qualitatively characterize the optimal fiscal plan.

Proposition 1. The Optimal Fiscal Plan
Suppose there is no initial debt and Assumptions 1-3 are satisfied, then:
(i) for $\theta_0 \in [1, \bar{\theta}]$*,* $\hat{s}_1 = \hat{s}_2 = ... \leq \bar{s}$ *and* $Pr_t^{t+1} = 1 \ \forall t \geq 1$*;*
(ii) for $\theta_0 > \bar{\theta}$*,* $\hat{s}_1 > \hat{s}_2 > ... > \bar{s}$ *and* $Pr_t^{t+1} < 1 \ \forall t \geq 1$*.*

According to Proposition 1, if $\theta_0 \leq \bar{\theta}$ the planner prefers to stay in the "safe" region in which default risk is zero in every period. If $\theta_0 > \bar{\theta}$ then the planner enters the "crisis" region in which default risk is always positive.

In the safe region, the only motive for the planner is to smooth public spending over time. If the planner deviates from fixed budget surpluses by marginally increasing budget surplus in one period by $\beta \triangle s$ and marginally decreasing budget surplus in the next period by $\triangle s$, the planner's welfare strictly decreases because the per-period utility is strictly concave. Moreover, this deviation strictly decreases the present value of budget surpluses because interest rates move against changes in budget surpluses. The opposite deviation - a decrease in budget surplus in the first period and an increase in the second - has exactly the same implication.

In the crisis region, there is also the saving motive to decrease default risk. If a planner marginally increases s_t and marginally decreases s_{t+1}, the effect on $W_t(\boldsymbol{s}_t)$ is negligible as it depends on both s_t and s_{t+1}. However, $W_{t+1}(\boldsymbol{s}_{t+1})$ increases as it depends on s_{t+1} but not s_t. This increase reduces the default risk in period $t+1$ and, thus, relaxes the dynamic budget constraint. However, the opposite perturbation of budget surpluses has the opposite adverse effect on the default risk. The optimal fiscal policy is, thus, to pay larger fraction of debt in earlier periods to benefit from lower long-term risk of default.

4 Optimal Maturity Structure

In this section I show that generally there exists a unique maturity structure which makes the optimal fiscal plan discussed in the previous session to be consistent over time.

Suppose that the planner designs an optimal fiscal plan $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ and no default decision is triggered till period $T > 0$. Then suppose that in period T the planner receives an unexpected option to redesign the existing fiscal plan. How the outstanding debt maturity structure affects the planner's incentive to deviate from the original fiscal plan?

The planner's problem is analogous to (10):

$$\hat{W}_T(\boldsymbol{b}_{T-1}) = \max_{\boldsymbol{s}_T} u(1-\tau+s_T) + \theta_T \omega(\tau - s_T) + \beta \cdot \mathbb{E} \max\left\{W_{T+1}(\boldsymbol{s}_T),\, V_{T+1}^{def}\right\} \tag{16}$$

$$\text{subject to } s_t \in (-(1-\tau),\, \tau),\, \forall t \geq T \text{ and}$$

$$\sum_{t=T}^{\infty} \beta^{t-T} u'_t s_t \prod_{k=T+1}^{t} F(W_k(\boldsymbol{s}_k)) \geq \sum_{t=T}^{\infty} \beta^{t-T} u'_t b_{T-1}^t \prod_{k=T+1}^{t} F(W_k(\boldsymbol{s}_k))$$

Denote by $\hat{\boldsymbol{s}}_T(\boldsymbol{b}_{T-1}) = (\hat{s}_T(\boldsymbol{b}_{T-1}),\, \hat{s}_{T+1}(\boldsymbol{b}_{T-1}),\, ...)$ the optimal fiscal plan designed by a planner in period T as a function of debt portfolio $\boldsymbol{b}_{T-1}$. The optimal fiscal plan satisfies the first-order condition similar to (15):

$$\frac{\theta_{t+1}\omega'_{t+1} - u'_{t+1}}{\theta_t \omega'_t - u'_t} = \frac{\frac{\partial\left(u'_{t+1}\cdot(s_{t+1} - b_{T-1}^{t+1})\right)}{\partial s_{t+1}} + \frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}\left(S_{t+1} - B_{T-1,\,t+1}\right)}{\frac{\partial\left(u'_t\cdot(s_t - b_{T-1}^t)\right)}{\partial s_t}} \tag{17}$$

where $B_{T-1,\,t+1} = \sum_{k=0}^{\infty} \beta^k Pr_t^{t+k} u'_{t+k} b_{T-1}^{t+k},\ t \geq T$.

Note that the optimality conditions (15) and (17) are quite similar. The left-hand sides of the equations - the marginal rates of substitution between contingent budget surpluses in period t and $t+1$ - are identical for both planners in period 0 and period T. The reason is that the preferences of a government does not change over time in this model.

However, the right-hand sides of (15) and (17) - showing a rate at which a planner can reallocate budget surpluses over time keeping budget constraint constant - are different. In both cases the planners choose optimal fiscal plan taking into account how perturbations in budget surpluses s_t and s_{t+1} affect the market value of outstanding debt through changes in risk-free interest rates and default risks. Therefore, ex ante optimal fiscal plan and ex post optimal fiscal plan are not identical for any maturity structure at T.

How can a government in an environment with lack of commitment discipline future governments to follow the ex ante optimal fiscal plan? Lemma 2 establishes conditions under which the ex ante optimal fiscal plan is a solution to the optimality condition (17).

Lemma 2. Optimal Maturity Structure

The ex ante optimal fiscal plan $\hat{\boldsymbol{s}}_0(\boldsymbol{b}_{-1})$ satisfies the optimality condition (17) of a planner in period T if the maturity structure of outstanding debt $\boldsymbol{b}_{T-1}$ satisfies:

$$\frac{\theta_{t+1}\hat{\omega}'_{t+1}-\hat{u}'_{t+1}}{\theta_t\hat{\omega}'_t-\hat{u}'_t}=\frac{\frac{\partial\left(\hat{u}'_{t+1}\cdot(b^{t+1}_{-1}-b^{t+1}_{T-1})\right)}{\partial s_{t+1}}+\frac{\frac{\partial \hat{Pr}^{t+1}_t}{\partial s_{t+1}}}{\hat{Pr}^{t+1}_t}\left(B_{-1,\,t+1}-B_{T-1,\,t+1}\right)}{\frac{\partial\left(\hat{u}'_t\cdot(b^t_{-1}-b^t_{T-1})\right)}{\partial s_t}} \tag{18}$$

where $\hat{\omega}'_{t+1}=\omega(\tau-\hat{s}_{t+1}(\boldsymbol{b}_{-1}))$, $\hat{u}'_{t+1}=u(1-\tau+\hat{s}_{t+1}(\boldsymbol{b}_{-1}))$, $\hat{Pr}^{t+1}_t=F\left(W_{t+1}(\hat{\boldsymbol{s}}_{t+1}(\boldsymbol{b}_{-1}))\right)$.

Intuition is as follows. Consider new debt issued between periods 0 and $T-1$ ($b^t_{T-1}-b^t_{-1}$, $\forall t\geq T$). All the changes in the stocks of debt should be such that he planner at T has no incentive to deviate from the ex ante optimal allocation. The latter is possible only if the market value of this new debt issued is not affected by perturbations in s_t and s_{t+1} keeping welfare constant which is given by equation (18).

Next I turn to the analysis of the shape of the maturity structure. We know from the literature that in an environment with no default the optimal maturity structure is approximately flat (see Lucas and Stokey (1983), Debortoli et al. (2017). If, alternatively, we study a model with default but risk-neutral lenders then the optimal debt policy is to issue only short-term debt (see Aguiar et al. (2016). The abovementioned findings are also the solutions to this model.

The interesting case is thus the profile of the maturity structure if borrowers are risk-averse and default risk is present. Let suppose that the initial debt is zero to abstract away from potential effect of initial debt on the optimal fiscal plan and debt management. Consider $\theta_0 > \bar{\theta}$ so that the economy is in the crisis region with strictly positive probability of default. According to Proposition 2, the maturity structure has a decaying profile.

Proposition 2. Decaying Profile of the Maturity Structure.

Suppose that initial debt is zero and $\theta_0 > \bar{\theta}$. Then the optimal maturity structure has a decaying profile:

$$b_0^1 > b_0^2 > ... > 0.$$

The are two reasons for the decaying profile of maturity of debt. The first reason is the asymmetry of responses of short-term and long-term interest rates to perturbations in fiscal plans. Consider again a decrease in s_t and an increase in s_{t+1} keeping welfare constant. The optimality conditions requires this deviation from the fiscal plan to have no effect on the market value of debt. The perturbation affects the marginal utility of consumption at t and $t+1$, thus, the market value of debt maturing at t and $t+1$ alters due to changes in risk-free interest rates. In addition, the perturbation affects default risk in period $t+1$ as the welfare W_{t+1} decreases due to an increase in s_{t+1}. Note that the default risk in period t remains unchanged - the perturbation does not affect initial welfare W_0 so that a decrease in s_t is offset by an increase in s_{t+1}. As a consequence, the market value of long-term debt maturing in $t+1$, $t+2$ and so on is affected by the increase in default risk in period $t+1$. While the market value of short-term debt maturing in t and before is not affected by changes in default probabilities.

The second reason of the decaying maturity profile is the decreasing stream of budget surpluses given by Proposition 1 as well as the assumption of the decreasing marginal utility of consumption of the borrowers. As the private consumption is higher in an earlier period than in the subsequent period, risk-free interest rate is more sensitive to changes in consumption in later periods.

To sum it up, the long-term interest rates are more elastic and easier manipulated by a government. A perturbation in fiscal policy leads to more substantial changes in the market value of long-term debt than the short-term debt due to more responsive risk-free interest rates as well as more responsive default probabilities. Therefore, the government issues more short-term debt and less long-term debt so that changes in the market value of short-term debt and long-term debt cancel each other.

An important observation is that the maturity structure does not (directly) depends on the levels of risk-free interest rates and default risks. The term structure of debt rather depends on responsiveness of risk-free interest rates and risk premiums to marginal changes in fiscal policy. Indeed, consider an economy with some constant risk of default which does not depend on the level or term structure of debt[3]. Suppose that the planner optimally decides to finance initial budget deficit with equal budget surpluses every period. Then, according to equation (18), term structure of debt would be exactly flat violating the conclusion of Proposition 2. Hence, this is not the presence of default risk which skews the maturity profile toward the short end but rather the continuous dependence of default probabilities on fiscal policies conducted by the government.

5 Quantitative Exercises

In this section I numerically solve for the optimal maturity and discuss how it is affected by the debt-to-GDP ratio, risk aversion, sensitivity of default risk to fiscal policies and initial debt.

5.1 Functional Forms and Parameters

Throughout this section I assume that conditional on no prior default, the government's per-period payoff is

$$\frac{c^{1-\gamma_C}-1}{1-\gamma_C}+\kappa\frac{g^{1-\gamma_G}-1}{1-\gamma_G}$$

where $\gamma_C=\gamma_G=2$, $\tau=0.4$ and κ satisfies $(1-\tau)^{-\gamma_C}=\kappa\tau^{-\gamma_G}$. I set $\beta=0.96$ so that the risk-free interest rate is approximately 4%.

I assume that the distribution of the outside option V_t^{def} can be approximated by the following function

$$f(v)=\begin{cases}\alpha_0+\alpha_1 v+\alpha_2 v^2 & v\in(V_{min},V_{max})\\ 0 & otherwise\end{cases}$$

where α_0, α_1 and α_2 satisfy the following conditions:

$$f(V_{max})=0,$$

[3] a discrete distribution of the outside option value can lead to (locally) constant default probabilities

$$\int_{V_{min}}^{V_{max}} f(v)dv = 1,$$

$$f(v) > 0\,\forall v \in (V_{min}, V_{max}).$$

Importantly, when modeling the distribution of the outside option we are primarily interesting in the right tail of the distribution. Most countries have relatively low default risks below 10%. In addition, at some high level of debt the issuance of new debt might start to decrease the value of issued debt as the revenue from additional bond issuance is outweighed by bond price decline.

The computational algorithm consists of two major steps. First, I solve for the optimal fiscal plan which is defined is Section 3. I start by guessing s_0 and solve for s_1, s_2, ..., s_T where T is a large number given the optimality condition (15) and that the budget constraint holds with strict inequality. My assumption is that in the long-run the difference between present value of budget surpluses and outstanding debt vanishes. If the initial value of s_0 is too small, then the sequence of budget surpluses is insufficient to repay outstanding debt. If the initial value of s_0 is too high, the government starts accumulating huge assets. After solving for the optimal path of budget surpluses, I solve for the optimal maturity structure. The optimal maturity structure satisfies condition (18) and the budget constraint (3).

5.2 Fiscal Policy and Maturity Profile in a Benchmark Case

First suppose that there is no initial shock to the taste parameter for public spending, i.e. $\theta_0 = 1$. Suppose that the total nominal initial debt is 1 so that the debt-to-GDP ratio is 100%. Let assume that initial maturity is exponentially decaying in the following form:

$$b_{-1}^{t+1} = (1-\delta)b_{-1}^{t} \tag{19}$$

where δ is the decay rate. Higher δ implies shorter maturity. If $\delta = 1$ then the initial debt can consist of only one-period bonds. If $\delta = 0$ then the initial maturity structure is flat.

In addition, as not all countries have very long debt with maturity of 50 or 100 years, and even if countries have such long debt, its share in total debt is tiny, let suppose that the maturity of outstanding debt is limited by 30 years and there is no debt with higher maturity. I choose such a number of years because the empirical data on maturity structure of debt available on Bloomberg is limited by 30 years. For the benchmark case I set $\delta = 0.13$ as the average maturity of initial debt is then approximately 7 years - close to the average maturity of sovereign debt of many countries.

Figure 1: The optimal maturity structure in a benchmark case

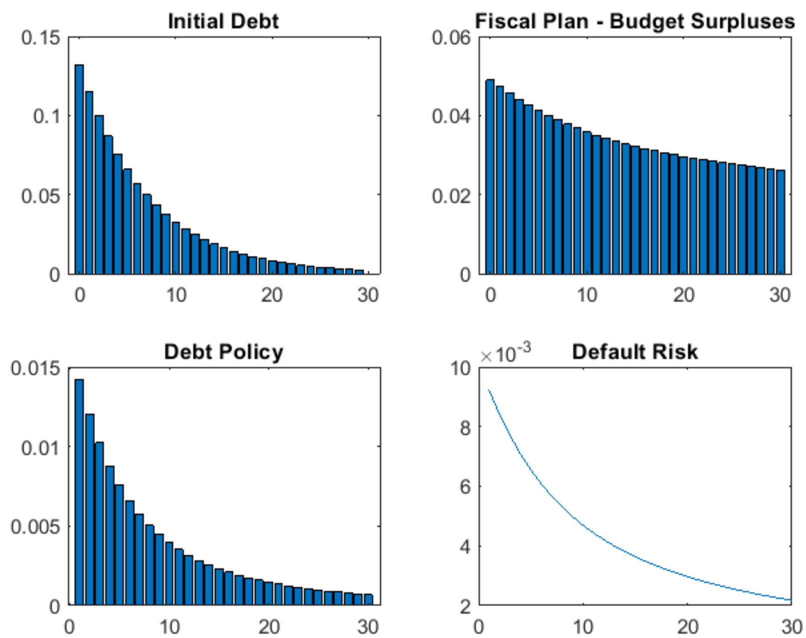


Figure 1 displays the maturity structure of initial debt over time, optimal fiscal plan, optimal debt policy and default risk in every period for the first 30 periods. As discussed in Section 3, the optimal fiscal plan is skewed to the left end as the government has an incentive to repay more debt in earlier periods to decrease long-term default risk. Term structure of net issued debt is strictly decaying. Average maturity of newly issued debt is 8.3 years. Overall, the shape of newly issued and initial debt look very similar. However, as I discuss in Section 5.4, the shape of the initial debt is not the main reason for a government to mimic it and optimal maturity profile remains decaying even if the initial maturity structure is short or flat. Notice that the default risk is the very first period is a bit below than 1% which is a realistic number for many economies. Over time, as the government decreases debt-to-GDP ratio, the default risk declines.

5.3 Shock to the Taste Parameter for Public Spendings

In this exercise, I study how debt and fiscal policy react to a shock in public spending. If θ_0 goes up, the government prefers to increase public spending in this period which in turn leads to higher borrowings and potentially higher default risk.

Figure 2: The effect of shocks to the taste parameter for public spending

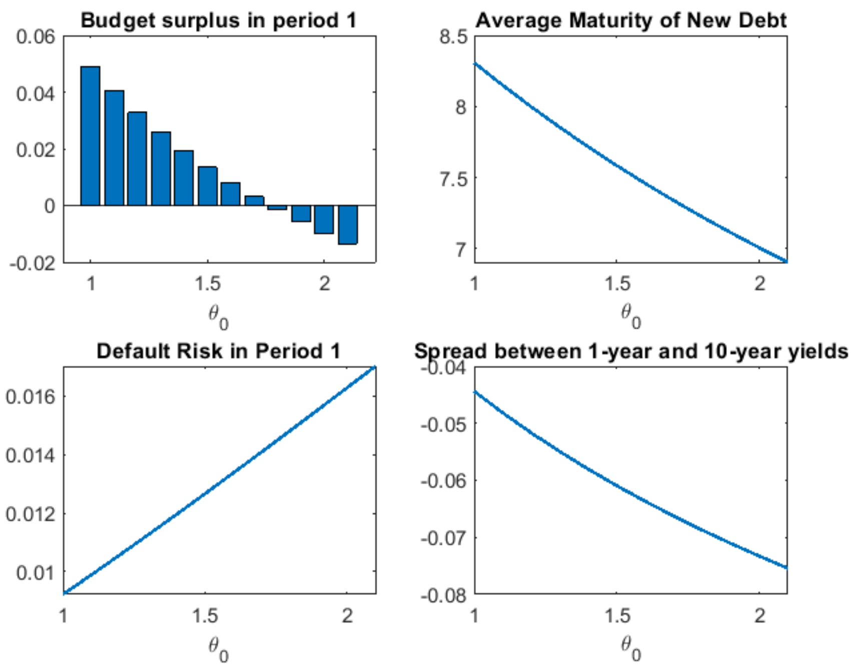
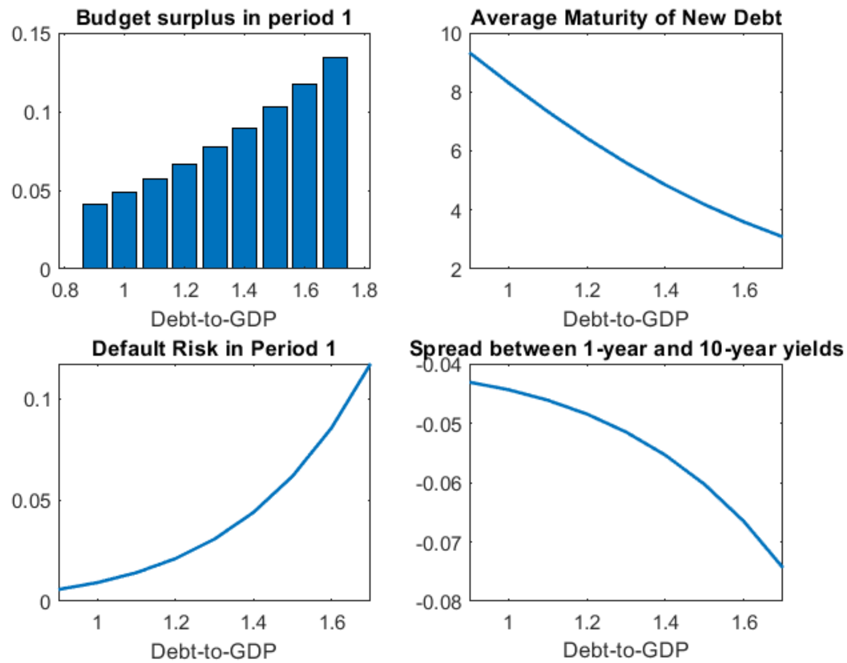


Figure 2 shows budget surplus and average maturity of issued debt as well as default risk in period 1 and spread between one-year and ten-year yields for different values of θ_0. The case $\theta_0 = 1$ corresponds to the exercise considered in the previous subsection. For a higher θ_0 the budget surplus in the first period decreases and eventually becomes negative. The default risk increases from 1% to approximately 1.7%. As the government borrows more and faces higher default risk the average maturity of issued debt declines from 8.3 years to 7 years.

5.4 Debt-to-GDP Ratio

Figure 3 shows budget surplus in period 1 and average maturity of issued debt for different debt-to-GDP ratio keeping maturity of initial debt constant at 7 years.

Figure 3. The effect of initial debt-to-GDP ratio



For a higher initial debt there is a higher incentive to pay debt which is reflected in increasing budget surplus. In addition, countries with higher debt-to-GDP ratio face higher default risk. In this exercise as debt-to-GDP ratio increases from 80% to 170% the default risk rises from less than 1% to more than 10%. Such drastic changes in default probabilities have significant effect on sovereign choice of maturity of debt: the average maturity of debt decreases from 9 years to less than 4 years.

5.5 Initial Maturity Structure

In this subsection I study how term composition of initial debt affects the maturity choice of the government. I start with two very different schemes of initial debt: only short-term debt and flat maturity structure.

Figure 4. Fiscal policies if initial debt is short

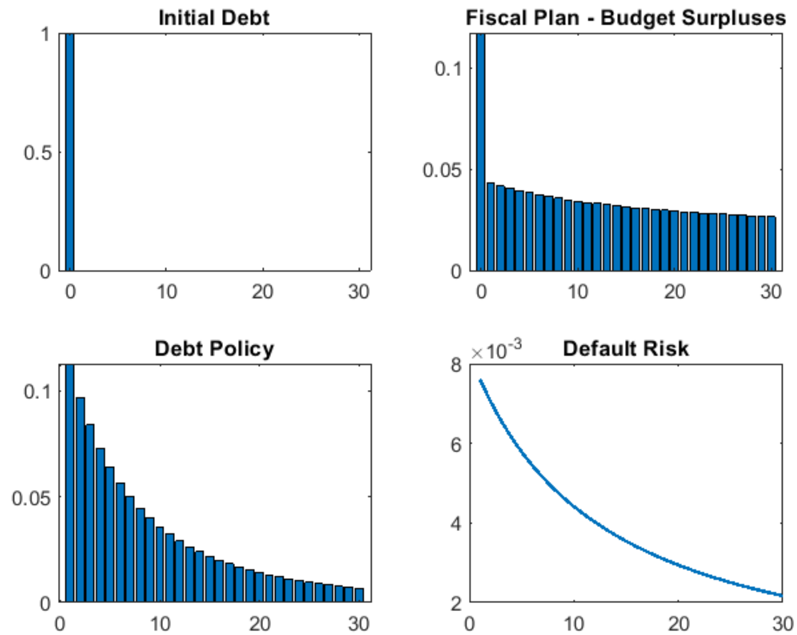


Figure 4 displays optimal fiscal plan, optimal debt policy and default risk over time if the government inherits only one-period bonds. The path of budget surpluses is decaying, however, the surplus in the initial period is much larger comparing to the subsequent ones. The reason is that (as the government roll-overs a huge stock of debt) a marginal increase in private consumption today decreases all risk-free interest rates making borrowing much cheaper. As a higher budget surplus decreases total debt it also leads to lower default risks. Even though the initial debt is only short-term, the maturity profile of net debt issued is strictly decaying.

Figure 5. Fiscal policies if initial debt is flat

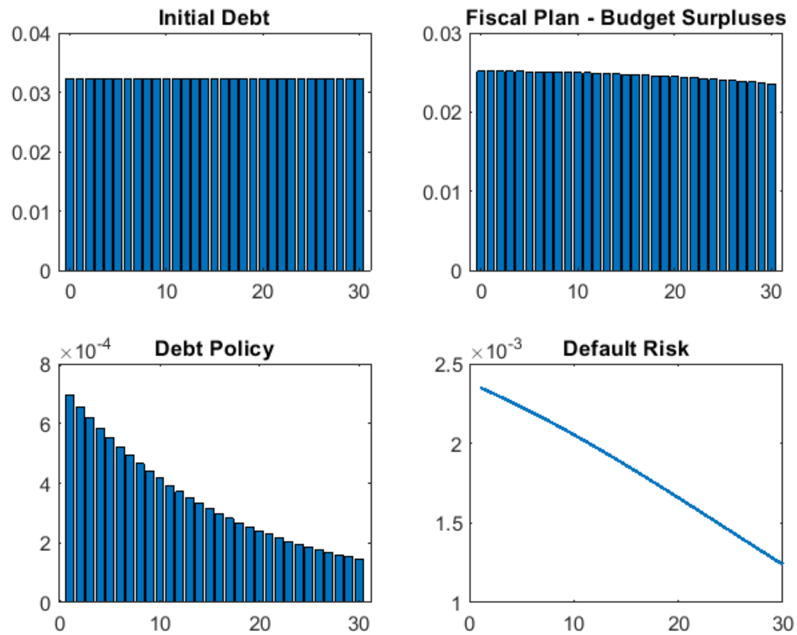


Figure 5 shows the same exercise but for the case with flat maturity structure. More specifically, I suppose that the government has to pay constant amount of payments in the first 30 years and nothing afterwards. We see that budget surpluses are almost perfectly smoothed over time. As short-term debt is much lower compared to the previous example, the total amount of issued debt is much smaller now. Nonetheless, the shape of the maturity profile is still decaying.

Figure 6. Effect of initial maturity

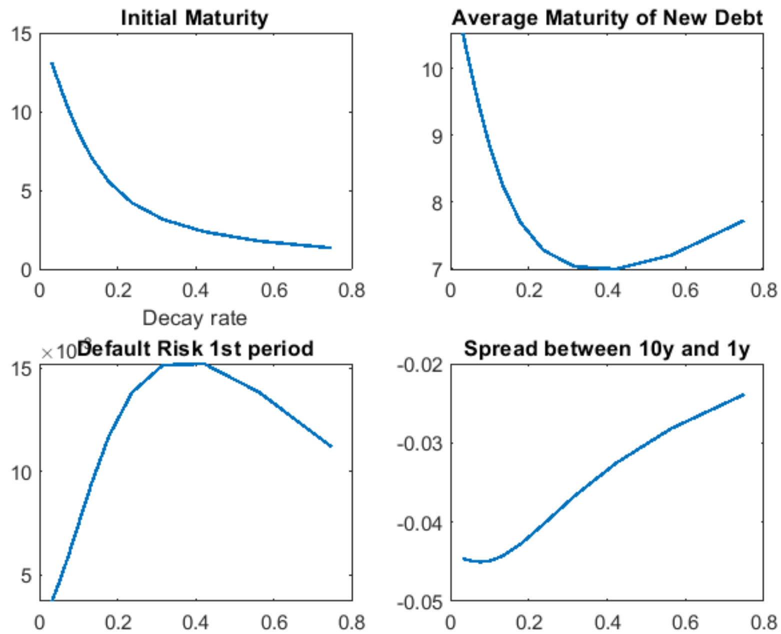


Figure 6 displays how initial maturity affects the maturity of issued debt. For this exercise I consider different decay rates from almost 0 to 0.75 which correspond to the average maturity of initial debt from almost 15 years till almost 2 years.

The effect of initial maturity on the term structure of issued debt is non-monotonic. First, maturity of issued debt declines with lower maturity of initial debt. However, once decay rate exceeds 0.4 the maturity starts to increase.

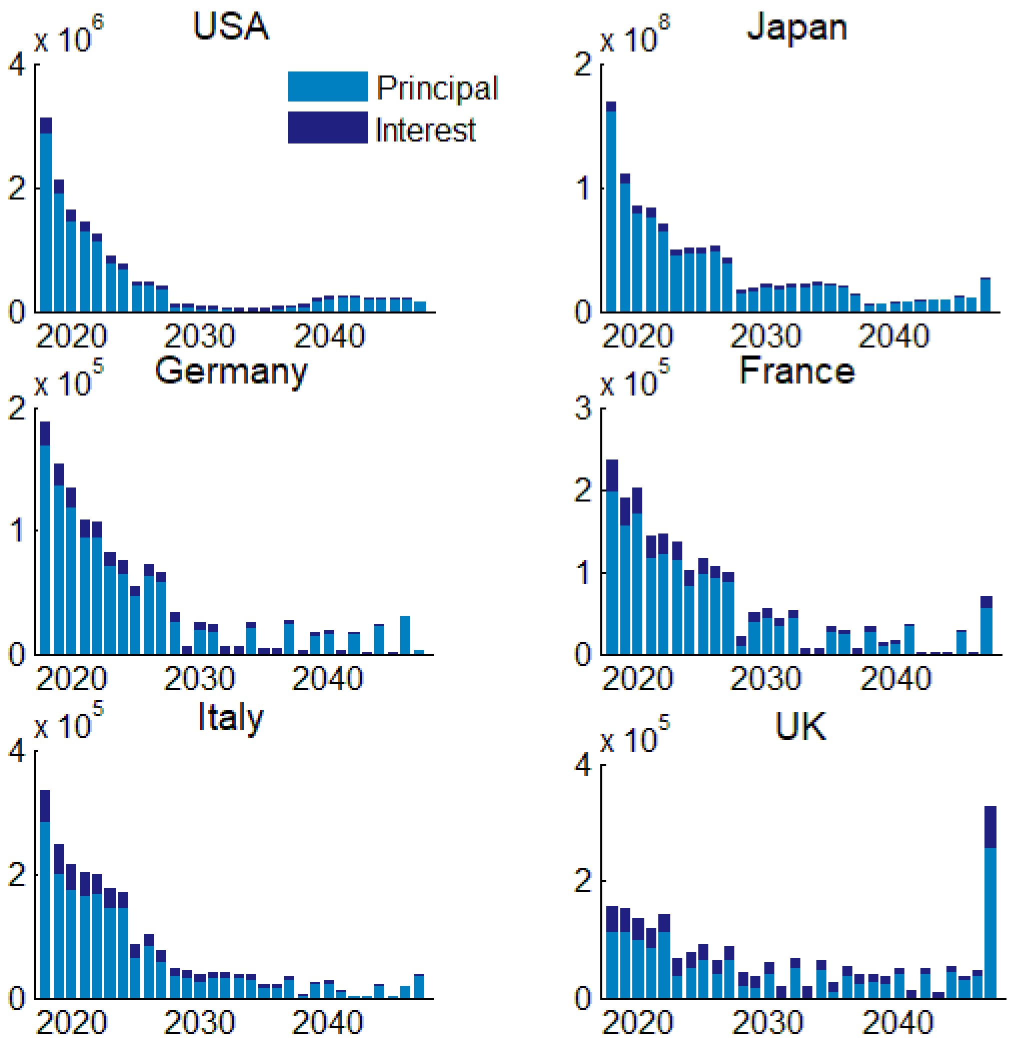
Recall that according to (18) the maturity structure depends on sensitivity of risk-free interest rates and default premiums. For a low decay rate the optimal fiscal policy is smooth which in turn implies low default probability. As the short-term and long-term risk-free interest rate sensitivities are approximately the same and default probability sensitivity is low - the optimal maturity structure is closer to flat corresponding to high average maturity of debt. For lower initial maturity the optimal fiscal plan becomes less smooth and while default risk is higher. This makes sensitivity of long-term risk-free interest rates and default probability to be higher resulting in steeper maturity profile and lower average maturity of issued debt. Finally, for even lower maturity of initial debt the saving motive starts to dominate which leads to a lower default risk. The sensitivity of default risk decreases so that the issuance of long-term debt does not increase long-term interest rates that much.

5.6 The Maturity Structure of Developed Countries

Figure 7 displays the maturity structure of marketable bonds for the following countries: the US, Japan, Germany, France, Italy and the UK. The US debt is in millions USD, the debts of the German, Italian and French governments are in millions EUR, the UK debt is in millions GBP and the Japanese debt is in millions JPY. The data was collected on 17th of October, 2017 and each bar shows the principal and interest payments owed by a government that has to be paid by the government in a given year as of October 17, 2017. I, thus, skip the payments due in the end of October, November and December 2017 and start with 2018. In each panel the first bar represents the total payments owed by a government due in 2018, the second bar represents the total payments due in 2019 and so on.

The exception is the very last bar in each panel that includes payments due in 2047 and all future years. The last bar is somewhat higher for Italy, Japan and France, but most importantly it represents a considerable part of debt for the United Kingdom. The main reason is that the British Government actively issued consol bonds during the Industrial Revolution (see Mokyr, 2011). Due to this aggregation of debt I ignore the last bar in the discussion of maturity data.

Figure 7: Maturity Structure of the USA, Japan, Germany, France, Italy and the UK (Source: Bloomberg)



The maturity statistics is broadly consistent with the predictions of the model. First of all, debt is skewed toward the short-term end. Even though the countries issue bonds maturing in 30 years and later, the average maturity for the US, Japan, Germany, France and Italy is 5.79, 7.74, 6.8, 7.83 and 6.8 years respectively. It is worth noting that the maturity structure of the UK government debt is much flatter, and the average maturity is 14.97 years. For each country the stock of debt maturing in one year is the largest[4]. In 2018

[4]recall that I ignore the last bar which aggregates total payments due in 2047 and later years.

the US government has to pay (or roll-over) more than 20% of the total debt. The debt to be paid by the US government in the next five years constitutes 62% of the total debt. Similarly, total amount of debt maturing in the next 5 years constitutes approximately 50% of total debt for Japan, Germany, France and Italy[5].

Moreover, maturity structure has a decaying profile as predicted by the model. The US maturity structure exhibits decaying profile for 15 years: payments due in 2018-2033 are strictly decreasing. Then the term structure does display some increasing trend, however, note that the total debt due in 2034 and all later years is lower than the debt due in 2018. Debt term structures of Japan, Germany, France and Italy have similar patterns. Even much flatter UK debt has a tendency to decline over maturity date: The total debt maturing in 1-5 years amounts to approximately 32%, while the total debt to be paid in 6-10 years constitutes less than 18%.

One of the reasons maturity structure is not perfectly decaying is that the number of issuance is limited and most of them are short-term. For example, the number of French debt active issuances is only 95[6]. There is no principal payments due at 2033, 2034, 2037 and some subsequent years. The number of issuances can be limited due to some fixed costs or other frictions.

Discussion

The model predictions of optimal maturity structure of government debt seem to be consistent both in terms of the shape of the maturity profile and the average maturity of issued debt. In addition, the model predicts that in periods with higher default risk the maturity of issued debt shortens which is also consistent with empirical observations (see Arellano and Ramanayanan, 2012). The model also can explain much longer maturity of British debt by the fact that long initial maturity leads to small and quite long issuance of new debt which makes the term structure of the total debt to be slowly evolving over time.

6 Summary

This paper shows that in an environment with endogenous risk-free interest rates and endogenous default premiums the optimal maturity structure has a decaying profile. An important assumption is that marginal changes in fiscal policies lead to a marginal change in the risk-free interest rate and the default risk. In this model, fiscal policy is time consistent in a sense

[5]it is 48%, 53%, 46% and 52% for Japan, Germany, France and Italy respectively.

[6]Source: Bloomberg

that the government pursues the ex ante optimal fiscal plan conditional on no default. The model allows to analyze the maturity profile of sovereign debt with infinitely many maturities and numerical exercises show consistency of model predictions with empirical evidence.

The main conclusion of the model is that the skewness of debt profile is given by asymmetry in short-term and long-term interest rates. The long-term interest rates are more responsive to perturbations in fiscal policies mainly due to higher sensitivity of long-term default risk. In addition, the maturity of debt depends on relative sensitivity of short-term and long-term risk-free interest rates and long-term default risk. If long-term risk-free interest rate is relative more sensitive to changes in fiscal policies than short-term interest rates, or default risk is more sensitive than risk-free interest rate, than the maturity of debt is shorter.

There are several interesting avenues for future research. First, this paper assumes that the government cannot default within the period once new debt has been issued. The optimal debt policy under lack of commitment and positive default risk implies issuance of a large stock of short-term debt. This in turn increases the likelihood of self-fulfilling debt crisis if the latter is possible. Thus, allowing for self-fulfilling debt crises could lead to an interesting trade-off between short-term and long-term debt in such an environment. Second, the government is assumed to be able to default on its debt, but partial default is not allowed in this model. Therefore, it would be interesting to investigate the optimal fiscal and debt policies if the government issues nominal debt that can be inflated away rather than real debt.

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Appendix A. Proofs.

Lemma A.1. *For $k \geq t$*

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}} = -\beta^k \cdot Pr_t^{t+k} \cdot \left(\theta_{t+k}\omega'_{t+k} - u'_{t+k}\right).$$

Proof.

Recall that $W_t(\boldsymbol{s}_t)$ is defined recursively as

$$W_t(\boldsymbol{s}_t) = u_t + \theta_t\omega_t + \beta \cdot \mathbb{E}\max\left\{W_{t+1}(\boldsymbol{s}_{t+1}),\, V_{t+1}^{def}\right\}.$$

.

Note that

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_t} = -\left(\theta_t\omega'_t - u'_t\right)$$

because s_t does not affect budget surpluses or default probabilities in future periods. I first show that

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}} = \beta \cdot Pr_t^{t+1} \cdot \frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}}$$

for $k \geq t$ where $Pr_t^{t+1} = F(W_{t+1}(\boldsymbol{s}_{t+1}))$.

Decompose the last component of $W_t(\boldsymbol{s}_t)$ as

$$\mathbb{E}\max\left\{W_{t+1}(\boldsymbol{s}_{t+1}),\, V_{t+1}^{def}\right\} = (1 - Pr_t^{t+1}) \cdot \mathbb{E}[V_{t+1}^{def}|V_{t+1}^{def} > W_{t+1}(\boldsymbol{s}_{t+1})] + Pr_t^{t+1} \cdot W_{t+1}(\boldsymbol{s}_{t+1})$$

where

$$\mathbb{E}[V_{t+1}^{def}|V_{t+1}^{def} > W_{t+1}(\boldsymbol{s}_{t+1})] = \frac{1}{1 - F(W_{t+1}(\boldsymbol{s}_{t+1}))} \cdot \int\limits_{W_{t+1}(\boldsymbol{s}_{t+1})}^{V^{max}} v dF(v)$$

is the conditional expected value of outside value option if the latter is greater than $W_{t+1}(\boldsymbol{s}_{t+1})$. Then

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}} = \beta \cdot \frac{\partial (1 - Pr_t^{t+1}) \cdot \mathbb{E}[V_{t+1}^{def}|V_{t+1}^{def} > W_{t+1}(\boldsymbol{s}_{t+1})]}{\partial s_{t+k}} + \beta \cdot \frac{\partial Pr_t^{t+1} \cdot W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}},$$

$$\frac{\partial(1-Pr_t^{t+1})\cdot\mathbb{E}[V_{t+1}^{def}|V_{t+1}^{def}>W_{t+1}(\boldsymbol{s}_{t+1})]}{\partial s_{t+1}}=\frac{\partial}{\partial s_{t+1}}\int\limits_{W_{t+1}(\boldsymbol{s}_{t+1})}^{V^{max}} vdF(v)=$$

$$=-W_{t+1}(\boldsymbol{s}_{t+1})\cdot f(W_{t+1}(\boldsymbol{s}_{t+1}))\cdot\frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}},$$

$$\frac{\partial Pr_t^{t+1}(\boldsymbol{s}_{t+1})\cdot W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}}=Pr_t^{t+1}\cdot\frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}}+$$

$$+W_{t+1}(\boldsymbol{s}_{t+1})\cdot f\left(W_{t+1}(\boldsymbol{s}_{t+1})\right)\cdot\frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}}.$$

Thus yielding

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}}=\beta\cdot Pr_t^{t+1}\cdot\frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+k}}.$$

Iterating forward leads to

$$\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_{t+k}}=\beta^k\cdot\Pi_{j=1}^k Pr_{t+j-1}^{t+j}\cdot\frac{\partial W_{t+k}(\boldsymbol{s}_{t+k})}{\partial s_{t+k}}=$$

$$=-\beta^k\cdot Pr_t^{t+k}\cdot\left(\theta_{t+k}\omega'_{t+k}-u'_{t+k}\right).$$

■

Lemma A.2. *Suppose $\hat{\boldsymbol{s}}_0$ is an optimal fiscal plan and $\hat{s}_t$ is an element of $\hat{\boldsymbol{s}}_0$, $t\geq 1$. Then if $\hat{s}_t\geq 0$*

$$\frac{\partial u'_t s_t}{\partial s_t}|_{s_t=\hat{s}_t}\geq 0$$

Proof

The proof is by contradiction. Consider an effect of a marginal decrease in $\hat{s}_t$ on the market value of budget surpluses:

$$\frac{\partial S_0}{\partial s_t}|_{s_0=\hat{s}_0}=\frac{\partial\sum_{k=0}^{\infty}\beta^k u'_k\cdot Pr_0^k\cdot s_k}{\partial s_t}|_{s_0=\hat{s}_0}=$$

$$=\sum_{k=1}^{t}\beta^k\hat{u}'\hat{s}_k\cdot\frac{\partial Pr_0^k(\hat{\boldsymbol{s}}_0)}{\partial\hat{s}_t}|_{s_0=\hat{s}_0}+\beta^t\hat{Pr}_0^{\,t}\frac{\partial u'_t s_t}{\partial s_t}|_{s_t=\hat{s}_t}\tag{20}$$

The first term of (20) represents a decrease in the present value of budget surpluses before period t due to changes in default probabilities. The second term is the change in market value of budget surplus at period t due to change in risk-free interest rate and budget surplus itself.

The first term is negative. As $\hat{s}_t$ goes up, $W_k(\hat{\boldsymbol{s}}_k)$ for $k = 1, ..., t$ (weakly) decreases which in turn increases risk of default in those periods:

$$\frac{\partial Pr_0^k(\hat{\boldsymbol{s}}_0)}{\partial \hat{s}_t}|_{s_0=\hat{s}_0} = \sum_{k=1}^{t} \frac{f(W_k(\hat{\boldsymbol{s}}_k))}{F(W_k(\hat{\boldsymbol{s}}_k))} \cdot Pr_0^k(\hat{\boldsymbol{s}}_0) \cdot \frac{\partial W_k(\boldsymbol{s}_k)}{\partial s_t}|_{s_0=\hat{s}_0} \leq 0$$

Suppose for a contradiction that $\frac{\partial u_t' s_t}{\partial s_t}|_{s_t=\hat{s}_t}$ is also decreasing so that the second term of (20) is negative. Then a marginal decrease in $\hat{s}_t$ would cause an increase in the present value of budget surpluses. Therefore, a marginal decrease in $\hat{s}_t$ keeping all other surpluses constant is feasible. However, this deviation also strictly increases the government's welfare $W_0(\hat{\boldsymbol{s}}_0)$ contradicting the optimality of $\hat{\boldsymbol{s}}_0$.

■

Lemma A.3. *If $u'(1-\tau+s) \cdot s$ is increasing in s for $s \in (-(1-\tau), \tau)$ then $\frac{\partial u'(1-\tau+s) \cdot s}{\partial s}$ is strictly decreasing in s.*

Proof

Recall from Assumption 3 that $u(c) = \frac{c^{1-\gamma_C}}{1-\gamma_C}$, $\gamma_C > 0$. Then

$$\frac{\partial u'(1-\tau+s) \cdot s}{\partial s} = (1-\tau+s)^{-\gamma_C - 1} \cdot (1-\tau+s-\gamma_C \cdot s)$$

$\frac{\partial u'(1-\tau+s) \cdot s}{\partial s} \geq 0$ implies $1-\tau+s-\gamma_C \cdot s \geq 0$, therefore,

$$\frac{\partial^2 u'(1-\tau+s) \cdot s}{\partial s^2} = 2 \cdot u''(1-\tau+s) + (-1-\gamma_C) u''(1-\tau+s) \cdot \frac{s}{1-\tau+s} =$$

$$= u''(1-\tau+s) \cdot (1-\tau+(1-\tau+s-\gamma_C \cdot s)) < 0.$$

■

Proof of Lemma 1.

Following Lemma A.1 the left-hand side of (14) can be rewritten as

$$\frac{\frac{\partial}{\partial s_{t+1}}W_0(\boldsymbol{s}_0)}{\frac{\partial}{\partial s_t}W_0(\boldsymbol{s}_0)} = \frac{\beta^{t+1}\cdot Pr_0^{t+1}}{\beta^t \cdot Pr_0^t}\cdot\frac{\frac{\partial}{\partial s_{t+1}}W_{t+1}(\boldsymbol{s}_{t+1})}{\frac{\partial}{\partial s_t}W_t(\boldsymbol{s}_t)} = \frac{\beta\prod_{i=1}^{t}F(W_i(\boldsymbol{s}_i))}{\prod_{i=1}^{t+1}F(W_i(\boldsymbol{s}_i))}\cdot\frac{\frac{\partial}{\partial s_{t+1}}W_{t+1}(\boldsymbol{s}_{t+1})}{\frac{\partial}{\partial s_t}W_t(\boldsymbol{s}_t)}$$

$$= \beta Pr_t^{t+1}\frac{\theta_{t+1}\omega'_{t+1}-u'_{t+1}}{\theta_t\omega'_t-u'_t} \tag{21}$$

Next

$$\frac{\partial}{\partial s_t}\left(S_0 - B_{-1,0}\right) = \beta^t Pr_0^t\cdot\frac{\partial\left(u'_t\cdot(s_t-b_{-1}^t)\right)}{\partial s_t}+$$

$$+\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\beta^j Pr_0^j\frac{\frac{\partial Pr_i^{i+1}}{\partial s_t}}{Pr_i^{i+1}}u'_j\cdot(s_j-b_{-1}^j) \tag{22}$$

The second term of (22) shows how value of contingent budget surpluses and outstnding debt change due to a change in default risk. Note that a marginal change in contingent budget surplus at period t affects default risk in periods 1, 2, ...,t but not in $t+1$, $t+2$ and so on.

Note that for $i\leq t$

$$\frac{\partial Pr_i^{i+1}}{\partial s_t} = f(W_i(\boldsymbol{s}_i))\cdot\frac{\partial W_i(\boldsymbol{s}_i)}{\partial s_t} = \beta^{t-i}Pr_i^t f(W_i(\boldsymbol{s}_i))\cdot\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_t} \tag{23}$$

and for $i>t$ the above derivative is just 0.

Plugging in (22) to (23) leads to

$$\frac{\partial}{\partial s_t}S_0 = \beta^t Pr_0^t\cdot\frac{\partial\left(u'_t\cdot(s_t-b_{-1}^t)\right)}{\partial s_t}+\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_t}\sum_{j=0}^{t}\sum_{i=0}^{j}\beta^{t-i}Pr_i^t f(W_i(\boldsymbol{s}_i))\beta^j Pr_0^j u'_j\cdot(s_j-b_{-1}^j)+$$

$$+\frac{\partial W_t(\boldsymbol{s}_t)}{\partial s_t}\sum_{j=t+1}^{\infty}\sum_{i=0}^{t}\beta^{t-i}Pr_i^t f(W_i(\boldsymbol{s}_i))\beta^j Pr_0^j u'_j\cdot(s_j-b_{-1}^j) \tag{24}$$

Updating (24) one period ahead yields

$$\frac{\partial}{\partial s_{t+1}}S_0 = \beta^{t+1}Pr_0^{t+1}\cdot\frac{\partial\left(u'_{t+1}\cdot(s_{t+1}-b_{-1}^{t+1})\right)}{\partial s_{t+1}}+$$

$$+\beta Pr_t^{t+1}\frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+1}}\sum_{j=0}^{t}\sum_{i=0}^{j}\beta^{t-i}Pr_i^t f(W_i(\boldsymbol{s}_i))\beta^j Pr_0^j u'_j\cdot(s_j-b_{-1}^j)+$$

$$+\beta Pr_t^{t+1}\frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+1}}\sum_{j=t+1}^{\infty}\sum_{i=0}^{t}\beta^{t-i}Pr_i^t f(W_i(\boldsymbol{s}_i))\beta^j Pr_0^j u_j'\cdot(s_j-b_{-1}^j)+$$

$$+\frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+1}}\sum_{j=t+1}^{\infty}f(W_{t+1}(\boldsymbol{s}_{t+1}))\beta^j Pr_0^j u_j'\cdot(s_j-b_{-1}^j) \tag{25}$$

Notice that the ratio of the sum of the second term and the third term of (25) to the corrsponding sum of (24) is $\beta Pr_t^{t+1}\cdot\frac{\frac{\partial}{\partial s_{t+1}}W_{t+1}(\boldsymbol{s}_{t+1})}{\frac{\partial}{\partial s_t}W_t(\boldsymbol{s}_t)}$ which is exactly the left-hand side of (14). Therefore, the right-hand side of (14) can be simplified as follows

$$\frac{\frac{\partial}{\partial s_{t+1}}S_0(\boldsymbol{s}_0)}{\frac{\partial}{\partial s_t}S_0(\boldsymbol{s}_0)}=\frac{\beta^{t+1}Pr_0^{t+1}\cdot\frac{\partial\left(u_{t+1}'\cdot(s_{t+1}-b_{-1}^{t+1})\right)}{\partial s_{t+1}}+\frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+1}}\sum_{j=t+1}^{\infty}f(W_{t+1}(\boldsymbol{s}_{t+1}))\beta^j Pr_0^j u_j'\cdot(s_j-b_{-1}^j)}{\beta^t Pr_0^t\cdot\frac{\partial\left(u_t'\cdot(s_t-b_{-1}^t)\right)}{\partial s_t}}=$$

$$=\beta Pr_t^{t+1}\frac{\frac{\partial\left(u_{t+1}'\cdot(s_{t+1}-b_{-1}^{t+1})\right)}{\partial s_{t+1}}+\frac{1}{Pr_t^{t+1}}f(W_{t+1}(\boldsymbol{s}_{t+1}))\frac{\partial W_{t+1}(\boldsymbol{s}_{t+1})}{\partial s_{t+1}}\sum_{j=t+1}^{\infty}\beta^{j-t-1}Pr_{t+1}^j u_j'\cdot(s_j-b_{-1}^j)}{\frac{\partial\left(u_t'\cdot(s_t-b_{-1}^t)\right)}{\partial s_t}}=$$

$$=\beta Pr_t^{t+1}\frac{\frac{\partial\left(u_{t+1}'\cdot(s_{t+1}-b_{-1}^{t+1})\right)}{\partial s_{t+1}}+\frac{\frac{\partial Pr_t^{t+1}}{\partial s_{t+1}}}{Pr_t^{t+1}}\left(S_{t+1}-B_{-1,t+1}\right)}{\frac{\partial\left(u_t'\cdot(s_t-b_{-1}^t)\right)}{\partial s_t}} \tag{26}$$

Combining (21) and (26) leads to (15). ■

Lemma A.4.

Suppose there is no initial debt and Assumptions 1-3 are satisfied. Then the optimal fiscal plan $\hat{\boldsymbol{s}}_0=(\hat{s}_0,\,\hat{s}_1,\,...)$ *has the following properties:*

(i) $\hat{s}_t\geq 0\ \forall t\geq 1$*;*

(ii) $\hat{s}_t\geq\hat{s}_{t+1}$ *with strict equality only if* $Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})=1$*;*

(iii) $Pr_{t+1}^{t+2}(\hat{\boldsymbol{s}}_{t+2})<1$ *if* $Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1})<1\ \forall t\geq 0$.

Proof

(i) The proof is by contradiction. Suppose $\hat{s}_t<0$ for some time periods. Note that by Lemma A.1 and Assumption 1 (i) $\frac{\partial W_0(s_0)}{\partial s_t}<0$ if $s_t<0\ \forall t\geq 1$. Therefore, setting all negative budget surpluses to zero (except for $t=0$) strictly increases planner's value $W_0(\boldsymbol{s}_0)$.

Further, note that the budget constraint (9) is now relaxed. Market value of all negative budget surpluses $\forall t\geq 1$ turn to be zero. Market value of all positive budget surpluses is (weakly) increased because probability of repaying debt is weakly increased.

(ii) Optimal fiscal plan satisfies the fisrt-order condition (15). Given that the initial debt is zero and $\theta_t = 1$ for $t \geq 1$, the condition is

$$\frac{\omega'_{t+1} - u'_{t+1}}{\omega'_t - u'_t} = \frac{\frac{\partial}{\partial s_{t+1}} \left(u'_{t+1} \cdot s_{t+1} \right) + \frac{\frac{\partial}{\partial s_{t+1}} Pr_t^{t+1}}{Pr_t^{t+1}} \cdot S_{t+1}}{\frac{\partial}{\partial s_t} \left(u'_t \cdot s_t \right)} \tag{27}$$

Note that $\omega'_t - u'_t = \omega'(\tau - s_t) - u'(1 - \tau + s_t)$ is decreasing in s_t because

$$\frac{\partial}{\partial s} \left(\omega'(\tau - s) - u'(1 - \tau + s) \right) = -\omega''(\tau - s) - u''(1 - \tau + s) < 0$$

Therefore, the left-hand side of (27) is greater than one whenever $s_t > s_{t+1}$ and vice versa.

Suppose that $Pr_t^{t+1} = 1$ and, hence, $\frac{\partial}{\partial s_{t+1}} Pr_t^{t+1} = 0$ by Assumption 2 {{check and link}}. Given that $\frac{\partial}{\partial s} \left(u'(1 - \tau + s) \cdot s \right)$ is strictly increasing in s at $s = \hat{s}_t$, the right-hand side is greater than one if $s_t < s_{t+1}$ and vice versa. Thus, optimality condition (27) holds with equality only if $s_t = s_{t+1}$.

Now suppose that $Pr_t^{t+1} < 1$ and, thus, $\frac{\partial}{\partial s_{t+1}} Pr_t^{t+1} < 0$. Note that $S_{t+1} > 0$ because $s_t \geq 0 \ \forall t \geq 1$ as we showed in (i), and if $s_{t+1} = s_{t+2} = ... = 0$ then $Pr_t^{t+1} = 1$. If $s_t \leq s_{t+1}$ then the right-hand side of (27) is strictly less than one while the left-hand side equals one. Therefore, the optimality condition can hold only if $s_t > s_{t+1}$.

(iii) Suppose by contradiction that $Pr_t^{t+1}(\hat{\boldsymbol{s}}_{t+1}) < 1$ but $Pr_{t+1}^{t+2}(\hat{\boldsymbol{s}}_{t+2}) = 1$ for any $t \geq 0$. The latter equality implies that $\hat{s}_{t+1} = \hat{s}_{t+2}$ as showed in (iii).

As default risk in period 1 is positive and there is no default risk in period 2, the government's welfare in $t+1$ is lower than V_{max} and the government's welfare in $t+2$ is larger than V_{max}:

$$W_{t+2}(\hat{\boldsymbol{s}}_{t+2}) = u(1 - \tau + \hat{s}_{t+2}) + \beta \mathbb{E} \max \left\{ W_{t+3}(\hat{\boldsymbol{s}}_{t+3}),\, V^{def} \right\} > V_{max}$$

$$W_{t+1}(\hat{\boldsymbol{s}}_{t+1}) = u(1 - \tau + \hat{s}_{t+1}) + \beta \mathbb{E} \max \left\{ W_{t+2}(\hat{\boldsymbol{s}}_{t+2}),\, V^{def} \right\} =$$
$$= u(1 - \tau + \hat{s}_{t+1}) + \beta W_{t+2}(\hat{\boldsymbol{s}}_{t+2}) < V_{max}$$

However, it contradicts $\hat{s}_{t+1} = \hat{s}_{t+2}$ because

$$\mathbb{E} \max \left\{ W_{t+3}(\hat{\boldsymbol{s}}_{t+3}),\, V^{def} \right\} \leq V_{max}$$

$$\Rightarrow u(1-\tau+\hat{s}_{t+2}) \geq (1-\beta)V_{max}$$

$$\Rightarrow u(1-\tau+\hat{s}_{t+1}) \geq (1-\beta)V_{max}$$

$$\Rightarrow W_{t+1}(\hat{\boldsymbol{s}}_{t+1}) = u(1-\tau+\hat{s}_{t+1}) + \beta W_{t+2}(\hat{\boldsymbol{s}}_{t+2}) > V_{max}$$

■

Proof of Proposition 1.

If $\theta_0 < \bar{\theta}$, the planner finds it optimal to increase contingent budget surplus s_0. It in turn relaxes the budget constraint allowing the planner to decrease s_t $\forall t \geq 1$. As the default risk does not increase and remains zero, Lemma A.4 (iii) implies that $s_1 = s_2 = ...$

If $\theta_0 > \bar{\theta}$, the planner wants to decrease s_0. It requires the planner to increase at least one budget surplus in the future. Thus, the planner's welfare at $t = 1$ $W_1(\boldsymbol{s}_1)$ strictly decreases implying that $Pr_1^2 < 1$.

Suppose by contradiction that $W_1(\boldsymbol{s}_1)$ does not decrease, i.e., we can find a fiscal plan $\boldsymbol{s}'_1 = (s'_1, s'_2, ...)$ such that $W_1(\boldsymbol{s}'_1) \leq V_{max}$ while the present value of fiscal plan $\boldsymbol{s}'_1$ (denoted by S'_1) weakly geater than $-\frac{1}{\beta}u'(1-\tau+s'_0) \cdot s'_0$ where $s'_0 < \bar{s}_0 < 0$. $\theta_0 > \bar{\theta}$ implies that $s'_1 < \bar{s}$. Lemma A.4 (iii) states that for any optimal fiscal plan, $s_t \geq s_{t+1}$ $\forall t \geq 1$. Hence, $s'_t < \bar{s}$ $\forall t$ which contradicts the budget constraint

$$\beta S'_1 < \beta \bar{S} = -u'(1-\tau+\bar{s}_0) \cdot \bar{s}_0 < -u'(1-\tau+s'_0) \cdot s'_0.$$

Lemma A.4 (iv) implies that $Pr_1^2 < 1$ leads to $Pr_t^{t+1} < 1$ $\forall t > 1$. Using Lemma A.4 (iii) we thus prove that $s_1 > s_2 > ...$

In addition, $s_t > \bar{s}$ $\forall t \geq 1$ because if $s_k \leq \bar{s}$ for some period k then $s_t \leq \bar{s}$ $\forall t \geq k$ by Lemma A.4 (iii). It in turn implies that $W_t(\boldsymbol{s}_t) \geq V_{max}$ and $Pr_t^{t+1} = 1$ for $t > k$ contradicting Lemma A.4 (iv).

■

Proof of Lemma 2.

Rewrite (17) as follows:

$$\frac{\theta_{t+1}\omega'_{t+1} - u'_{t+1}}{\theta_t \omega'_t - u'_t} =$$

$$\frac{\frac{\partial\left(u'_{t+1}\cdot(s_{t+1}-b^{t+1}_{-1})\right)}{\partial s_{t+1}}+\frac{\frac{\partial Pr^{t+1}_t}{\partial s_{t+1}}}{Pr^{t+1}_t}\left(S_{t+1}-B_{-1,t+1}\right)+\frac{\partial\left(u'_{t+1}\cdot(b^{t+1}_{-1}-b^{t+1}_{T-1})\right)}{\partial s_{t+1}}+\frac{\frac{\partial Pr^{t+1}_t}{\partial s_{t+1}}}{Pr^{t+1}_t}\left(B_{-1,t+1}-B_{T-1,t+1}\right)}{\frac{\partial\left(u'_t\cdot(s_t-b^t_{-1})\right)}{\partial s_t}+\frac{\partial\left(u'_t\cdot(b^t_{-1}-b^t_{T-1})\right)}{\partial s_t}}$$

Note that

$$\frac{\theta_{t+1}\hat{\omega}'_{t+1}-\hat{u}'_{t+1}}{\theta_t\hat{\omega}'_t-\hat{u}'_t}=\frac{\frac{\partial\left(\hat{u}'_{t+1}\cdot(\hat{s}_{t+1}(\boldsymbol{b}_{-1})-b^{t+1}_{-1})\right)}{\partial s_{t+1}}+\frac{\frac{\partial \hat{Pr}^{t+1}_t}{\partial s_{t+1}}}{\hat{Pr}^{t+1}_t}\left(\hat{S}_{t+1}(\boldsymbol{b}_{-1})-B_{-1,t+1}\right)}{\frac{\partial\left(\hat{u}'_t\cdot(\hat{s}_t(\boldsymbol{b}_{-1})-b^t_{-1})\right)}{\partial s_t}}$$

where $\hat{S}_{t+1}(\boldsymbol{b}_{-1})=\sum_{k=0}^{\infty}\beta^k\hat{Pr}_t^{t+k}\hat{u}'_{t+k}\hat{s}_{t+k}(\boldsymbol{b}_{-1})$.

Combining the two eqautions above yileds (18).

■

Proof of Proposition 2.

Given that initial debt is zero and $\theta_t=1\ \forall t\geq 1$ we can rewrite (18) as follows:

$$\frac{\hat{\omega}'_{t+1}-\hat{u}'_{t+1}}{\omega'_t-u'_t}=\frac{\hat{u}''_{t+1}\cdot b^{t+1}_{T-1}+\frac{\frac{\partial \hat{Pr}^{t+1}_t}{\partial s_{t+1}}}{\hat{Pr}^{t+1}_t}B_{T-1,t+1}}{\hat{u}''_t\cdot b^t_{T-1}}. \tag{28}$$

Recall from Proposition 1 that $\theta_0>\bar{\theta}$ implies that $s_t>s_{t+1}>0\ \forall t\geq 1$. Given that $\omega'(\tau-s)-u'(1-\tau+s)$ is increasing in s and strictly positive if $s>0$, we conclude that

$$0<\frac{\hat{\omega}'_{t+1}-\hat{u}'_{t+1}}{\omega'_t-u'_t}<1. \tag{29}$$

First, I show that $B_{T-1,t+1}>0\ \forall T\geq 1,\ t\geq T$.

Consider a planner's problem at period T as defined by (16). The dynamic budget constraint implies that $S_T=B_{T-1,T}>0$. Note that

$$B_{T-1,t}=\hat{u}'_T b^t_{T-1}+\beta\hat{Pr}^t_{T-1}B_{T-1,t+1} \tag{30}$$

Let assume by contradiction that $B_{T-1,t+1}$ is negative for some $t>T$ and let assume t is the first time the market value of debt is negative. It is possible only if $\hat{u}'_T b^t_{T-1}>-\beta\hat{Pr}^t_{T-1}B_{T-1,t+1}$ implying that b^t_{T-1} is positive.

Then the optimality condition (28) holds only if $b^{t+1}_{T-1}>0$. It in turn results in $B_{T-1,t+2}<0$ from iterating forward equation (30). Following this logic leads to $b^j_{T-1}>0\ \forall j\geq t+1$, which contradicts $B_{T-1,\,t+1} < 0$.

To show $b^t_{T-1} > b^{t+1}_{T-1}$, note that $\hat{u}''_{t+1} < \hat{u}''_t < 0$ and $\frac{\partial \hat{Pr}_t^{t+1}}{\partial s_{t+1}} < 0$. Inequality (29) together with $B_{T-1,\,t+1} > 0$ $\forall T \geq 1$, $t \geq T$ yields

$$\hat{u}''_t \cdot b^t_{T-1} < \hat{u}''_{t+1} \cdot b^{t+1}_{T-1} + \frac{\frac{\partial \hat{Pr}_t^{t+1}}{\partial s_{t+1}}}{\hat{Pr}_t^{t+1}} B_{T-1,\,t+1}$$

$$\Rightarrow \hat{u}''_t \cdot b^t_{T-1} < \hat{u}''_{t+1} \cdot b^{t+1}_{T-1} \;\Rightarrow\; b^t_{T-1} > b^{t+1}_{T-1}$$

Finally, let show that $b^t_{T-1} > 0$ $\forall t \geq T$. Let $b^t_{T-1} < 0$ for some t. Then the optimality condition (28) and $B_{T-1,\,t+1} > 0$ imply $b^{t+1}_{T-1} < 0$ as well. Iterating forward yields $b^j_{T-1} < 0$ $\forall j \geq t$ contradicting $B_{T-1,\,t+1} > 0$.

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