WP 31_09

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TRADING FREQUENCY AND VOLATILITY CLUSTERING

Trading Frequency and Volatility Clustering

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July 2008

Abstract

Volatility clustering, with autocorrelations of the hyperbolic decay rate, is unquestionably one of the most important stylized facts of financial time series. This paper presents a market microstructure model, that is able to generate volatility clustering with hyperbolic autocorrelations through traders with multiple trading frequencies using Bayesian information updating in an incomplete market. The model illustrates that signal extraction, which is induced by multiple trading frequency, can increase the persistence of the volatility of returns. Furthermore, we show that the local temporal memory of the underlying time series of returns and their volatility varies greatly varies with the number of traders in the market.

Key Words: Trading frequency, Volatility clustering, Signal extraction, Hyperbolic decay

JEL No: G10, G11, D43, D82

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Over the last five decades, a broader picture of the time series features of asset prices has emerged. Among these features, the predictability of returns at high trading frequencies and persistence of volatility received significant attention both theoretically and empirically. This latter feature is also known as volatility clustering and is unquestionably one of the most important stylized facts of financial time series. Engle (1982, 2000) and Bollerslev (1986) have proposed (G)ARCH-family models, which has been shown to be capable of capturing conditional volatility parsimoniously. In addition, as documented by Mandelbrot (1963), the autocorrelations of volatility decay at a hyperbolic rate rather than exponentially. A few, however, have investigated the reasons and mechanisms behind such volatility persistence in market microstructure-type economic models, and have successfully generated volatility clustering.[1]

The microstructure model proposed in this paper provides a framework for generating volatility clustering of the returns with autocorrelations of hyperbolic decay. In addition, the proposed mechanism is capable of generating a linearly trending price and a negative correlation at the first lag of returns. The formation of volatility clustering is due to the combined effects of rational traders with multiple trading frequencies and their strategic interactions. It is natural to model traders with multiple trading frequencies, because not all traders trade at every possible opportunity.[2] The assumption that traders trade strategically is also plausible, since large investors are aware that their trades have an impact on the market price and take this effect into account. Note that no specific assumptions need to be made regarding the informational structure; information can be available either privately or publicly.

Specifically, we consider a discrete-time, multiperiod model in which traders trade a stock that has a limited risk absorption capacity (i.e., an upward sloping supply curve).[3] Traders are divided into two groups according to their trading frequency, while group size is random to prevent perfect signal extraction. Type A traders trade more frequently (trade every trading period) while type B traders trade less frequently (trade every other period).[4] In this model, traders may differ in the following respects: First, they may have different beliefs about fundamentals either due to different initial beliefs in the public signal environment or different realizations of signals in the private signal environment. Second, traders differ in trading strategies due to their different trading frequencies. Let the public signal environment with identical initial priors of fundamentals be a benchmark case. In such a benchmark case, the aggregate demand of type A traders depends on the presence of type B traders. Therefore, although there are no trades between groups, there will be an alternating pattern in price due to the multiple trading frequencies.

[1] For example, Brock and LeBaron (1996), Cabrales and Hoshi (1996). Granger and Machina (2006) presents general mechanisms of how a time-invariant system can exhibit volatility clustering, although they do not provide microeconomic models that could lead to the system proposed in their paper.

[2] One example is the futures market, where typical traders are hedgers and speculators. The speculators in futures market generally have shorter trading horizons. Another example is the comparison of intraday traders and mutual fund managers, since mutual fund managers cannot conduct intraday trading due to regulatory restrictions.

[3] To allow for the strategic interaction between traders, we do not allow an infinite supply of the asset, such that large orders would have no impact on price.

[4] For simplicity, we do not model the arrivals of traders endogenously, although our main results will apply with endogenous arrivals.

When traders of multiple trading frequency behave strategically, the volatility clustering can be generated for two reasons. First, multiple trading frequencies lead to price fluctuations, which generate a serial correlation in the magnitude of returns. The price fluctuations are partly due to the absence of infrequent traders, which lower the size of the aggregate demand in every other period. The price fluctuations are also partly due to the different strategies used by frequent traders depending on the presence of infrequent traders. Intuitively, frequent traders may behave like monopolists in the absence of infrequent traders and like oligopolists in their presence. We label this source of volatility clustering *the alternating effect*. Secondly, in the private signal case, each group of traders has its own set of signals. Given different trading frequencies, it is natural for traders in one group to infer the other group's signals from the price. Infrequent traders can infer the signals from the price in the period when they are absent, because the prices are entirely determined by the demands of frequent traders. Therefore, the past prices provide information that determines the current price. This feedback mechanism facilitates the formation of the volatility clustering (see, e.g., Brock and LeBaron (1996)). We label this source of volatility clustering *the signal extraction effect*.[5]

When there is strategic interaction between traders, then the group size of traders, i.e., the mean arrival of traders, has an impact on the optimal strategy of traders. The strategic competition is more intense with larger group sizes. At the limit, the strategic interaction among large groups of traders converges to the competitive outcome. This decreases the persistence of the magnitude of returns significantly. These results show that when group sizes are large, the volatility clustering becomes negligible. Thus, the strategic behavior is necessary for the presence and the persistence of volatility clustering in this model.

There are two additional stylized facts that could be generated in addition to volatility clustering, namely, linearly trending prices and a negative correlation at the first lag of returns. The former is mainly due to the optimal trading strategy used by the traders in equilibrium. There are two ways for traders to make a profit in our model. First, traders who hold shares of the stock receive the payoff on the terminal trading date. Traders adjust their optimal holdings according to the realization of their own signals and to the other group's signals that they have extracted. Second, traders will harvest capital gains if they can correctly anticipate the price movement. Because traders are informationally large in this model, they can strategically adjust their holdings across the remaining trading dates in order to take the advantage of their own impact on prices, which leads to trending prices.[6] The negative first-order autocorrelation of returns is consistent with the noisy rational expectation equilibrium (see e.g., Makarov and Rytchkov (2007)). The correlation between realized and expected returns can be shown to depend partly on the correlation between

[5] This mechanism can generate volatility clustering even in the public signal case where signal extraction is absent. On the other hand, we find that signal extraction without multiple trading frequencies cannot generate volatility clustering.

[6] For example, if they believe that the stock is undervalued in the current period, they may adjust their holdings over several periods instead of just increasing their current period holding which may could drive the price up sharply and diminish the future capital gains.

exogenous supply and the current price, which is supposed to be negative.

This model generates a number of interesting and testable implications that are absent from standard models of asset pricing with uniform trading frequency. For instance, the traders with more precise signals have a marginal effect on the evolution of the equilibrium. This seemingly counterintuitive result can be explained by the fact that traders strategically adjust their optimal holdings over all trading dates. Perhaps the most novel feature of the model is that traders with different trading frequencies have different levels of impact on equilibrium prices and returns, with infrequent traders having a larger effect. This naturally results from the fact that infrequent traders have fewer trading dates to smooth their adjustment of optimal holdings.[7] Furthermore, we show that signal extraction not only helps traders to infer the fundamentals more precisely but also provides them with more accurate guesses as to the behavior of the other type of traders. This leads to greater persistence in the magnitude of returns. Naturally, this provides both a feedback mechanism and a forward mechanism, both of which contribute to the formation of volatility clustering.

Overall, the main contributions of this paper are as follows. First, rational traders with multiple trading frequencies behaving strategically can generate volatility clustering, and this mechanism is robust with respect to different specifications of informational structure. The qualitative statistical properties of equilibrium including prices, returns and the magnitude of returns, are similar in the public and the private signal settings, with or without the same initial beliefs about fundamentals. Second, multiple trading frequency in the private signal environment can induce signal extraction, which contributes to the formation of volatility clustering and leads to hierarchical information. Hence, multiple trading frequencies within the private information environment provides theoretical justification for the existence of hierarchical information, where the infinite regress problem collapses (see, e.g., Townsend (1983), McNulty and Huffman (1996) and Bomfim (2001)). Third, return predictability is generated and is robust with respect to different informational structures.

Several papers have examined the role of multiple trading frequencies in different environments. For example, Christian and Jia (2005) try to determine the optimal trading frequency using a technical trading rule. Hauser *et al.* (2001) shows that the higher the aggregate trading frequency, the more efficient the price discovery in a non-dealer market. To our knowledge, however, no paper which links the multiple trading frequencies to volatility clustering and hierarchical information. Unlike Hauser *et al.* (2001), who focus on the trading frequency determined by the institution at the aggregate level, this study examines trading frequency at the microstructure level.

There are a number of ways to generate volatility clustering. For instance, Brock and LeBaron (1996) studied asymmetric information, the adaptive beliefs model of stock price and volume, in which volatility clustering is generated from traders experimenting with different belief updating systems, where experimenting is based on the past profits and expected future profits. Cabrales and Hoshi (1996) built a heterogeneous beliefs asset pricing model in which the persistence of distribution

[7] As stated earlier, the proposed mechanism can generate similar stylized facts in various informational settings which may lead to identification problems. This issue can be easily solved by examining the impulse responses of traders with different trading frequencies.

of wealth can lead to volatility clustering. The approach in Haan and Spear (1998) was to develop a heterogeneous agent, incomplete market model of interest rates, in which persistence of financial frictions leads to volatility clustering. de Fontnouvelle (2000) investigated a costly information model of asset trading, in which agents need to pay to acquire information, which leads to volatility clustering in price. Timmermann (2001) studied an imperfect information model of asset pricing, in which Bayesian updating of parameter estimates leads to volatility clustering.

Unlike Brock and LeBaron (1996), our model does not rely on the experimentation between different belief updating systems. Although the feedback mechanism, i.e., the signal extraction, can contribute to the formation of volatility clustering of returns in our model, it is not essential. In this model, volatility clustering is generated even in the public signal environment where signal extraction is absent. Unlike Timmermann (2001), Bayesian information updating is not sufficient to generate volatility clustering in our model. Our findings show that without strategic behavior, the volatility clustering is absent even with Bayesian information updating.

This paper is organized as follows. Section I describes the basic setting used in the paper. Section II starts with the benchmark environment of the public signal case. In this case, although the setting is simplistic in that all signals are assumed to be publicly available, the model can generate the three stylized facts: First, prices display long memory and an upward sloping trend. Second, returns are stationary and display a negative first-order correlation. Third, the variance of returns (or the magnitude of returns) displays volatility clustering with hyperbolic decay rate. Section III considers the different information structure of the private signal case, which yields additional interesting results. The private signal case possesses all the features of the public signal case. In addition, multiple trading frequencies lead to signal extraction behavior, which adds to the temporal memory of the volatility and generates an information hierarchy. In Section IV, Monte Carlo simulations are carried out to illustrate the stylized facts, which are consistent with the theoretical results. Section V considers two extensions: First, in addition to improving the understanding of fundamentals, signal extraction can also help traders to predict the behaviors of the other group; we label this *sophisticated signal extraction*. Hence, signal extraction not only provides a feedback mechanism but also a forward-looking mechanism that links the prices to future prices. The other extension is to allow heterogeneous priors, which is shown to have negligible effects on the main findings. We conclude afterwards.

I. Basic Setting

We model a hypothetical financial market in which there is a single trading asset. Assume that there are two groups of traders on the market, namely *Type A* and *Type B* traders. Type A traders come to the market every period (speculators), while Type B traders come every other period (fundamentalists). Figure 1 illustrates this multiple trading frequency market structure. During a trading period, traders receive a signal about the value of the underlying asset. For analytical tractability, we make two simplifying assumptions: trading dates are finite and traders maximize

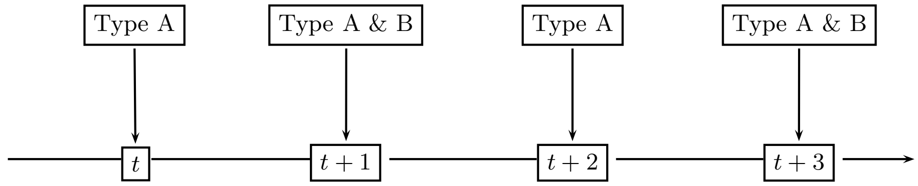


Figure 1: A FLOWCHART FOR MULTIPLE TRADING FREQUENCY. Type A traders (speculators) come to the market every trading date while type B traders (fundamentalists) come to the market every second trading date.

per period profit. The first assumption is innocuous. The myopic preference assumption may cause dissatisfaction, but it helps to avoid the large state variables problem which is endemic in the "forecasting the forecasts of others" literature. However, it turns out that it is not a serious problem to ignore hedging demand when the agent's maximization problem has a quadratic form and the signals are normally distributed.[8]

There is a single traded asset in the economy, with $T+1$ trading dates: $t = 0, 1, \ldots, T$. The asset pays $\widetilde{f}$ at $t = T$, where $\widetilde{f}$ is a normally distributed random variable.[9] The supply function of outstanding shares is $Q_s = p - \alpha$, where Q_s is aggregate liquidity supply and α captures the fixed cost of providing outstanding shares.[10] The supply function takes on a simplest form, because the supply side is not the main focus of this paper.

Two groups of traders, type A and type B, maximize a per-period objective function $E(W)$, taking into account their own influences on the equilibrium prices,[11] where W is the wealth of the trader. The type A traders come to the market at every date, but type B traders only come at $t = 0$, $t = 2, \ldots, t = T-1$ if T is odd, or $t = 1$, $t = 3, \ldots, t = T-1$ if T is even.

The arrivals of traders from both groups are assumed to be random. Random arrivals make it difficulty for traders to distinguish signals from the prices. The effective numbers of traders, n_t^A and n_t^B for type A and type B, are governed by an identical and independent (*iid*) normal distribution,[12] $N(n, \omega)$.

When $t = 0$, the two groups' prior beliefs about $\widetilde{f}$ are normally distributed, denoted by $N(\widehat{f}_0^A, \frac{1}{\tau_0})$ and $N(\widehat{f}_0^B, \frac{1}{\tau_0})$ where $\widehat{f}_0^A$ and $\widehat{f}_0^B$ can be different. Since we focus on the effect of trading frequency, the same prior beliefs are assumed, i.e., $\widehat{f}_0^A = \widehat{f}_0^B = \widehat{f}_0$. At $t = 1, 2, \ldots, T-1$, each trader type receives one signal: $S_t^A = \widetilde{f} + \epsilon_t^A$ and $S_t^B = \widetilde{f} + \epsilon_t^B$ for $t = 1, 2, \ldots, T-1$, where ϵ_t^A, ϵ_t^B are *iid* normally distributed $N(0, \frac{1}{\tau_\epsilon})$, where τ_ϵ is the precision of the signal.[13]

[8] We borrow the notation of Hong *et al.* (2006), who investigate the role of overconfidence in generating speculative bubbles.

[9] The payoff $\widetilde{f}$ can be interpreted as the liquidation value of the firm, and may be negative in the case of bankruptcy (considering the opportunity cost).

[10] The supply side is not modeled explicitly. The liquidity supply may come from noise traders or long term investors.

[11] Myopic preferences are adopted to avoid dynamic hedging problem and to obtain an analytically tractable solution.

[12] The arrival process of traders is independent of the economic variables. To avoid negative arrivals, we put a lower bound on the actual arrivals in the simulation study presented in Section IV.

[13] The inverse of the variance of the signal can be interpreted as the precision of the signal. If the variance of the

We start with the public signal case. Figure 2 illustrates the mechanism of price formation in the public signal case.

II. Public Signals

A. Equilibrium

We first solve for the beliefs of the two types of traders at time t. Using standard Bayesian updating formulas, these beliefs are easily characterized by the following proposition.

Proposition 1 *The beliefs of the two groups of traders at t are normally distributed as $N(f_t^A, \frac{1}{\tau_t})$ and $N(f_t^B, \frac{1}{\tau_t})$, where the precision is given by*

$$\tau_t = \tau_{t-1} + 2\tau_\epsilon \tag{1}$$

and the means are given by

$$\begin{aligned}\widehat{f}_t^A &= \widehat{f}_{t-1}^A + \frac{\tau_\epsilon}{\tau_t}(S_t^A + S_t^B - 2\widehat{f}_{t-1}^A)\\ \widehat{f}_t^B &= \widehat{f}_{t-1}^B + \frac{\tau_\epsilon}{\tau_t}(S_t^B + S_t^A - 2\widehat{f}_{t-1}^B).\end{aligned} \tag{2}$$

If the traders are with identical priors, i.e. $\widehat{f_0^A} = \widehat{f_0^B}$, then their beliefs remain the same over all the trading period, i.e., $\widehat{f}_t^A = \widehat{f}_t^B = \widehat{f}_t$ for $t = 1,2, \ldots, T-1$. Without a loss of generality, we solve the model when T is even. Given beliefs as in Proposition 1, we can solve for the equilibrium asset holdings x_{T-1}^A for type A and x_{T-1}^B for type B traders, and the prices p_{T-1} at $T-1$. With risk neutral preferences, we can write the utility maximization problem faced by the *ith* type A trader as

$$\max_{x_{T-1,i}^A} E[\widetilde{f} - (\alpha + \sum_{j=1}^{n_{T-1}^A} x_{T-1,j}^A + \sum_{j=1}^{n_{T-1}^B} x_{T-1,j}^B)]x_{T-1,i}^A$$

where the first-order condition is

$$\widehat{f}_{T-1} - \alpha - (n-1)x_{T-1,j}^A - nx_{T-1,j}^B - 2x_{T-1,i}^A \quad = 0.$$

Invoking symmetry, we have $x_{T-1,i}^A = x_{T-1,j}^B$ for all i,j in equilibrium, which leads to

$$x_{T-1}^A = \frac{\widehat{f}_{T-1} - \alpha}{2n+1}.$$

Hence, the Bayesian Nash equilibrium at $T-1$ can be characterized by

$$x_{T-1}^A \quad = \quad x_{T-1}^B = x_{T-1} = \frac{\widehat{f}_{T-1} - \alpha}{2n+1}$$

signal increases, the precision of the signal decreases.

$$p_{T-1} = \alpha + (n^A_{T-1} + n^B_{T-1})x_{T-1} = \alpha + (n^A_{T-1} + n^B_{T-1})\frac{\widehat{f}_{T-1} - \alpha}{2n+1} \quad (3)$$

Given the equilibrium price at $T-1$, we can use backward induction to derive the equilibrium holdings and price for $t = T-2$, $t = T-3$, ... , $t = 1$. If t is even, only type A traders arrive at market. Using an argument similar to the $T-1$ case, the optimal demand for the *ith* trader of type A is

$$\frac{E_t p_{t+1} - \alpha}{n+1}$$

where $E_t p_{t+1} = E[p_{t+1}|I_t]$, which is the conditional expectation of the next period price at time t given the information set I_t available at t, i.e., $I_t = \{p_{t-1}, p_{t-2}, p_{t-3}, \ldots, p_1, p_0\}$. Note that in period t, in order to determine the optimal holdings, traders need to forecast the next period price, p_{t+1}, and the optimal holdings of other traders.[14] If t is odd, both type A traders and type B traders come to the market. Using a similar argument as above, the optimal demand for the type A trader is

$$\frac{E_t p_{t+1} - \alpha}{2n+1}.$$

The equilibrium at $t = 0, 1, 2, \ldots, T-2$ differs from that at $t = T-1$ due to the myopic preference assumption. When $t = 0, 1, 2, \ldots, T-2$, traders only care about the price of the asset one period ahead because of their per period profit orientation. The optimal holding is determined by the forecast of the price in the next period which can be solved by backward induction and the expected holdings of other traders.

Proposition 2 *The Bayesian Nash equilibrium at t can be characterized by*
if t is odd

$$\begin{aligned} x^A_t &= x^B_t = x_t = \frac{2^{(T-t-1)/2} n^{(T-t-1)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-1)/2}}(\widehat{f}_t - \alpha) \\ p_t &= \alpha + (n^A_t + n^B_t)\frac{2^{(T-t-1)/2} n^{(T-t-1)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-1)/2}}(\widehat{f}_t - \alpha) \quad (3) \end{aligned}$$

if t is even

$$\begin{aligned} x^A_t &= \frac{2^{(T-t)/2} n^{(T-t-1)}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}(\widehat{f}_t - \alpha) \\ p_t &= \alpha + (n^A_t)\frac{2^{(T-t)/2} n^{(T-t-1)}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}(\widehat{f}_t - \alpha). \quad (3) \end{aligned}$$

Based on Proposition 1, Proposition 2 characterizes the Bayesian Nash equilibrium for all trading dates. The equilibrium price is a function of the remaining trading horizon and beliefs, which implies that the price dynamics are governed jointly by the trading horizon and the beliefs of traders.

[14] p_t is not determined when traders make their decisions at time t. The joint decisions of all market participants lead to the equilibrium price p_t.

B. The Properties of the Equilibrium

The previous section characterized the Bayesian Nash equilibrium in the public signal case. To examine how the mechanism proposed in the paper, namely, traders with the multiple trading frequencies behaving strategically, can generate the claimed stylized facts, including return predictability and volatility clustering, we need to study the properties of the equilibrium. The main properties of the equilibrium described are

1. Price series possess a linear trend and are linearly dependent.
2. Return series are stationary, and there is a negative autocorrelation of returns at first lag.
3. Return series display volatility clustering with hyperbolic decay autocorrelations.

Before elaborating on each of these properties, we will describe the equilibrium behavior of beliefs where the equilibrium price is a function of beliefs.

Beliefs

First, beliefs exhibit long memory as shown in Lemma 3.

Lemma 3

$$Cov(\widehat{f}_t, \widehat{f}_{t-j}) = \frac{1}{\tau_{t-j}} - \frac{1}{\tau_t}, \tag{4}$$

where τ_{t-j} is the precision of beliefs at $t-j$ for $j = 1, 2, 3, \ldots, t-1$.

Remarks:

1. Lemma 3 shows that beliefs have long memory and that the autocovariance function is a hyperbolic function of lags. To illustrate the nature of long memory and hyperbolic decay in beliefs, consider an impulse response experiment. For simplicity, assume that there is only one positive innovation at $t = 1$ and zero at other trading dates. Using Proposition 1, we know that the beliefs for traders at $t = 1, 2, \ldots, T-1$ are

$$\widehat{f}_t = \widehat{f}_0 + \frac{\tau_\epsilon}{\tau_t}(2t\epsilon_0 + \sum_{i=1}^{t} \epsilon_i^A + \sum_{i=1}^{t} \epsilon_i^B) \tag{5}$$

where $\epsilon_0 = \widetilde{f} - \widehat{f}_0$. Given the innovation at $t = 1$, the belief $\widehat{f}_1$ is updated and the effect of the innovation on the belief at $t = 1$ is τ_ϵ/τ_1. Then, the effect of the innovation on the belief at $t+1$, $\widehat{f}_{t+1}$ is τ_ϵ/τ_{t+1} and the effect of the innovation on the belief at $t+j$ is τ_ϵ/τ_{t+j}. Note that $\tau_{t+j} = \tau_0 + 2(t+j)\tau_\epsilon$; this leads to the persistence of beliefs, and the decay rate is hyperbolic.

2. Lemma 3 also characterizes the limiting behavior of the beliefs, which converge to the true value of the underlying asset $\widetilde{f}$ as $t \to \infty$. As $t \to \infty$, $\tau_\epsilon/\tau_t \to 0$, and $2t\tau_\epsilon/\tau_t \to 1$, Equation 5 is reduced to $\widehat{f}_t = \widehat{f}_0 + \epsilon_0 = \widehat{f}_0 + \widetilde{f} - \widehat{f}_0 = \widetilde{f}$. This implies that the beliefs of traders converge asymptotically to the true value of the underlying asset.

Prices

For simplicity, we assume that $\alpha = 0$ so that we can work with the logarithm of price series,[15]

$$\begin{aligned} \log p_t &= A_t + \frac{t}{2}B + \log \widehat{f}_t \quad \text{for } t \text{ is odd} \\ \log p_t &= C_t + \frac{t}{2}B + \log \widehat{f}_t \quad \text{for } t \text{ is even} \end{aligned} \tag{6}$$

where $A_t = \log(n_t^A + n_t^B) + \frac{T-1}{2}\log 2 + (T-1)\log n - \frac{T+1}{2}\log(2n+1) - \frac{T-1}{2}\log(n+1)$, $B = [\log(2n+1) + \log(n+1) - \log 2 - 2\log n]$, $C_t = \log(n_t^A) + \frac{T}{2}\log 2 + (T-1)\log n - \frac{T}{2}\log(2n+1) - \frac{T}{2}\log(n+1)$.

Remarks:

1. From Equation 6, price series contain three components: A_t and C_t, $\frac{t}{2}B$ and $\log \widehat{f}_t$. A_t and C_t are exogenous random variables that are determined by the *iid* arrival process. B is a positive constant that acts as a drift parameter. Therefore, $\log p_t$ has a deterministic linear upward sloping trend. As shown in Equation 4, $\widehat{f}_t$ is a long memory process. Therefore the price process also has a long memory which arises from the hyperbolic decay of the beliefs.

2. Intuitively, prices have long memory because of the embedded belief process. Price is a linear function of beliefs and preserves the linear dependence structure of beliefs. As a result, price series display long memory.

3. The deterministic trend originates from the strategic behaviors of traders. Traders will face a trade-off in deciding whether to increase their holdings. On the one hand, increasing their holdings today means that they can sell more at a higher price tomorrow. On the other hand, increasing their holdings will increase the cost of acquiring shares today. Without strategic behavior,[16] traders are not aware of their own impact on equilibrium price. They will therefore adjust their holdings until the price difference is zero, which leads to zero expected profit for every trading date. With strategic behavior, traders will exploit their monopolistic powers to prevent the increase of prices in the current period to order to make profits. When the remaining trading horizon is longer, the strategic behavior of traders has a larger cumulative impact on price, which implies that the prices increase over time.

[15] Monte Carlo simulations suggest that the results are not sensitive when α is nonzero.

[16] With the beliefs characterized in Proposition 1, we can solve for the equilibrium asset holdings x_{T-1}^A for type A and x_{T-1}^B for type B traders, and prices p_{T-1} at time $T-1$. With risk-neutral preferences, $E[\widetilde{f}] = p_{T-1}$ which implies that $p_{T-1} = \widehat{f}_{T-1}$. Using the market clear condition, we have $x_{T-1}^A = x_{T-1}^B = \frac{\widehat{f}_{T-1} - \alpha}{n_{T-1}^A + n_{T-1}^B}$. This implies that at $T-2$, when only type A traders come to the market, $E[p_{T-1}] = p_{T-2}$, which implies $p_{T-2} = \widehat{f}_{T-2}$ and $p_t = \widehat{f}_t$ for $t = 1, 2, \ldots, T-1$. Hence, prices follow a martingale process and there is no upward sloping trend embedded in the price series.

Returns

The gross returns[17] are defined by $r_t = p_t/p_{t-1}$ and the logarithm return $\log(r_t)$ at t is equal to

$$\log r_t = D_t + \log Z_t$$

where $D_t = \log(n+1) - \log(n) + \log(n_t^A + n_t^B) - \log(2n_{t-1}^A)$ and $Z_t = \widehat{f}_t/\widehat{f}_{t-1}$. D_t is an exogenous random variable which is determined by the arrival process, and Z_t is the ratio of beliefs of two types of traders. In order to understand the properties of r_t, we need to study the time series properties of D_t and Z_t. Since it is difficult to get a closed form for the autocovariance function of Z_t and D_t, we rely on Monte Carlo simulations.

Remarks:

1. The negative first-order autocorrelation of returns is mainly due to the change in the mean intensity of arrivals due to multiple trading frequencies. As shown in Figure 3, D_t possesses a negative first-order autocorrelation with the magnitude -0.25 and no statistically significant autocorrelations at higher lags. In the meantime, Z_t possesses no statistically significant autocorrelations at any lag. With the joint effects of Z_t and D_t, the Monte Carlo study suggests that r_t has a negative first-order autocorrelation.

Volatility Clustering

Let $\log(r_t) = D_t + \log Z_t$, while $\log Z_t = log(\widehat{f}_t) - \log(\widehat{f}_{t-1})$. It is easy to show that D_t is a stationary process. Hence we focus on the time series properties of Z_t. Z_t can be recursively written as[18]

$$Z_t = 1 + \frac{\tau_{t-1}}{\tau_t}\left(\frac{S_t^A + S_t^B}{S_{t-1}^A + S_{t-1}^B}\left(1 - \frac{\tau_{t-2}}{\tau_{t-1}}\frac{1}{Z_{t-1}}\right) - 2\right), \tag{7}$$

which is a nonlinear function of Z_{t-1}.

Remarks:

1. For expositional purposes, we define $Z_t = f_t(Z_{t-1})$. Using Equation 7, we have $Var(\log(r_t) = Var[D_t] + Var[Z_t] = Var[D_t] + (\frac{\partial f_t}{\partial Z_{t-1}})^2 Var[Z_{t-1}]$. Remember that Z_{t-1} is a function of r_{t-1}, i.e., $Z_{t-1} = f_t^{-1}(r_{t-1})$. This suggests that $Var(\log(r_t) = Var[D_t] + (\frac{\partial f_t}{\partial Z_{t-1}})^2 Var(f_t^{-1}(r_{t-1}))$. As in Granger and Machina (2006), when Z_t is a nonlinear function of Z_{t-1}, it is evidence of volatility clustering.

2. Volatility clustering is mainly due to the time series properties of Z_t. As shown in Figure 3, Var(D_t) possesses no statistically significant autocorrelations at any lag. In contrast, the

[17] Alternatively, one can define returns by $r_t = \log(p_t) - \log(p_{t-1})$. These two specifications do not alter our core findings.

[18] See derivations in Appendix B.

autocorrelations of $\text{Var}(Z_t)$ at the first ten lags are all statistically significant with magnitudes ranging from 0.05 to 0.27. In addition, $\text{Var}(Z_t)$ decay with a hyperbolic rate of 0.34.

Impact of Signal Precision

In addition to the stylized facts, this model generates interesting predictions that are absent from the standard asset pricing model. For example, this model predicts that the traders with more precise signals impose smaller effects on changing equilibrium prices. It is useful to consider an impulse response experiment. We start with an equilibrium where the beliefs of traders have already converged to the true value of the underlying asset, i.e., $\widehat{f}_0 = \widetilde{f}$. Suppose there is a large negative innovation in the signal at $t = 1$ with the magnitude $-\widetilde{f}$, i.e., $S_1^A = 0$ and $S_1^B = 0$. All other signals are equal to $\widetilde{f}$, i.e., $S_t^A = S_t^B = \widetilde{f}$. There is no noise in arrivals, i.e., $n_t^A = n_t^B = n$ and $\alpha = 0$. Then the beliefs, equilibrium prices, and returns can be computed according to Proposition 2

$$\begin{aligned}
\widehat{f}_t &= \widetilde{f}(1 - \frac{2\tau_\epsilon}{\tau_t}) = \frac{\tau_{t-1}}{\tau_t}\widetilde{f} \\
p_t &= \frac{2^{(T-t+1)/2} n^{T-t}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-1)/2}}\widehat{f}_t \quad \text{for } t \text{ is odd} \\
p_t &= \frac{2^{(T-t)/2} n^{T-t}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}\widehat{f}_t \qquad \text{for } t \text{ is even} \\
r_t &= \frac{\tau_{t-2}}{\tau_t}.
\end{aligned}$$

When the precision of signal τ_ϵ increases, the innovation in the signal has a smaller effect on returns. To see this, note that $r_t = \frac{\tau_{t-2}}{\tau_t}$ can be rewritten as $1 - \frac{4}{\frac{\tau_0}{\tau_\epsilon} + 2t}$. When τ_ϵ increases, r_t decreases. This implies that when traders have more precise signal, the change in price is smaller. This seemingly counterintuitive result is caused by the strategic adjustments of the optimal holdings of traders. When the signal is precise, traders tend to be reluctant to adjust their optimal holdings. This leads to traders with more precise signals imposing smaller effects on equilibrium prices and returns.

III. Private Signals

The model developed in the previous section is simple, yet capable of capturing the three stylized facts of financial data. Namely, prices display long memory and an upward sloping trend, returns are stationary and display a negative first-order correlation, and the variance of returns (magnitude of returns) displays volatility clustering with a hyperbolic decay rate. This section examines a variant of the model where the traders receive private signals instead of public signals. Signal extraction due to multiple trading frequencies in a private signal environment naturally contributes to the formation of volatility clustering. Intuitively, given different trading frequencies, it is natural for traders in one group to infer the other group's signals from the prices. Infrequent traders can infer the signals from the price in the period when they are absent, because the prices are entirely determined by the demands of the frequent traders. Therefore, the past prices provide information that determines the current price. This feedback mechanism facilitates the formation of the volatility clustering (see, e.g., Brock and LeBaron (1996)). Signal extraction generates other interesting findings as well, which provide insights into understanding private information trading in the market microstructure. For example, signal extraction in this model generates an information hierarchy among traders in an *ex ante* symmetric information setting.[19] In this model, infrequent traders can infer signals received by frequent traders exactly. In addition, infrequent traders infer more precise signals than frequent traders. The information hierarchy created by multiple trading frequencies suggests that the asymmetry of information diffusion may be endogenously determined by trading frequencies, rather than exogenously given.

A. Informational Structure

Note that at $t = 0, 2, \ldots, T-4, T-2$, only type A traders are present in the market. This implies that the equilibrium prices in such periods are entirely determined by the behavior and beliefs of type A traders. Therefore, type B traders can extract the beliefs of the type A traders by inverting the equilibrium price function of the beliefs.

We assume that traders know the exact number of traders who came to the market in the last period, i.e., at time t, the information set for the traders is $I_t = \{P_{t-1}, N^A_{t-1}, N^B_{t-1}\}$, where $P_{t-1} = \{p_{t-1}, p_{t-2}, \ldots, p_1, p_0\}$, and $N^A_{t-1} = \{n^A_{t-1}, n^A_{t-2}, \ldots, n^A_0\}$ and $N^B_{t-1} = \{n^B_{t-1}, n^B_{t-2}, \ldots, n^B_0\}$.

B. Bayesian Nash Equilibrium

To describe signal extraction behavior and the evolution of beliefs, we first characterize the Bayesian Nash equilibrium. Given a sequence of beliefs after signal extraction, $\widehat{f}^A_t$ and $\widehat{f}^B_t$ for type A traders and type B traders respectively, we can solve for the equilibrium at time $T-1$.

With risk-neutral preferences, we can characterize the utility maximization problem faced by the *ith* type A trader as

$$\max_{x^A_{T-1,i}} E[\widetilde{f} - (\alpha + \sum_{j=1}^{n^A_{T-1}} x^A_{T-1,j} + \sum_{j=1}^{n^B_{T-1}} x^B_{T-1,j})]x^A_{T-1,i}.$$

The first-order condition is

$$\begin{aligned} &\widehat{f}^A_{T-1} - \alpha - (n-1)x^A_{T-1,j} - nx^B_{T-1,j} - 2x^A_{T-1,i} = 0 \\ &\qquad \Rightarrow x^A_{T-1,i} = \frac{\widehat{f}^A_{T-1} - \alpha - nx^B_{T-1,j}}{n+1}. \end{aligned}$$

Hence, the Bayesian Nash equilibrium at $T-1$ can be characterized by

$$x^A_{T-1} \quad = \quad \frac{(n+1)\widehat{f}^A_{T-1} - n\widehat{f}^B_{T-1} - \alpha}{2n+1}$$

[19]A symmetric information setting means that *ex ante*, all traders will receive the same number of signals per period, which are drawn from *iid* distribution.

$$
\begin{aligned}
x_{T-1}^B &= \frac{(n+1)\widehat{f}_{T-1}^B - n\widehat{f}_{T-1}^A - \alpha}{2n+1} \\
p_{T-1} &= \alpha + n_{T-1}^A x_{T-1}^A + n_{T-1}^B x_{T-1}^B.
\end{aligned}
$$

Given the price in the $T-1$ period, we can derive the equilibrium for $t = 1, 2, 3, \ldots, T-2$:

Proposition 4 *The Bayesian Nash equilibrium at t can be characterized by*
if t is odd:

$$
\begin{aligned}
x_t^A &= \frac{2^{(T-t-1)/2} n^{(T-t-1)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-3)/2}}(\widehat{f}_t^A - \alpha) - \frac{2^{(T-t-1)/2} n^{(T-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-1)/2}}(\widehat{f}_t^B - \alpha) \\
x_t^B &= \frac{2^{(T-t-1)/2} n^{(T-1-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-3)/2}}(\widehat{f}_t^B - \alpha) - \frac{2^{(T-t-1)/2} n^{(T-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-1)/2}}(\widehat{f}_t^A - \alpha) \\
p_t &= \alpha + n_t^A x_t^A + n_t^B x_t^B
\end{aligned}
$$

if t is even:

$$
\begin{aligned}
x_t^A &= \frac{2^{(T-t)/2} n^{(T-1-t)}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}(\widehat{f}_t^A - \alpha) \\
p_t &= \alpha + (n_t^A)\frac{2^{(T-t)/2} n^{(T-1-t)}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}(\widehat{f}_t^A - \alpha).
\end{aligned}
$$

C. Signal Extraction

Using Proposition 4, we can describe the signal extraction behavior of traders. At $t = T-1$, type B traders know the actual arrivals n_{T-2}^A and n_{T-2}^B of both types of traders, and price p_{T-2} in the last period. From Proposition 8, $p_{T-2} = \alpha + (n_t^A)\frac{2n}{(2n+1)(n+1)}(\widehat{f}_{T-2}^A - \alpha)$. Therefore, the type B traders can invert the price formula to get $\widehat{f}_{T-2}^A = \alpha + \frac{(2n+1)(n+1)}{(2n)n_{T-2}^A}(p_{T-2} - \alpha)$. Hence, at $T-1$, type B traders know type A traders' belief at $T-2$, $\widehat{f}_{T-2}^A$. A similar analysis can be applied to type A traders and to other trading dates.

Proposition 5 *Type A traders know the exact beliefs of type B traders every other trading period, and type B traders know the exact beliefs of type A traders every trading period, i.e.,*

1. *When t is odd, type B traders know $\widehat{f}_{t-1}^A$, $\widehat{f}_{t-2}^A$, $\ldots$, $\widehat{f}_0^A$ while type A traders only know $\widehat{f}_{t-2}^B$, $\widehat{f}_{t-4}^B$, $\ldots$, $\widehat{f}_0^B$.*
2. *When t is even, type B traders know $\widehat{f}_{t-1}^A$, $\widehat{f}_{t-2}^A$, $\ldots$, $\widehat{f}_0^A$ while type A traders only know $\widehat{f}_{t-1}^B$, $\widehat{f}_{t-3}^B$, $\ldots$, $\widehat{f}_0^B$.*

Furthermore, traders can recover the private signals received by the other group by knowing the history of the beliefs. For instance, type B traders know the full history of $\widehat{f}_t^A$, and they understand that the difference in the beliefs is due to the signals. By inverting the Bayesian updating formula, they can even infer the private signals received by type A traders in addition to their beliefs.

Assuming that the identical initial prior beliefs $\widehat{f}_0^A = \widehat{f}_0^B$ are common knowledge, then the beliefs of each type of traders can be determined as follows:

Proposition 6 *The beliefs of two groups of traders at t are normally distributed, denoted by* $N(f_t^A, \frac{1}{\tau_t})$ *and* $N(f_t^B, \frac{1}{\tau_t})$*, where the precision for type A traders is given by*

$$\begin{aligned}
\tau_1^A &= \tau_0^A + \tau_\epsilon \\
\tau_2^A &= \tau_1^A + 2\tau_\epsilon \\
\tau_t^A &= \tau_{t-1}^A + \tau_\epsilon, \qquad \textit{for } t \textit{ is } 2k-1,\ k \geq 2 \\
\tau_t^A &= \tau_{t-1}^A + \tau_\epsilon + \widehat{\tau}_t^A, \qquad \textit{for } t \textit{ is } 2k,\ k \geq 2
\end{aligned}$$

where $\widehat{\tau}_t^A = \frac{1+(\tau_{t-1}^B)^2}{(1+\tau_{t-1}^B)^2}\tau_\epsilon$.

The mean for type A traders is given by

$$\begin{aligned}
\widehat{f}_1^A &= \widehat{f}_0^A + \frac{\tau_\epsilon}{\tau_1}(S_1^A - \widehat{f}_0^A) \\
\widehat{f}_2^A &= \widehat{f}_1^A + \frac{\tau_\epsilon}{\tau_2^A}(S_2^A + S_1^B - 2\widehat{f}_1^A) \\
\widehat{f}_t^A &= \widehat{f}_{t-1}^A + \frac{\tau_\epsilon}{\tau_t^A}(S_t^A - \widehat{f}_{t-1}^A) \qquad \textit{for } t = 2k-1,\ k \geq 2 \\
\widehat{f}_t^A &= \widehat{f}_{t-1}^A + \frac{\tau_\epsilon}{\tau_t^A}(S_t^A - \widehat{f}_{t-1}^A) + \frac{\widehat{\tau}_t^A}{\tau_t^A}(\widehat{S}_t^A - \widehat{f}_{t-1}^A) \qquad \textit{for } t = 2k,\ k \geq 2
\end{aligned}$$

where $\widehat{S}_t^A = \frac{\tau_{t-1}^B S_{t-2}^B}{1+\tau_{t-1}^B} + \frac{S_{t-1}^B}{1+\tau_{t-1}^B}$.

The precision for type B traders is given by

$$\begin{aligned}
\tau_1^B &= \tau_0^B + \tau_\epsilon \\
\tau_t^B &= \tau_{t-1}^B + 2\tau_\epsilon \qquad \textit{for } t \geq 2
\end{aligned}$$

and the mean for type B traders is given by

$$\begin{aligned}
\widehat{f}_1^B &= \widehat{f}_0^B + \frac{\tau_\epsilon}{\tau_1^B}(S_1^B - \widehat{f}_0^B) \\
\widehat{f}_t^B &= \widehat{f}_{t-1}^B + \frac{\tau_\epsilon}{\tau_t^B}(S_t^B + S_{t-1}^A - 2\widehat{f}_{t-1}^B) \qquad \textit{for } t \geq 2
\end{aligned}$$

Remarks:

1. Type B traders know the exact private signals received by type A traders, where type A traders only have estimates of the type B traders' private signals ($\widehat{S}_t^A$). This implies that there is an informational hierarchy among traders. Type B traders know all the signals that type A traders know. Notice that we start from an *ex ante* symmetric setting.

2. Type B traders extract higher precision signals: the signals extracted by type B traders are of precision τ_ϵ, while the signals extracted by type A traders are of precision $\widehat{\tau}_t^A = \frac{1+(\tau_{t-1}^B)^2}{(1+\tau_{t-1}^B)^2}\tau_\epsilon$. It is easy to see that $(1+\tau_{t-1}^B)^2 > 1+(\tau_{t-1}^B)^2$. Therefore, $\widehat{\tau}_t^A = \frac{1+(\tau_{t-1}^B)^2}{(1+\tau_{t-1}^B)^2}\tau_\epsilon < \tau_\epsilon$.[20]

3. Signal extraction facilitates the formation of volatility clustering by changing the formation of the beliefs. It is easy to see that due to signal extraction, there is a correlation between the traders' beliefs which is absent in the public signal case. The price is linear in beliefs and preserves the correlation between the traders' beliefs in its own correlation across trading dates.[21]

D. Properties of Equilibrium

The previous section characterized the Bayesian Nash equilibrium in the public signal case. In order to examine how the mechanism proposed in the paper, namely, traders with the multiple trading frequencies behaving strategically can generate return predictability and volatility clustering, we need to study the properties of the equilibrium. In addition, we are going to examine the difference between the equilibria in the private signal case and the public signal case. The main properties of the equilibrium in the private signal case are

1. Price series possess a linear trend and are linearly dependent.
2. Return series are stationary and there is a negative autocorrelation of returns at first lag.
3. Return series display volatility clustering with hyperbolic decay autocorrelations.

Before elaborating on each of these properties, we will describe the equilibrium behavior of beliefs where the equilibrium price is a function of beliefs.

Beliefs

First, beliefs exhibit long memory as shown in Lemma 7.

Lemma 7 *The autocovariance functions of beliefs $\widehat{f}_t^A$ and $\widehat{f}_t^B$ can be characterized by*

$$\begin{aligned} Cov(\widehat{f}_t^A, \widehat{f}_{t-j}^A) &= \frac{1}{\tau_{t-j}^A} - \frac{1}{\tau_t^A} \\ Cov(\widehat{f}_t^B, \widehat{f}_{t-j}^B) &= \frac{1}{\tau_{t-j}^B} - \frac{1}{\tau_t^B} \end{aligned}$$

[20] An exception occurs at $t = 2$. This is the only trading date where type A traders can extract an exact signal received by type B traders. This exception originates from the assumption that initial beliefs are identical. When we relax this assumption in Section V, type A traders are no longer able to extract exact signals.

[21] In addition to the beliefs channel, signal extraction can facilitate the formation of volatility clustering by providing a forward looking mechanism. As discussed in Section V, when traders use signal extraction not only to improve the understanding of fundamentals (via the beliefs channel) but also to predict the behaviors of the other group of traders, the effect of signal extraction on the formation of volatility clustering is stronger.

$$Cov(\widehat{f}_t^A, \widehat{f}_{t-j}^B) = \frac{\sum_{k=1}^{[t/2]} \widehat{\tau}_k^A}{\tau_t^A \tau_{t-j}^B},$$

where $[x]$ is the integer part of x.

Remarks:

1. As in the public signal case, the signal innovation will have an impact on the beliefs even at long lags and the decay rate is hyperbolic. To see this, rewrite the equations for the beliefs in the form

$$\begin{aligned} \widehat{f}_t^A &= \widehat{f}_0 + \frac{\tau_\epsilon}{\tau_t^A}(t\epsilon_0 + \sum_{i=1}^{t} \epsilon_i^A) + \frac{\sum_{k=1}^{t/2} \widehat{\tau}_t^A \epsilon_0}{\tau_t^A} + \frac{\sum_{k=1}^{t/2} \widehat{\tau}_t^A \widehat{\epsilon}_{2k}^A}{\tau_t^A} \\ \widehat{f}_t^B &= \widehat{f}_0 + \frac{\tau_\epsilon}{\tau_t^B}((2t-1)\epsilon_0 + \sum_{i=1}^{t-1} (\epsilon_i^A + \epsilon_i^B) + \epsilon_t^B) \end{aligned} \quad (8)$$

   where $\epsilon_0 = \widetilde{f} - \widehat{f}_0$ and $\widehat{\epsilon}_t^A = \widehat{S}_t^A - \widetilde{f}$. Using a similar impulse response experiment as in the public signal case, one can show that the beliefs have long memory and that the decay rate is hyperbolic.

2. It then follows that when $t \to \infty$, the beliefs of each group converge asymptotically to the true value of the underlying asset.

3. The correlation between beliefs is nonzero. It originates from the signal extraction behavior. For instance, as shown in Proposition 6, type B traders know the private signals received by type A traders in the last period. Hence, there is a common signal incorporated in the beliefs of type B traders in the current trading period and type A traders in the last trading period. This imposes a correlation structure on the beliefs of the traders. It turns out that the correlation structure of the beliefs is preserved in the prices, since the price is linear in beliefs.

Prices

Next, we examine the properties of prices. The prices characterized in Proposition 4 can be rewritten as

$$\begin{aligned} p_t &= \alpha + (n_t^A - \frac{n}{n+1} n_t^B) K_t (\widehat{f}_t^A - \alpha) + (n_t^B - \frac{n}{n+1} n_t^A) K_t (\widehat{f}_t^B - \alpha) && \text{for } t \text{ is odd} \\ p_t &= \alpha + \frac{2^{1/2}(2n+1)^{1/2}}{(n+1)^{3/2}} n_t^A K_t (\widehat{f}_t^A - \alpha) && \text{for } t \text{ is even} \end{aligned}$$

where $K_t = \frac{2^{(T-t-1)/2} n^{(T-t-1)}}{(2n+1)^{(T-t+1)/2} (n+1)^{(T-t-3)/2}}$.

For simplicity, we assume that $\alpha = 0$ and that there is no noise in the arrivals, i.e, $n_t^A = n_t^B = n$. Therefore, the prices can be characterized by:

$$
\begin{aligned}
p_t &= \frac{n}{n+1} K_t(\widehat{f}_t^A + \widehat{f}_t^B) && \text{for } t \text{ is odd} \\
p_t &= \frac{2^{1/2}(2n+1)^{1/2} n}{(n+1)^{3/2}} K_t(\widehat{f}_t^A) && \text{for } t \text{ is even}
\end{aligned} \tag{9}
$$

where $K_t = \frac{2^{(T-t-1)/2} n^{(T-t-1)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-3)/2}}$.

Remarks:

1. As in the public signal case, (log) price has a linear trend over time: $\log K_t$ is linear in t. Therefore K_t increases exponentially.[22] Again, the deterministic trend arises from strategic behaviors of traders. Due to differences in beliefs, they may adjust their optimal holdings at different rates. This further contributes to the alternating pattern in prices.

2. The prices display long memory because of the embedded belief process. Likewise, the price is linear in beliefs and preserves the dependence structure of the beliefs. Formally, the autocovariance function of prices can be characterized by

$$
\begin{aligned}
\mathrm{Cov}(p_t, p_{t-2j-1}) &= \frac{n}{n+1} \frac{(n+1/2)^{1/2} n}{(n+1)^{3/2}} K_t K_{t-2j-1} \mathrm{Cov}(\widehat{f}_t^A + \widehat{f}_t^B, \widehat{f}_{t-2j-1}^A) \\
\mathrm{Cov}(p_t, p_{t-2j}) &= \frac{n^2}{n+1^2} K_t K_{t-2j} \mathrm{Cov}(\widehat{f}_t^A + \widehat{f}_t^B, \widehat{f}_{t-2j}^A + \widehat{f}_{t-2j}^B)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathrm{Cov}(\widehat{f}_t^A + \widehat{f}_t^B, \widehat{f}_{t-j}^A + \widehat{f}_{t-j}^B) &= \frac{1}{\tau_{t-j}^A} - \frac{1}{\tau_t^A} + \frac{1}{\tau_{t-j}^B} - \frac{1}{\tau_t^B} + \frac{\sum_{k=1}^{[t/2]} \widehat{\tau}_k^A}{\tau_t^A \tau_{t-j}^B} + \frac{\sum_{k=1}^{[t/2]} \widehat{\tau}_k^A}{\tau_{t-j}^A \tau_t^B} \\
\mathrm{Cov}(\widehat{f}_t^A + \widehat{f}_t^B, \widehat{f}_{t-j}^A) &= \frac{1}{\tau_{t-j}^A} - \frac{1}{\tau_t^A} + \frac{\sum_{k=1}^{[t/2]} \widehat{\tau}_k^A}{\tau_t^A \tau_{t-j}^B}.
\end{aligned}
$$

Returns

Next, we examine the return series. Define $r_t = p_t / p_{t-1}$. Since it is difficult to get a closed form for the autocovariance function, we rely on Monte Carlo simulations. Monte Carlo simulations show that the autocorrelation of r_t at the first lag is -0.25 and is statistically significant, while the autocorrelations at other lags are statistically not different from zero.

[22] $\log K_t = (T-1)/2 \log 2 + (T-1)/2 \log n - (T-1)/2 \log(2n+1) - (T-3)\log(n+1) + t/2[\log(2n+1) + \log(n+1) - \log 2 - 2\log n]$.

Volatility Clustering

Let $r_t = X_t + Y_t$, where $X_t = \frac{\widehat{f}_t^A}{2\widehat{f}_{t-1}^A}$ and $Y_t = \frac{\widehat{f}_t^B}{2\widehat{f}_{t-1}^A}$. X_t can be rewritten as

$$X_t = 1 + \frac{\tau_{t-1}}{\tau_t}\left(\frac{S_t^A}{S_{t-1}^A}\left(1 - \frac{\tau_{t-2}}{\tau_{t-1}}\frac{1}{X_{t-1}}\right) - 1\right)$$

Remarks:

1. For expositional purposes, we define $X_t = m_t(X_{t-1})$. Using the same methods of analysis as in the public signal case, we obtain

$$\begin{aligned} \mathrm{Var}(r_t) &= \mathrm{Var}(Y_t) + \mathrm{Var}(X_t) + 2\mathrm{Cov}(X_t, Y_t) \\ &= \mathrm{Var}(Y_t) + \left(\frac{\partial m_t}{\partial X_{t-1}}\right)^2 \mathrm{Var}(X_{t-1}) + 2\mathrm{Cov}(X_t, Y_t) \end{aligned}$$

   Remember that X_{t-1} is a function of r_{t-1}, i.e., $X_{t-1} = m_t^{-1}(r_{t-1})$. This suggests that $Var(r_t) = \mathrm{Var}(Y_t) + (\frac{\partial m_t}{\partial X_{t-1}})^2 \mathrm{Var}(m_t^{-1}(r_{t-1})) + 2\mathrm{Cov}(X_t, Y_t)$. As in Granger and Machina (2006), when X_t is a nonlinear function of X_{t-1}, it is evidence of volatility clustering.

2. Signal extraction adds terms to the expression for volatility clustering. $\mathrm{Cov}(X_t, Y_t)$ is the covariance between the beliefs ratios of type A traders, $\frac{\widehat{f}_t^A}{2\widehat{f}_{t-1}^A}$ and $\frac{\widehat{f}_t^B}{2\widehat{f}_{t-1}^A}$. As shown in Lemma 7, the covariance between beliefs is positive. This shows that signal extraction facilitates the formation of volatility clustering.

Role of Multiple Trading Frequency

Through a multiple trading frequency mechanism, this model generates interesting predictions that are absent from the standard asset pricing model. We already discussed one example in the public signal case, namely, that signal precision has an impact on the equilibrium prices and returns. With signal extraction, this model is capable of generating other interesting predictions.

1. Traders with different trading frequencies have different levels of impact on equilibrium prices and returns. To see this, it is useful to consider a heuristic argument. Let us compare the cumulative impact of one innovation in the signal for type A traders or type B traders. As shown in Equation 8, the cumulative impulse response of signal for type A traders is $\sum_{i=1}^{t} \frac{\tau_\epsilon}{\tau_t^A} + \sum_{i=1}^{t} \frac{\tau_\epsilon}{\tau_t^B}$. In contrast, the cumulative impulse response of signal for type B traders is $\sum_{i=1}^{t} \frac{\widehat{\tau}_t^A}{\tau_t^A} \frac{1}{1+\tau_{t-1}^B} + \sum_{i=1}^{t} \frac{\tau_\epsilon}{\tau_t^B}$. Note that $\widehat{\tau}_t^A < \tau_\epsilon$ and $\frac{1}{1+\tau_{t-1}^B} < 1$. Given that $\widehat{\tau}_t^A$ is positive every other period, the cumulative impulse response of the infrequent traders to a signal is smaller. Notice that the smaller cumulative impulse response is due to the fact that type A traders are not capable of doing signal extraction every trading period and their signal precision is lower.

2. The timing schedule of signals has an impact on equilibrium prices and returns. It arises from the fact that type A traders are only able to extract exactly the signal of type B traders on $t = 2$. If the bad news arrives at a later time, type A traders incorporate it into their beliefs differently.

IV. Simulations

This section illustrates how the model is able to generate the claimed stylized facts, namely, linearly trending prices, negative first-order autocorrelation of returns and volatility clustering. In the simulations, the parameters are set at $\alpha = 100$, $T = 150$, $n = 50$, $\widetilde{f} = 150$, $\tau_0 = 10$, $\tau_\epsilon = 3$ and $\omega = 400$. We also report the robustness of our findings with different sets of parameter values.

A. Public Signals

The simulation results illustrate that (log) prices are linearly trending, returns are stationary and possess a negative first lag autocorrelation, which is consistent with the literature (see, e.g., Dacorogna *et al.* (2001)). In addition, as shown in Figure 4, the autocorrelation function of the variance of returns decays hyperbolically. Table III reports the average autocorrelations of r_t and $\text{Var}(r_t)$ across 100 simulations in the public signal case. It shows that for r_t, there is a statistically significant negative autocorrelation at the first lag with a magnitude of -0.48. There are no statistically significant autocorrelations at other lags. For $\text{Var}(r_t)$, all autocorrelations are statistically significant, with magnitudes ranging from 0.07 to 0.347.

The fixed cost of providing liquidity, α, mainly affects the position of the prices. Varying α does not alter the pattern of the prices and returns. In addition, changing α has negligible effects on the dependence structure of returns and the variance of returns. Changes in α mainly change the position of prices but not the slope of price series. Because the fixed cost is constant over time, the return series will be independent of α. Figure 5 shows the simulation results for $\alpha = 80$, $\alpha = 150$ and $\alpha = 200$. The simulation results are consistent with our intuition. The dependence structure of the returns and the variance of returns are almost the same for different α values.

Next, we examine the mean arrivals of traders of each group, n. As a common property of the Cournot game, as n increases, the prices become more volatile. In the mean time, the returns display less dependence structure, and less dependence structure at the second moment (the variance of returns). Intuitively, as n increases, the strategic outcome converges to the competitive outcome, which involves less strategic behavior and reduces the dependence structure of the returns and the variance of returns. Figure 6 shows the autocorrelation function of returns and the variance of returns when $n = 50$, $n = 100$, $n = 200$. The simulations show that as n increases, the variance of returns displays less dependence structure. When $n = 200$, the autocorrelation coefficients of the variance of returns at all but the first lag are very small. In contrast, when $n = 50$, the variance of returns displays evidence of volatility clustering.

We continue to study the effects of the variance of arrivals, ω. Figure 7 suggests that ω has a negligible effect on the dependence structure of return and the variance of returns. It mainly affects the volatility of prices. If ω is lower, then the prices become smoother (less volatile). This is consistent with our intuition that ω is the parameter characterizing the arrival process only, which is independent of the updating procedure. This means that ω should not have any explanatory power in the dependence structure as opposed to n, which can affect the dependence structure of the variance of returns by affecting the interaction between traders.

Finally, we investigate the effects of τ_ϵ, the precision of the signal. When the precision of the signal is high, it tends to put more weight on the signals rather than the beliefs of the last period. Therefore, the potential high precision signal τ_ϵ can reduce the dependence structure. The simulation results suggest that the magnitude of this effect is small.

B. Private Signals

The simulation results are quite similar to the public signal case. Price series have an upward sloping trend and there is a dependence structure in returns, while the returns are stationary. The first-order autocorrelation coefficient of the returns is negative. As in Figure 8, the autocorrelation function of the variance of returns exhibits hyperbolic decay, which is evidence of volatility clustering. Table IV reports the average autocorrelations of r_t and $\text{Var}(r_t)$ in 100 simulations. It shows that for r_t, the autocorrelation at the first lag is -0.25, which is statistically significant, while the autocorrelations at other lags are statistically indistinguishable from zero. For $\text{Var}(r_t)$, all the autocorrelations are statistically significant. The magnitude ranges from 0.1 to 0.29. Compared to the public signal case, the persistent structure of $\text{Var}(r_t)$ is of a similar magnitude.

We also present the results of the experiments in terms of parameters in Figure 9, Figures 10 and 11. The model behavior resembles the public signal case. The dependence structures of returns and the variance of returns, which are captured by the autocorrelation function, are similar when changing the carrying cost, α. Changing the mean intensity of arrival n has negligible effects on dependence structure of returns. As n increases, however, the variance of returns becomes less persistent, which is reflected in smaller magnitude of the autocorrelation coefficients. Different values of the standard deviation of arrivals ω have negligible effects on the dependence structure of returns and the variance of returns.

To summarize, this simulation provides an illustration of the robustness of the proposed mechanism. In both the public and the private signal cases, the simulation demonstrates the three stylized facts. First, the prices display long memory and an upward sloping trend. Second, the returns are stationary and display a negative first-order correlation. Third, the variance of returns (the magnitude of returns) display volatility clustering and the decay rate is hyperbolic. Furthermore, the simulations suggest that there is an inverse relationship between the mean intensity of the arrivals and the persistence structure of the variance of returns. In both cases, as the mean intensity of the arrivals n increases, the magnitudes of the autocorrelations become smaller, indicating that the variance of returns is less persistent. As guided by the theoretical framework, it is reasonable to believe that this diminishing effect of mean intensity of the arrivals is due to less strategic behaviors of traders, as the Bayesian Nash equilibrium converges to the competitive equilibrium. This framework suggests that strategic behavior contributes to the persistent structure in the variance of returns.

V. Extensions

A. Sophisticated Guess

Previous analysis shows that signal extraction facilitates the formation of volatility clustering through a beliefs channel. It is interesting to consider additional mechanisms led by signal extraction. In additional to the beliefs channel, traders seek the short-run profit opportunities by predicting the other party's behavior using the extracted beliefs. Intuitively, this forward looking mechanism will bring more persistence to the prices and the magnitude of returns.

For expositional purposes, first consider the Bayesian Nash equilibrium at time $T-1$. Time $T-1$ is the last trading date. Therefore, traders only care about the true value of the underlying asset. The profit maximization problem faced by the *ith* type A trader can be characterized by

$$\max_{x^A_{T-1,i}} E[\widetilde{f} - (\alpha + \sum_{j=1}^{n^A_{T-1}} x^A_{T-1,j} + \sum_{j=1}^{n^B_{T-1}} x^B_{T-1,j})]x^A_{T-1,i}.$$

Therefore the equilibrium at time $T-1$ is the same as in the signal extraction case we studied in Section III

$$\begin{aligned} x^A_{T-1} &= \frac{(n+1)\widehat{f}^A_{T-1} - n\widehat{f}^B_{T-1} - \alpha}{2n+1} \\ x^B_{T-1} &= \frac{(n+1)\widehat{f}^B_{T-1} - n\widehat{f}^A_{T-1} - \alpha}{2n+1} \\ p_{T-1} &= \alpha + n^A_{T-1}x^A_{T-1} + n^B_{T-1}x^B_{T-1}. \end{aligned}$$

At time $T-2$, only type A traders come to the market, and they care about the per period profit instead, i.e., they care about the price in the next period p_{T-1} instead of $\widetilde{f}$. Therefore, the profit maximization problem faced by the *ith* type A trader at time $T-2$ is

$$\max_{x^A_{T-2,i}} E^A[p_{T-1} - (\alpha + \sum_{j=1}^{n^A_{T-2}} x^A_{T-2,j} + \sum_{j=1}^{n^B_{T-2}} x^B_{T-2,j})]x^A_{T-2,i}$$

which implies that

$$x^A_{T-2} = \frac{E^A[p_{T-1}] - \alpha}{n+1}.$$

By backward induction,

$$E^A[p_{T-1}] - \alpha \quad = \quad E^A[n^A_{T-1}x^A_{T-1} + n^B_{T-1}x^B_{T-1}]$$

$$
\begin{aligned}
&= nE^A[\frac{n}{2n+1}(\widehat{f}^A_{T-1} + \widehat{f}^B_{T-1} - 2\alpha)] \\
&= \frac{n^2}{2n+1}(E^A[\widehat{f}^A_{T-1}] + E^A[\widehat{f}^B_{T-1}]).
\end{aligned}
$$

At $t = T-2$, the best guess for $\widehat{f}^A_{T-1}$ is $\widehat{f}^A_{T-2}$, the current mean of the beliefs. Without signal extraction, the best guess for $\widehat{f}^B_{T-1}$ is also $\widehat{f}^A_{T-2}$, because there is no further information on the beliefs of type B traders. With signal extraction, type A traders may be able to find a better guess instead of their own belief, for instance, $\widehat{f}^B_{T-3}$. One direct impact of this new guess is that signal extraction behavior can alter the optimal holdings of traders and the equilibrium price.

Given a sequence of beliefs after signal extraction, $\widehat{f}^A_t$ and $\widehat{f}^B_t$ for type A and type B traders, we can characterize the Bayesian Nash equilibrium as:

Proposition 8 *The Bayesian Nash equilibrium at t can be characterized by*

$$
\begin{aligned}
x^A_t &= \frac{2^{(T-t-3)/2} n^{(T-t-1)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-3)/2}}(\widehat{f}^A_t + \widehat{f}^B_{t-2} - 2\alpha) \\
&\quad - \frac{2^{(T-t-3)/2} n^{(T-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-1)/2}}(\widehat{f}^A_{t-1} + \widehat{f}^B_t - 2\alpha) \\
x^B_t &= \frac{2^T n^{(T-t-1)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-3)/2}}(\widehat{f}^B_t + \widehat{f}^A_{t-1} - 2\alpha) \\
&\quad - \frac{n^{(T-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-1)/2}}(\widehat{f}^A_t + \widehat{f}^B_{t-2} - 2\alpha) \\
p_t &= \alpha + n^A_t x^A_t + n^B_t x^B_t
\end{aligned}
$$

if t is odd;

$$
\begin{aligned}
x^A_t &= \frac{2^{(T-t-2)/2} n^{(T-t-1)}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}(\widehat{f}^A_t + \widehat{f}^B_{t-1} - 2\alpha) \\
p_t &= \alpha + (n^A_t) x^A_t
\end{aligned}
$$

if t is even;

Figure 13 demonstrates the simulation result and Table V demonstrates the average autocorrelations of r_t and $\mathrm{Var}(r_t)$ across 100 simulations. We can see that the variance of r_t displays a more persistent structure which is reflected by larger magnitude of autocorrelations ranging from 0.168 to 0.383. This suggests that signal extraction can affect the formation of prices and induce more persistence in the variance of returns.

B. Heterogeneous Priors

Previous analysis assumed that the prior beliefs are the same for both types of traders. In this section, we extend the models studied in the previous sections to allow for the heterogeneous priors, i.e. $\widehat{f}^A_0 \neq \widehat{f}^B_0$.

Public Signals

We start with the public signal case, in which both traders receive the same signal but they will hold different beliefs. We will solve the Bayesian Nash equilibrium in a similar way to the private signal case. Formally, the beliefs of both groups can be characterized by

Proposition 9 *The beliefs of two groups of traders at* t *are normally distributed, denoted by* $N(f_t^A, \frac{1}{\tau_t})$ *and* $N(f_t^B, \frac{1}{\tau_t})$*,where the precision is given by*

$$\tau_t = \tau_{t-1} + 2\tau_\epsilon$$

and the means are given by

$$\begin{aligned}
\widehat{f}_t^A &= \widehat{f}_{t-1}^A + \frac{\tau_\epsilon}{\tau_t}(S_t^A + S_t^B - 2\widehat{f}_{t-1}^A) \\
\widehat{f}_t^B &= \widehat{f}_{t-1}^B + \frac{\tau_\epsilon}{\tau_t}(S_t^B + S_t^A - 2\widehat{f}_{t-1}^B).
\end{aligned}$$

We can characterize the Bayesian Nash equilibrium at t as:

Proposition 10 *The Bayesian Nash equilibrium at* t *can be characterized by:*

$$\begin{aligned}
x_t^A &= \frac{2^{(T-t-1)/2} n^{(T-1-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-3)/2}}(\widehat{f}_t^A - \alpha) - \frac{2^{(T-t-1)/2} n^{(T-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-1)/2}}(\widehat{f}_t^B - \alpha) \\
x_t^B &= \frac{2^{(T-t-1)/2} n^{(T-1-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-3)/2}}(\widehat{f}_t^B - \alpha) - \frac{2^{(T-t-1)/2} n^{(T-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t-1)/2}}(\widehat{f}_t^A - \alpha) \\
p_t &= \alpha + n_t^A x_t^A + n_t^B x_t^B
\end{aligned}$$

if t *is odd;*

$$\begin{aligned}
x_t^A &= \frac{2^{(T-t)/2} n^{(T-1-t)}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}(\widehat{f}_t^A - \alpha) \\
p_t &= \alpha + (n_t^A)\frac{2^{(T-t)/2} n^{(T-1-t)}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}(\widehat{f}_t^A - \alpha).
\end{aligned}$$

if t *is even.*

Remarks:

1. Monte Carlo simulations confirm that heterogeneous beliefs in the public signal case will not change qualitative features of our main results. From Proposition 10, we can see that the Bayesian Nash equilibrium is quite similar to the private signal case. This makes sense because with heterogeneous priors, the beliefs are different at each trading date even with public signals.

Private Signals

Note that with heterogeneous beliefs, type A traders cannot extract exact signals at any trading date as opposed to the homogeneous priors case, in which they could at $t = 2$. This is the only modification of the equilibrium in the private signals case with heterogeneous priors. Instead, type A traders can only get a composite signal combined with the initial belief of type B traders $\widehat{f}_0^B$ and their belief $\widehat{f}_1^B$ at $t = 1$. For $t = 3, 4, 5, \ldots, T-1$, the signal extraction behavior of both types is identical to the homogeneous priors case. Formally,

Proposition 11 *The beliefs of the two groups of traders at t are normally distributed, denoted by $N(f_t^A, \frac{1}{\tau_t})$ and $N(f_t^B, \frac{1}{\tau_t})$, where the precision for type A traders is given by*

$$\begin{aligned}
\tau_1^A &= \tau_0^A + \tau_\epsilon \\
\tau_t^A &= \tau_{t-1}^A + \tau_\epsilon, \qquad \textit{for } t \textit{ is } 2k-1,\ k \geq 2 \\
\tau_t^A &= \tau_{t-1}^A + \tau_\epsilon + \widehat{\tau}_t^A, \qquad \textit{for } t \textit{ is } 2k,\ k \geq 1
\end{aligned}$$

where $\widehat{\tau}_t^A = \frac{1+(\tau_{t-1}^B)^2}{(1+\tau_{t-1}^B)^2}\tau_\epsilon$. The mean for type A is given by

$$\begin{aligned}
\widehat{f}_1^A &= \widehat{f}_0^A + \frac{\tau_\epsilon}{\tau_1}(S_1^A - \widehat{f}_0^A) \\
\widehat{f}_t^A &= \widehat{f}_{t-1}^A + \frac{\tau_\epsilon}{\tau_t^A}(S_t^A - \widehat{f}_{t-1}^A), \qquad \textit{for } t \textit{ is } 2k-1,\ k \geq 2 \\
\widehat{f}_t^A &= \widehat{f}_{t-1}^A + \frac{\tau_\epsilon}{\tau_t^A}(S_t^A - \widehat{f}_{t-1}^A) + \frac{\widehat{\tau}_t^A}{\tau_t^A}(\widehat{S}_t^A - \widehat{f}_{t-1}^A), \qquad \textit{for } t \textit{ is } 2k,\ k \geq 1
\end{aligned}$$

where $\widehat{S}_t^A = \frac{\tau_{t-1}^B S_{t-2}^B}{1+\tau_{t-1}^B} + \frac{S_{t-1}^B}{1+\tau_{t-1}^B}$. The precision for type B traders is given by

$$\begin{aligned}
\tau_1^B &= \tau_0^B + \tau_\epsilon \\
\tau_t^B &= \tau_{t-1}^B + 2\tau_\epsilon \qquad \textit{for } t \geq 2
\end{aligned}$$

and the mean for type B traders is given by

$$\begin{aligned}
\widehat{f}_1^B &= \widehat{f}_0^B + \frac{\tau_\epsilon}{\tau_1^B}(S_1^B - \widehat{f}_0^B) \\
\widehat{f}_t^B &= \widehat{f}_{t-1}^B + \frac{\tau_\epsilon}{\tau_t^B}(S_t^B + S_{t-1}^A - 2\widehat{f}_{t-1}^B) \qquad \textit{for } t \geq 2
\end{aligned}$$

Figure 12 depicts the average autocorrelations of r_t and $\mathrm{Var}(r_t)$ across 100 simulations in the public signal case and in the private signal case with heterogeneous priors. It shows that the magnitudes of the average autocorrelations are similar to those in the cases with homogeneous beliefs. This implies that the changing from the homogeneous priors to the heterogeneous priors have negligible effects on the persistence structure of the returns and the variance of returns.

In summary, heterogeneous beliefs affect the equilibrium in a minor way. In the public signal case, traders have different beliefs. The differences in beliefs are constant over time and equal to the difference in priors. In the private signal case, it changes the belief updating of type A traders at only one trading date, $t = 2$. As a result, the qualitative properties of prices, returns, and volatility of returns do not change.

VI. Conclusions

This paper has developed a discrete-time multiperiod model of volatility clustering due to the combined effects of rational traders with multiple trading frequencies and their strategic interactions. First, multiple trading frequencies lead to an alternating pattern in prices which generates a serial correlation in the magnitude of the returns. Secondly, signal extraction provides a feedback mechanism, which induces a correlation between the past prices and the current price. This facilitates the formation of the volatility clustering. In addition, the proposed mechanism is capable of generating linearly trending prices and a negative correlation at the first lag of returns.

We also find that the number of traders has an impact on the formation of volatility clustering. This is a consequence of the fact that when the mean intensity of arrivals increases, the strategic competition outcome will converge to the competitive outcome. Hence, the effect of strategic interaction diminishes. Monte Carlo simulations show that in all settings, as the mean arrivals of traders increase, the variance of returns becomes less persistent. In the extreme, when the number of traders is sufficiently large, the model predicts that there is only one statistically significant first-order autocorrelation of variance of returns, while other autocorrelation coefficients are statistically indistinguishable from zero.

This model yields several interesting predictions. First, traders with more precise signals have a smaller impact on the evolution of equilibrium prices and returns. Secondly, traders with different trading frequencies impose different levels of impact on the equilibrium prices and returns. Frequent traders respond to the signals in smaller magnitudes and this is transformed into a smaller cumulative impact on the evolution of the equilibrium.

We show that an information hierarchy can be generated in an *ex ante* symmetric setting through signal extraction. Thus when trading frequencies are different, signal extraction can endogenously determine the information diffusion. The informational advantages of the traders may be due to their trading frequency. One potentially interesting avenue for future research is to endogenize the trading frequencies in a more general model where trading frequencies and an information hierarchy are simultaneously determined. Doing so would allow us to address additional issues, such as the microstructure impact of trading from information diffusion. For instance, in the context of this model, trading frequency is exogenously determined and leads to signal extraction which generates volatility clustering and information hierarchy. It is not obvious which factors make traders choose to trade less frequently. We leave the clarification of these issues for the future work.

A Proofs

Proof to Proposition 1

Proof: With identical initial beliefs $\widehat{f}_0^A = \widehat{f}_0^B$, and invoking DeGroot (1970), this leads to $\widehat{f}_t^A = \widehat{f}_t^B$ for $t = 1, 2, \ldots, T-1$:

$$\begin{aligned}
\tau_t &= \tau_{t-1} + 2\tau_\epsilon \\
\widehat{f}_t^A &= \widehat{f}_{t-1}^A + \frac{\tau_\epsilon}{\tau_t}(S_t^A + S_t^B - 2\widehat{f}_{t-1}^A) \\
\widehat{f}_t^B &= \widehat{f}_1^B + \frac{\tau_\epsilon}{\tau_t}(S_t^B + S_t^A - 2\widehat{f}_{t-1}^B).
\end{aligned}$$

Q.E.D

Proof to Proposition 2

Proof: The proof is by induction. The Bayesian Nash equilibrium at $T-1$ is characterized by

$$\begin{aligned}
x_{T-1}^A &= x_{T-1}^B = x_{T-1} = \frac{\widehat{f}_{T-1} - \alpha}{2n+1} \\
p_{T-1} &= \alpha + (n_{T-1}^A + n_{T-1}^B)x_{T-1} \\
&= \alpha + (n_{T-1}^A + n_{T-1}^B)\frac{\widehat{f}_{T-1} - \alpha}{2n+1}.
\end{aligned}$$

Therefore, the equilibrium characterization holds when $t = T-1$. Suppose the Proposition holds at $t = 2k$, where k is a integer such that $2k \leq T-1$, i.e,

$$\begin{aligned}
x_{2k}^A &= \frac{2^{(T-2k)/2} n^{(T-2k-1)}}{(2n+1)^{(T-2k)/2}(n+1)^{(T-2k)/2}}(\widehat{f}_{2k} - \alpha) \\
p_{2k} &= \alpha + (n_{2k}^A)\frac{2^{(T-2k)/2} n^{(T-2k-1)}}{(2n+1)^{(T-2k)/2}(n+1)^{(T-2k)/2}}(\widehat{f}_{2k} - \alpha).
\end{aligned}$$

Therefore at $t = 2k-1$, the ith type A traders face the maximization problem which can be characterized by

$$max_{x_{2k-1,i}^A} E[p_{2k} - (\alpha + \sum_{j=1}^{n_{2k-1}^A} x_{2k-1,j}^A + \sum_{j=1}^{n_{2k-1}^B} x_{2k-1,j}^B)]x_{2k-1,i}^A.$$

First-order condition to the above utility maximization problem is:

$$E[p_{2k}] - \alpha - (n-1)x_{2k-1,j}^A - nx_{2k-1,j}^B - 2x_{2k-1,i}^A = 0.$$

Invoking the symmetry result, we will have $x_{2k-1}^A = x_{2k-1}^B$ in equilibrium, then Equation 10 can be rewritten as:

$$x_{2k-1,i}^A = \frac{E[p_{2k}] - \alpha}{2n+1}.$$

At $t = 2k-1$,

$$\begin{aligned} E[p_{2k}] &= E[\alpha + (n_{2k}^A)\frac{2^{(T-2k)/2}n^{(T-2k)}}{(2n+1)^{(T-2k)/2}(n+1)^{(T-2k)/2}}(\widehat{f}_{2k} - \alpha)] \\ &= \alpha + n\frac{2^{(T-2k)/2}n^{(T-2k)}}{(2n+1)^{(T-2k)/2}(n+1)^{(T-2k)/2}}E(\widehat{f}_{2k} - \alpha) \\ &= \alpha + \frac{2^{(T-2k)/2}n^{(T-2k+1)}}{(2n+1)^{(T-2k)/2}(n+1)^{(T-2k)/2}}(\widehat{f}_{2k-1} - \alpha). \end{aligned}$$

Therefore, the optimal holdings for type A traders and type B traders at $t = 2k-1$ can be characterized by

$$x_{2k-1,i}^A = x_{2k-1,i}^B = \frac{E[p_{2k}] - \alpha}{2n+1} = \frac{2^{(T-2k)/2}n^{(T-2k+1)}}{(2n+1)^{(T-2k+2)/2}(n+1)^{(T-2k)/2}}(\widehat{f}_{2k-1} - \alpha).$$

Further, the price at $t = 2k-1$ is

$$p_{2k-1} = \alpha + (n_{2k-1}^A + n_{2k-1}^B)\frac{2^{(T-2k)/2}n^{(T-2k)}}{(2n+1)^{(T+2k+2)/2}(n+1)^{(T-2k)/2}}(\widehat{f}_{2k-1} - \alpha).$$

At $t = 2k-2$, the type A traders face the maximization problem which can be characterized by

$$max_{x_{2k-2,i}^A} \quad E[p_{2k-1} - (\alpha + \sum_{j=1}^{n_{2k-2}^A} x_{2k-2,j}^A]x_{2k-2,i}^A.$$

First-order condition to the above utility maximization problem is:

$$E[p_{2k-1}] - \alpha - (n-1)x_{2k-1,j}^A - 2x_{2k-1,i}^A = 0.$$

Invoking the symmetry result, we will have $x_{2k-1}^A = x_{2k-1}^B$ in equilibrium, then Equation 10 can be rewritten as:

$$x_{2k-2,i}^A = \frac{E[p_{2k-1}] - \alpha}{n+1}.$$

Similarly, at $t = 2k-2$

$$\begin{aligned} E[p_{2k-1}] &= E[\alpha + (n_{2k-1}^A + n_{2k-1}^B)\frac{2^{(T-2k)/2}n^{(T-2k)}}{(2n+1)^{(T-2k+2)/2}(n+1)^{(T-2k)/2}}(\widehat{f}_{2k-1} - \alpha)] \\ &= \alpha + 2n\frac{2^{(T-2k)/2}n^{(T-2k)}}{(2n+1)^{(T-2k+2)/2}(n+1)^{(T-2k)/2}}E(\widehat{f}_{2k-1} - \alpha) \\ &= \alpha + \frac{2^{(T-2k+2)/2}n^{(T-2k+1)}}{(2n+1)^{(T-2k+2)/2}(n+1)^{(T-2k)/2}}(\widehat{f}_{2k-2} - \alpha). \end{aligned}$$

Therefore, the optimal holding for type A traders and type B traders at $t = 2k-2$ can be characterized by

$$x_{2k-2,i}^A = \frac{E[p_{2k-1}] - \alpha}{n+1}$$

$$
\begin{aligned}
&= \frac{2^{(T-2k+2)/2} n^{(T-2k+1)}}{(2n+1)^{(T-2k+2)/2}(n+1)^{(T-2k+2)/2}}(\widehat{f}_{2k-2}-\alpha) \\
&= \frac{2^{(T-t)/2} n^{(T-t-1)}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}(\widehat{f}_t-\alpha).
\end{aligned}
$$

Furthermore, the price at $t = 2k-2$ is

$$
p_t = \alpha + (n_t^A)\frac{2^{(T-t)/2} n^{(T-t-1)}}{(2n+1)^{(T-t)/2}(n+1)^{(T-t)/2}}(\widehat{f}_t-\alpha).
$$

Q.E.D

Proof to Proposition 5

Proof : At t, both types of traders know actual arrivals of both types of traders' last period, N_{t-1}^A and N_{t-1}^B, and price in the last period, p_{t-1}. When t is odd, $t-1$ is even. From Proposition 4,

$$
p_{t-1} = \alpha + (n_{t-1}^A)\frac{2^{(T-t+1)/2} n^{(T-t)}}{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t+1)/2}}(\widehat{f}_{t-1}^A-\alpha). \tag{10}
$$

Type B can infer $\widehat{f}_{t-1}^A$ according to

$$
\widehat{f}_{t-1}^A = \alpha + \frac{(2n+1)^{(T-t+1)/2}(n+1)^{(T-t+1)/2}}{2^{(T-t+1)/2} n^{(T-t)}(n_{t-1}^A)}(p_{t-1}-\alpha), \tag{11}
$$

while type A cannot extract any information about $\widehat{f}_{t-1}^B$. When t is even, $t-1$ is odd. From Proposition 4,

$$
p_{t-1} = \alpha + (n_{t-1}^A - \frac{n}{n+1}n_{t-1}^B)K_{t-1}(\widehat{f}_{t-1}^A-\alpha) + (n_{t-1}^B - \frac{n}{n+1}n_{t-1}^A)K_{t-1}(\widehat{f}_{t-1}^B-\alpha) \tag{12}
$$

where $K_{t-1} = \frac{2^{(T-t)/2} n^{(T-t)}}{(2n+1)^{(T-t+2)/2}(n+1)^{(T-t-2)/2}}$. Type B can infer $\widehat{f}_{t-1}^A$ according to

$$
\widehat{f}_{t-1}^A = \alpha + \frac{(p_{t-1}-\alpha)}{(n_{t-1}^A - \frac{n}{n+1}n_{t-1}^B)K_{t-1}} - \frac{(n_{t-1}^B - \frac{n}{n+1}n_{t-1}^A)K_{t-1}(\widehat{f}_{t-1}^B-\alpha)}{(n_{t-1}^A - \frac{n}{n+1}n_{t-1}^B)K_{t-1}}. \tag{13}
$$

And type A can also infer $\widehat{f}_{t-1}^B$ in a similar way:

$$
\widehat{f}_{t-1}^B = \alpha + \frac{(p_{t-1}-\alpha)}{(n_{t-1}^B - \frac{n}{n+1}n_{t-1}^A)K_{t-1}} - \frac{(n_{t-1}^A - \frac{n}{n+1}n_{t-1}^B)K_{t-1}(\widehat{f}_{t-1}^A-\alpha)}{(n_{t-1}^B - \frac{n}{n+1}n_{t-1}^A)K_{t-1}}. \tag{14}
$$

Hence,

1. When t is odd, type B traders know $\widehat{f}_{t-1}^A$, $\widehat{f}_{t-2}^A$, ... , $\widehat{f}_0^A$ while type A only know $\widehat{f}_{t-2}^B$, $\widehat{f}_{t-4}^B$, ... , $\widehat{f}_0^B$.
2. When t is even, type B traders know $\widehat{f}_{t-1}^A$, $\widehat{f}_{t-2}^A$, ... , $\widehat{f}_0^A$ while type A only know $\widehat{f}_{t-1}^B$, $\widehat{f}_{t-3}^B$, ... , $\widehat{f}_0^B$.

Q.E.D

Proof to Proposition 6

Proof: This is done by induction. Notice that at $t = 1$, there is no need to do signal extraction because there are no signals for traders at $t = 0$. Therefore at $t = 1$, both types of traders will update their belief according to their own signals, i.e:

$$\begin{aligned}
\tau_1^A &= \tau_0^A + \tau_\epsilon \\
\tau_1^B &= \tau_0^B + \tau_\epsilon \\
\widehat{f}_1^A &= \widehat{f}_0^A + \frac{\tau_\epsilon}{\tau_1}(S_1^A - \widehat{f}_0^A) \\
\widehat{f}_1^B &= \widehat{f}_0^B + \frac{\tau_\epsilon}{\tau_1}(S_1^B - \widehat{f}_0^B).
\end{aligned}$$

At $t = 2$, type B traders know $\widehat{f}_1^A$, and type A traders know $\widehat{f}_1^B$. For type B traders, they can infer the signal type A traders received at $t = 1$, $S_1^A = (\widehat{f}_1^A - \widehat{f}_0^A)\frac{\tau_1^A}{\tau_\epsilon} + \widehat{f}_0^A$. In the meantime, type A traders are also able to infer the exact signal type B received at $t = 1$, $S_1^B = (\widehat{f}_1^B - \widehat{f}_0^B)\frac{\tau_1^B}{\tau_\epsilon} + \widehat{f}_0^B$ when $t = 2$. Therefore type A traders update their beliefs incorporating their own signal S_2^A and the signal extracted S_1^B, while type B traders update their beliefs incorporating their own signal S_2^B and the signal extracted S_1^A, i.e:

$$\begin{aligned}
\tau_2^A &= \tau_1^A + 2\tau_\epsilon \\
\tau_2^B &= \tau_1^B + 2\tau_\epsilon \\
\widehat{f}_2^A &= \widehat{f}_1^A + \frac{\tau_\epsilon}{\tau_2^A}(S_2^A + S_1^B - 2\widehat{f}_1^A) \\
\widehat{f}_2^B &= \widehat{f}_1^B + \frac{\tau_\epsilon}{\tau_2^B}(S_2^B + S_1^A - 2\widehat{f}_1^B).
\end{aligned}$$

At $t = 3$, type B traders know $\widehat{f}_2^A$ and $\widehat{f}_1^A$, while type A traders have no further information. Similarly, type B traders can infer exact signal $S_2^A = (\widehat{f}_2^A - \widehat{f}_1^A)\frac{\tau_2^A}{\tau_\epsilon} + 2\widehat{f}_1^A - S_1^B$ while type A traders cannot, i.e:

$$\begin{aligned}
\tau_3^A &= \tau_2^A + \tau_\epsilon \\
\tau_3^B &= \tau_2^B + 2\tau_\epsilon \\
\widehat{f}_3^A &= \widehat{f}_2^A + \frac{\tau_\epsilon}{\tau_3^A}(S_3^A - \widehat{f}_2^A) \\
\widehat{f}_3^B &= \widehat{f}_2^B + \frac{\tau_\epsilon}{\tau_3^B}(S_3^B + S_2^A - 2\widehat{f}_2^B).
\end{aligned}$$

At $t = 4$, type B can infer $S_3^A = (\widehat{f}_3^A - \widehat{f}_1^A)\frac{\tau_3^A}{\tau_\epsilon} + \widehat{f}_3^A$, while type A knows $\widehat{f}_3^B$ and $\widehat{f}_1^B$. Type A traders understand that

$$\widehat{f}_3^B = \widehat{f}_2^B + \frac{\tau_\epsilon}{\tau_3^B}(S_3^B + S_2^A - 2\widehat{f}_2^B)$$

$$
\begin{aligned}
&= \widehat{f}_1^B + \frac{\tau_\epsilon}{\tau_2^B}(S_2^B + S_1^A - 2\widehat{f}_1^B) + \frac{\tau_\epsilon}{\tau_3^B}(S_3^B + S_2^A - 2\widehat{f}_2^B) \\
&= \widehat{f}_1^B + \frac{\tau_\epsilon}{\tau_2^B}(S_2^B + S_1^A - 2\widehat{f}_1^B) + \frac{\tau_\epsilon}{\tau_3^B}(S_3^B + S_2^A - 2\widehat{f}_1^B - \frac{2\tau_\epsilon}{\tau_2^B}(S_2^B + S_1^A - 2\widehat{f}_1^B)).
\end{aligned}
$$

Therefore, type A traders cannot exact the exact signal anymore. They can only know $\tau_\epsilon S_2^B + \frac{\tau_\epsilon}{\tau_3^B} S_3^B$, which is normally distributed. They can always normalize this combined signal to update their belief. Let $\widehat{S}_4^A = \frac{\tau_3^B S_2^B}{1+\tau_3^B} + \frac{S_3^B}{1+\tau_3^B}$ be the normalized signal, where $\widehat{S}_4^A$ is normally distributed with mean $\widetilde{f}$, precision $\widehat{\tau}_4^A = \frac{(1+\tau_3^B)^2}{(1+\tau_3^B)^2}\tau_\epsilon$. Therefore,

$$
\begin{aligned}
\tau_4^A &= \tau_3^A + \tau_\epsilon + \widehat{\tau}_4^A \\
\tau_4^B &= \tau_3^B + 2\tau_\epsilon \\
\widehat{f}_4^A &= \widehat{f}_3^A + \frac{\tau_\epsilon}{\tau_4^A}(S_4^A - \widehat{f}_3^A) + \frac{\widehat{\tau}_4^A}{\tau_4^A}(\widehat{S}_4^A - \widehat{f}_3^A) \\
\widehat{f}_2^B &= \widehat{f}_1^B + \frac{\tau_\epsilon}{\tau_2^B}(S_2^B + S_1^A - 2\widehat{f}_1^B).
\end{aligned}
$$

Suppose the proposition holds at $t = 2k$.[23] This implies

$$
\begin{aligned}
\widehat{f}_{2k}^A &= \widehat{f}_{2k-1}^A + \frac{\tau_\epsilon}{\tau_{2k}^A}(S_{2k}^A - \widehat{f}_{2k-1}^A) + \frac{\widehat{\tau}_{2k}^A}{\tau_{2k}}(\widehat{S}_{2k}^A - \widehat{f}_{2k-1}^A) \\
\widehat{f}_{2k}^B &= \widehat{f}_{2k-1}^B + \frac{\tau_\epsilon}{\tau_{2k}^B}(S_{2k}^B + S_{2k-1}^A - 2\widehat{f}_{2k-1}^B)
\end{aligned}
$$

where $\widehat{S}_{2k}^A = \frac{\tau_{2k-1}^B S_{2k-2}^B}{1+\tau_{2k-1}^B} + \frac{S_{2k-1}^B}{1+\tau_{2k-1}^B}$.

At $t = 2k+1$, type B traders know $\widehat{f}_{2k}^A$, $\widehat{f}_{2k-1}^A$, while type A traders have no further information. Notice that $\widehat{S}_{2k}^A$ is a function of S_{2k-2}^B and S_{2k-1}^B. This implies type B traders know $\widehat{S}_{2k}^A$. Therefore, type B traders can infer exact signal

$$
S_{2k}^A = \frac{\tau_{2k}^A}{\tau_\epsilon}(\widehat{f}_{2k}^A - \widehat{f}_{2k-1}^A - \frac{\widehat{\tau}_{2k}^A}{\tau_{2k}^A}(\widehat{S}_{2k}^A - \widehat{f}_{2k-1}^A)) + \widehat{f}_{2k-1}^A \tag{15}
$$

while type A traders cannot infer anything new. Therefore, at $t = 2k+1$,

$$
\begin{aligned}
\tau_{2k+1}^A &= \tau_{2k}^A + \tau_\epsilon \\
\tau_{2k+1}^B &= \tau_{2k}^B + 2\tau_\epsilon \\
\widehat{f}_{2k+1}^A &= \widehat{f}_{2k}^A + \frac{\tau_\epsilon}{\tau_{2k+1}^A}(S_{2k+1}^A - \widehat{f}_{2k}^A) \\
\widehat{f}_{2k+1}^B &= \widehat{f}_{2k}^B + \frac{\tau_\epsilon}{\tau_{2k+1}^B}(S_{2k+1}^B + S_{2k}^A - 2\widehat{f}_{2k}^B).
\end{aligned}
$$

In other words, the proposition holds at $t = 2k+1$. To complete the proof, we also need to examine the beliefs updating at $t = 2k+2$. At $t = 2k+2$, type B traders know $\widehat{f}_{2k+1}$ and $\widehat{f}_{2k}$. Therefore,

[23] We have already shown that it holds at $t = 4$.

they can infer $S^A_{2k+1} = \frac{\tau^A_{2k+1}}{\tau_\epsilon}(\widehat{f}^A_{2k+1} - \widehat{f}^A_{2k}) + \widehat{f}^A_{2k}$. Therefore, for type B traders, the belief updating at $t = 2k+2$ is characterized as

$$\begin{aligned}
\tau^B_{2k+2} &= \tau^B_{2k+1} + 2\tau_\epsilon \\
\widehat{f}^B_{2k+2} &= \widehat{f}^B_{2k+1} + \frac{\tau_\epsilon}{\tau^B_{2k+2}}(S^B_{2k+2} + S^A_{2k+1} - 2\widehat{f}^B_{2k+1})
\end{aligned}$$

while type A traders only know $\widehat{f}_{2k+1}$ and $\widehat{f}_{2k-1}$. They understand the belief updating of type B at $t = 2k+1$ is

$$\begin{aligned}
\widehat{f}^B_{2k+1} &= \widehat{f}^B_{2k} + \frac{\tau_\epsilon}{\tau^B_{2K+1}}(S^B_{2k+1} + S^A_{2k} - 2\widehat{f}^B_{2k}) \\
&= \widehat{f}^B_{2k-1} + \frac{\tau_\epsilon}{\tau^B_{2k}}(S^B_{2k} + S^A_{2k-1} - 2\widehat{f}^B_{2k-1}) + \frac{\tau_\epsilon}{\tau^B_{2k+1}}(S^B_{2k+1} + S^A_{2k} - 2\widehat{f}^B_{2k}) \\
&= \widehat{f}^B_{2k-1} + \frac{\tau_\epsilon}{\tau^B_{2k}}(S^B_{2k} + S^A_{2k-1} - 2\widehat{f}^B_{2k-1}) \\
&\quad + \frac{\tau_\epsilon}{\tau^B_{2k+1}}(S^B_{2k+1} + S^A_{2k} - 2(\widehat{f}^B_{2k-1} + \frac{\tau_\epsilon}{\tau^B_{2k}}(S^B_{2k} + S^A_{2k-1} - 2\widehat{f}^B_{2k-1})).
\end{aligned}$$

Therefore, type A can infer a composite signal $\widehat{S}^A_{2k+2} = \frac{\tau^B_{2k+1} S^B_{2k}}{1+\tau^B_{2k+1}} + \frac{S^B_{2k+1}}{1+\tau^B_{2k+1}}$, with the precision $\widehat{\tau}^A_{2k+2} = \frac{1+(\tau^B_{2k+1})^2}{(1+\tau^B_{2k+1})^2}\tau_\epsilon$. In summary, the belief updating of type A traders at $t = 2k+2$ is

$$\begin{aligned}
\tau^A_{2k+2} &= \tau^A_{2k+1} + \tau_\epsilon + \widehat{\tau}^A_{2k+2} \\
\tau^B_{2k+2} &= \tau^B_{2k+1} + 2\tau_\epsilon \\
\widehat{f}^A_{2k+2} &= \widehat{f}^A_{2k+1} + \frac{\tau_\epsilon}{\tau^A_{2k+2}}(S^A_{2k+2} - \widehat{f}^A_{2k+1}) + \frac{\widehat{\tau}^A_{2k+2}}{\tau^A_{2k+2}}(\widehat{S}^A_{2k+2} - \widehat{f}^A_{2k+1}) \\
\widehat{f}^B_{2k+2} &= \widehat{f}^B_{2k+1} + \frac{\tau_\epsilon}{\tau^B_{2k+2}}(S^B_{2k+2} + S^A_{2k+1} - 2\widehat{f}^B_{2k+1}).
\end{aligned}$$

Hence, the proposition holds at $t = 2k+2$ and this completes the proof.

B Derivations

Derivation of Equation 7

We start with Z_t. By definition, $Z_t = \frac{\widehat{f}_t}{\widehat{f}_{t-1}}$, which can be written as

$$Z_t = 1 + \frac{\tau_\epsilon}{\tau_t}(\frac{S^A_t + S^B_t}{\widehat{f}_{t-1}} - 2) \tag{16}$$

Note that $\frac{1}{Z_{t-1}} = \frac{\widehat{f}_{t-2}}{\widehat{f}_{t-1}}$, which can be written as

$$\frac{1}{Z_{t-1}} = \frac{\tau_{t-1}}{\tau_{t-2}} - \frac{\tau_\epsilon}{\tau_{t-2}\widehat{f}_{t-1}}(S^A_{t-1} + S^B_{t-1})$$

which can be used to solve $\frac{1}{\widehat{f}_{t-1}}$

$$\frac{1}{\widehat{f}_{t-1}} = \frac{(\frac{\tau_{t-1}}{\tau_\epsilon} - \frac{\tau_{t-2}}{\tau_\epsilon}\frac{1}{Z_{t-1}})}{(S^A_{t-1} + S^B_{t-1})}.$$

Substitute Equation 17 into Equation 16, we get the expression.

Lags	D_t			$\mathrm{Var}(D_t)$		
	Mean	Std	p-value	Mean	Std	p-value
1	-0.255	0.073	0	0.018	0.041	0
2	0.006	0.082	0.444	0.007	0.060	0.252
3	-0.008	0.060	0.203	-0.004	0.046	0.338
5	-0.002	0.064	0.770	-0.003	0.052	0.560
6	-0.004	0.070	0.608	-0.007	0.047	0.174
7	-0.009	0.076	0.233	-0.003	0.050	0.549
8	0.003	0.068	0.693	-0.006	0.043	0.198
9	-0.005	0.070	0.507	0.006	0.067	0.356
10	-0.006	0.069	0.419	-0.005	0.044	0.239

Table I.: Monte Carlo study of D_t (arrival component). First column reports the mean of autocorrelations. Second column reports the variance of autocorrelations. Third column reports the p-value of t test for null hypothesis that the mean equals zero. The columns four to six are the corresponding results for variance of D_t.

Lags	Z_t			$\mathrm{Var}(Z_t)$		
	Mean	Std	p-value	Mean	Std	p-value
1	-0.024	0.172	0.290	0.271	0.152	0
2	0.009	0.157	0.624	0.220	0.179	0
3	-0.036	0.161	0.043	0.200	0.141	0
4	0.001	0.171	0.938	0.144	0.160	0
5	0.006	0.160	0.641	0.130	0.155	0
6	0.001	0.173	0.924	0.108	0.164	0
7	-0.002	0.157	0.865	0.097	0.177	0
8	0.008	0.164	0.486	0.079	0.167	0
9	0.015	0.172	0.126	0.071	0.166	0
10	0.002	0.163	0.871	0.054	0.174	0

Table II.: Monte Carlo study of Z_t (belief component). First column reports the mean of autocorrelations. Second column reports the variance of autocorrelations. Third column reports the p-value of t test for null hypothesis that the mean equals zero. The columns four to six are the corresponding results for variance of Z_t.

Lags	r_t			$\mathrm{Var}(r_t)$		
	Mean	Std	p-value	Mean	Std	p-value
1	-0.489	0.058	0	0.347	0.165	0
2	-0.005	0.097	0.579	0.094	0.153	0
3	0.001	0.105	0.138	0.099	0.155	0
4	-0.002	0.109	0.893	0.091	0.152	0
5	-0.006	0.088	0.414	0.083	0.164	0
6	0.013	0.098	0.534	0.083	0.154	0
7	-0.011	0.093	0.187	0.082	0.162	0
8	0.008	0.110	0.308	0.086	0.152	0
9	-0.004	0.099	0.291	0.079	0.155	0
10	-0.003	0.105	0.958	0.071	0.160	0

Table III.: AUTOCORRELATIONS OF r_t AND $\mathrm{VAR}(r_t)$ IN PUBLIC SIGNAL CASE. First column reports the average of the autocorrelations across 100 simulations. Second column reports the standard deviation of the autocorrelations across 100 simulations. Third column reports the p-value of t test for the null hypothesis that the mean equals zero. The columns four to six are the corresponding results for $\mathrm{Var}(r_t)$.

Lags	r_t			$\mathrm{Var}(r_t)$		
	Mean	Std	p-value	Mean	Std	p-value
1	-0.461	0.077	0	0.293	0.151	0
2	-0.018	0.131	0.173	0.125	0.171	0
3	0.017	0.114	0.134	0.121	0.159	0
4	-0.001	0.119	0.905	0.125	0.157	0
5	0.010	0.127	0.413	0.128	0.160	0
6	-0.009	0.131	0.519	0.119	0.160	0
7	0.016	0.115	0.173	0.114	0.172	0
8	-0.014	0.125	0.280	0.102	0.162	0
9	0.013	0.122	0.291	0.114	0.158	0
10	0.001	0.110	0.902	0.109	0.170	0

Table IV.: ACFS OF r_t AND $\mathrm{VAR}(r_t)$ IN SIGNAL EXTRACTION CASE. First column reports the mean of autocorrelations. Second column reports the variance of autocorrelations. Third column reports the p-value of t test for null hypothesis that the mean equals zero. The columns four to six are the corresponding results for $\mathrm{Var}(r_t)$.

Lags	r_t			$\mathrm{Var}(r_t)$		
	Mean	Std	p-value	Mean	Std	p-value
1	-0.460	0.099	0	0.383	0.154	0
2	-0.023	0.166	0.169	0.214	0.160	0
3	0.024	0.142	0.090	0.204	0.165	0
4	-0.003	0.144	0.860	0.202	0.163	0
5	0.011	0.150	0.479	0.198	0.163	0
6	-0.006	0.151	0.698	0.188	0.176	0
7	-0.019	0.135	0.221	0.179	0.187	0
8	0.019	0.144	0.188	0.166	0.165	0
9	0.002	0.140	0.179	0.175	0.153	0
10	0.001	0.136	0.913	0.168	0.178	0

Table V.: ACFs of r_t and Var(r_t) when traders use sophisticated guess. First column reports the mean of autocorrelations. Second column reports the variance of autocorrelations. Third column reports the p-value of t test for null hypothesis that the mean equals zero. The column four to six are the corresponding results for Var(r_t).

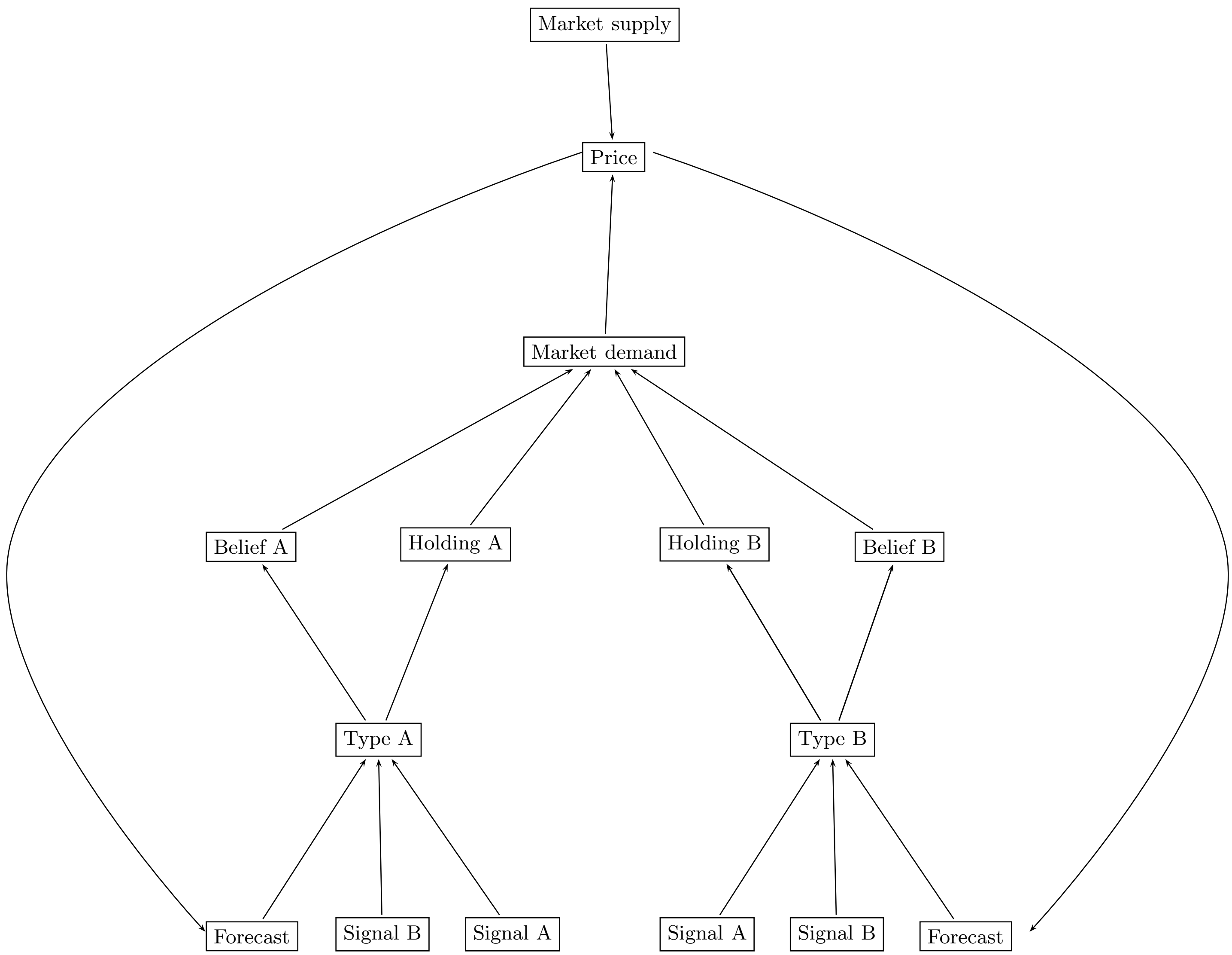
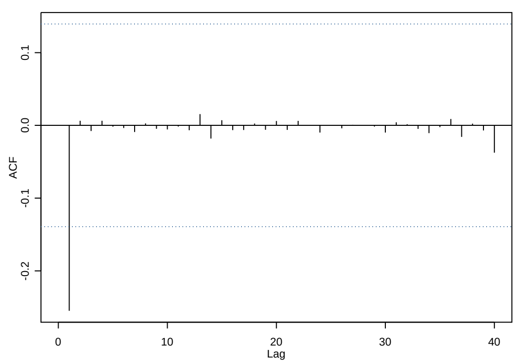
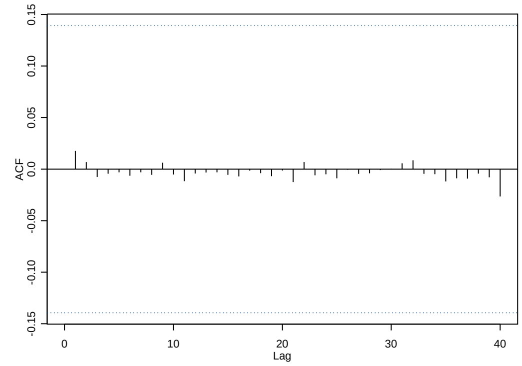


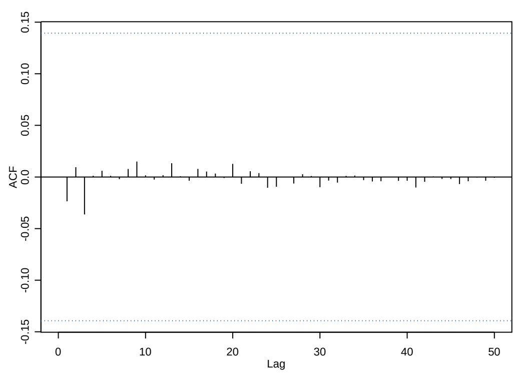
Figure 2: A FLOWCHART FOR THE PUBLIC SIGNAL CASE: HOW THE PRICE IS GENERATED? In the public signal case, each trader receives a signal, observes the signal of other traders, and forecasts the next period's price at the beginning of the current trading period. Based on this information, each trader updates his belief about the value of the underlying asset and adjusts his optimal holdings which are aggregated into the market demand. Combined with the market supply, the price is determined.



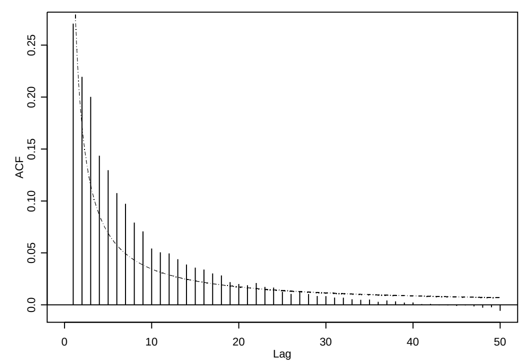
(a) ACF of simulated D_t



(b) ACF of simulated $\mathrm{Var}(D_t)$

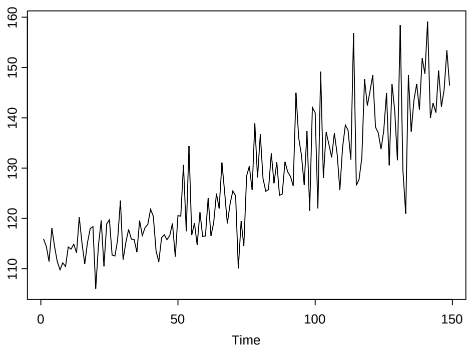


(c) ACF of simulated Z_t

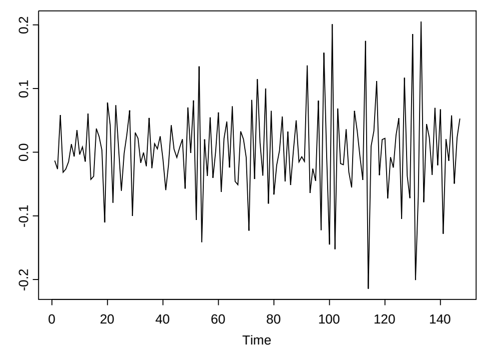


(d) ACF of simulated $\mathrm{Var}(Z_t)$

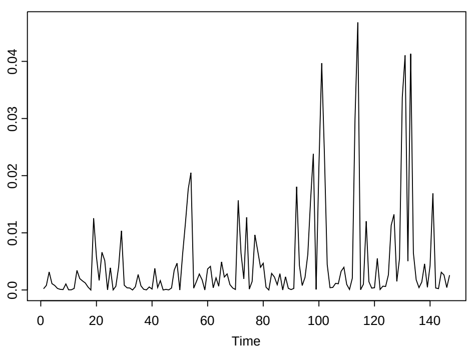
Figure 3: Monte Carlo Study of autocorrelation function (ACF) of D_t and Z_t. (a) Average autocorrelations of D_t across 100 simulations. (b) Average autocorrelations of $\mathrm{Var}(D_t)$. (c) Average autocorrelations of Z_t. (d) Average autocorrelations of $\mathrm{Var}(Z_t)$. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



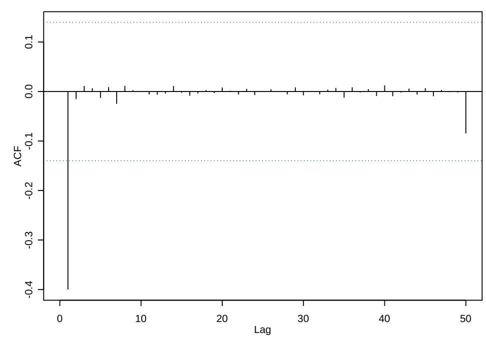
(a) Simulated prices



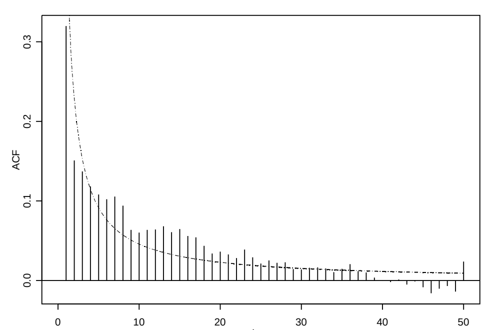
(b) Simulated returns



(c) Simulated variance of return

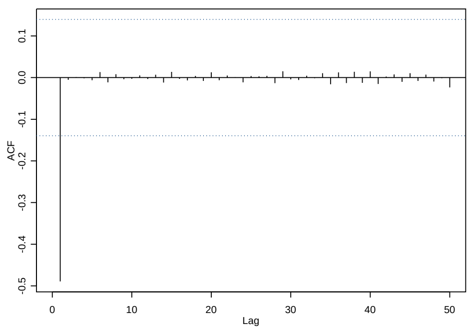


(d) ACF of simulated returns

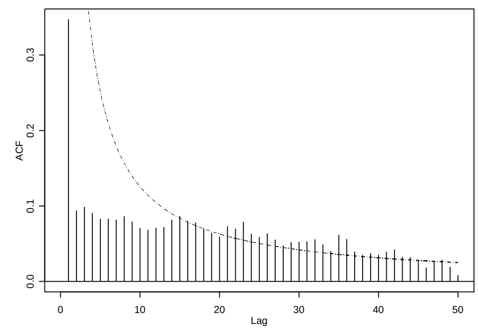


(e) ACF of simulated the variance of returns

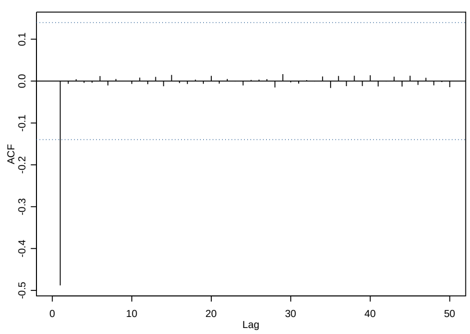
Figure 4: Simulation results in the public signal case. (a) Time series of simulated prices. (b) Time series of simulated returns. (c) Time series of simulated variance of returns. (d) Average autocorrelations of returns across 100 simulations (e) Average autocorrelations of variance of returns. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



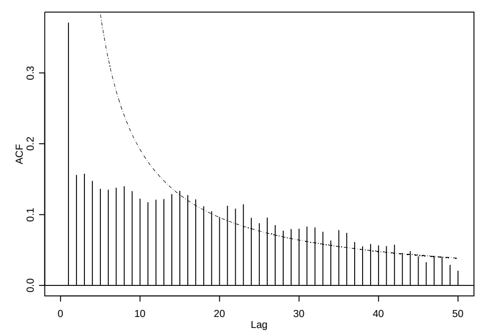
(a) ACF of returns for $\alpha = 80$



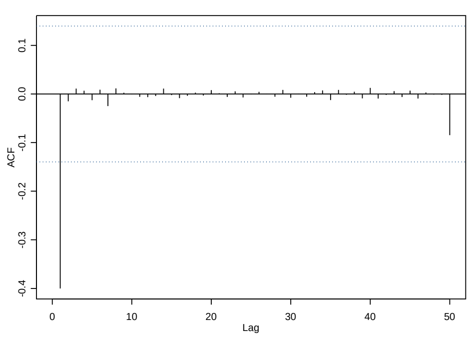
(b) ACF of variance of returns for $\alpha = 80$



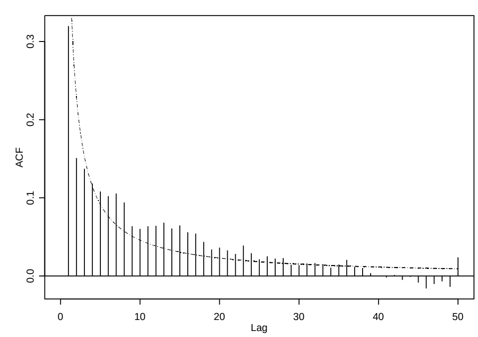
(c) ACF of returns when $\alpha = 150$



(d) ACF of variance of returns for $\alpha = 150$

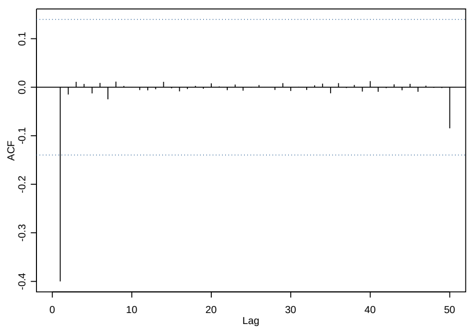


(e) ACF of returns for $\alpha = 200$

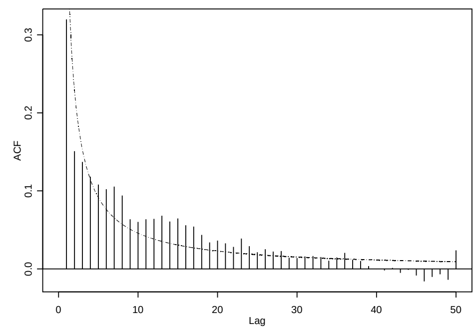


(f) ACF of variance of returns for $\alpha = 200$

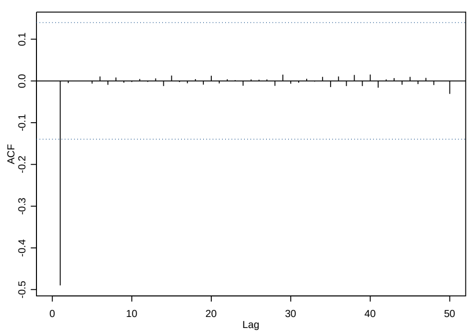
Figure 5: Dependence structure (autocorrelation function (ACF) of r_t and $\mathrm{Var}(r_t)$) and carrying cost in the public signal case. (a) Average autocorrelations of returns across 100 simulations for $\alpha = 80$. (b) Average autocorrelations of variance of returns for $\alpha = 80$. (c) Average autocorrelations of returns for $\alpha = 150$. (d) Average autocorrelations of variance of returns for when $\alpha = 150$. (e) Average autocorrelations of returns for $\alpha = 200$. (f) Average autocorrelations of variance of returns for $\alpha = 200$. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



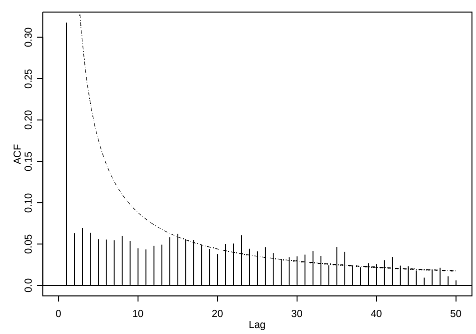
(a) ACF of returns for $n = 50$



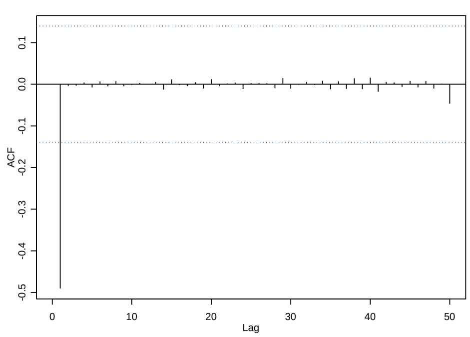
(b) ACF of variance of returns for $n = 50$



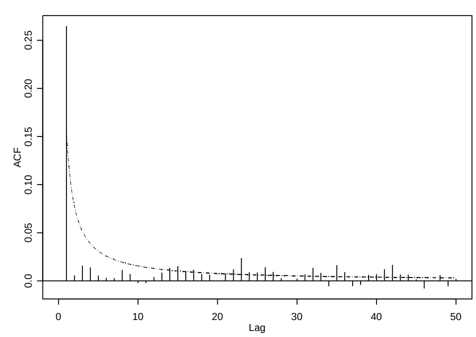
(c) ACF of returns for $n = 100$



(d) ACF of variance of returns for $n = 100$

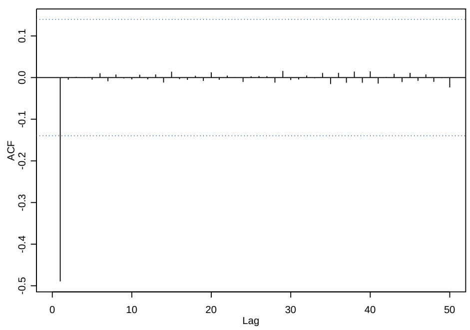


(e) ACF of returns for $n = 200$

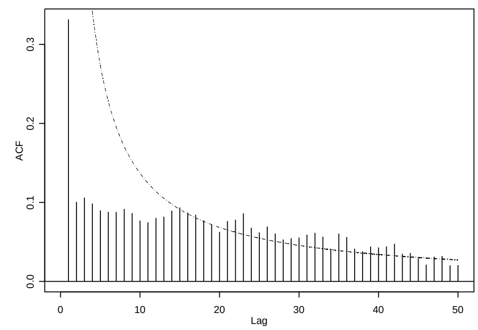


(f) ACF of variance of returns for $n = 200$

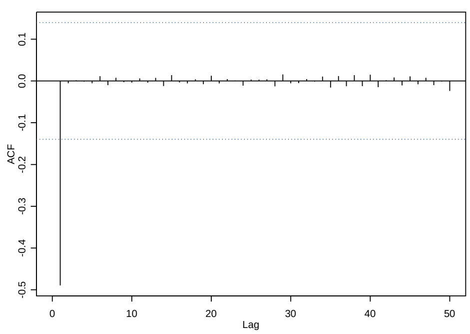
Figure 6: Dependence structure (autocorrelation function (ACF) of r_t and $\mathrm{Var}(r_t)$) and mean arrivals in the public signal case. (a) Average autocorrelations of returns across 100 simulations for $n = 50$. (b) Average autocorrelations of variance of returns for $n = 50$. (c) Average autocorrelations of returns for $n = 100$. (d) Average ACF of variance of returns for $n = 100$. (e) Average autocorrelations of returns when $n = 200$. (f) Average autocorrelations of variance of returns for $n = 200$. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



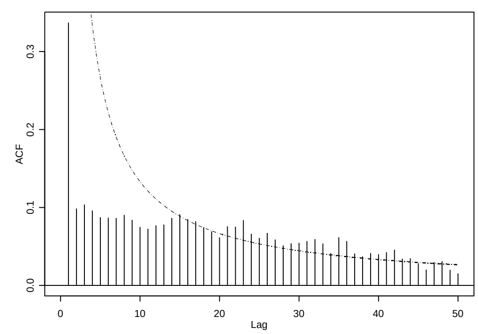
(a) ACF of returns for $\sqrt{\omega} = 10$



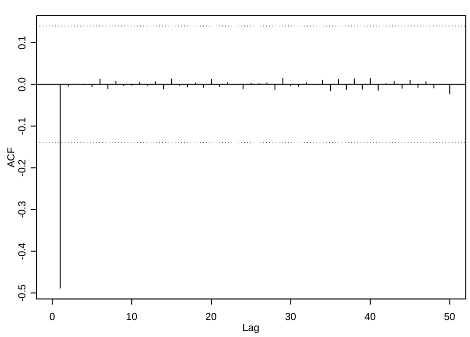
(b) ACF of variance of returns for $\sqrt{\omega} = 10$



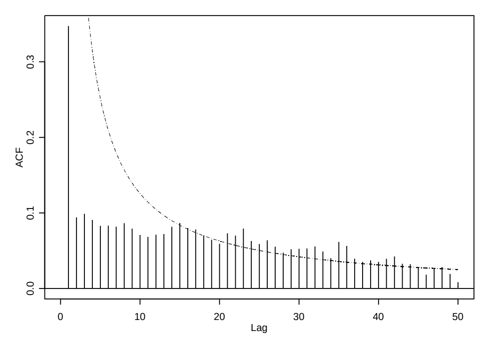
(c) ACF of returns for $\sqrt{\omega} = 20$



(d) ACF of variance of returns for $\sqrt{\omega} = 20$

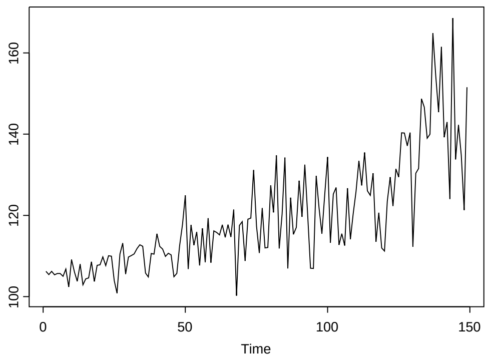


(e) ACF of returns for $\sqrt{\omega} = 30$

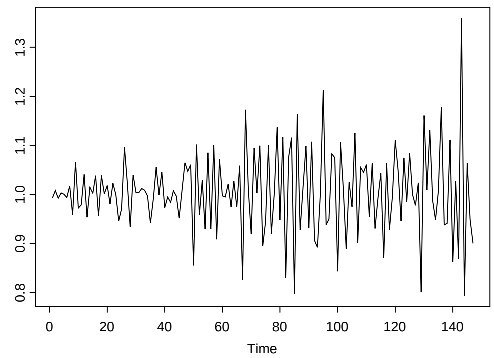


(f) ACF of variance of returns for $\sqrt{\omega} = 30$

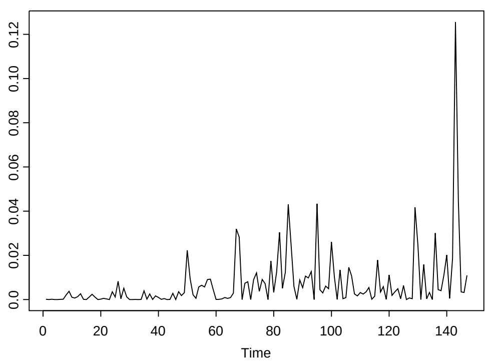
Figure 7: Dependence structure (autocorrelation function (ACF) of r_t and $\mathrm{Var}(r_t)$) and standard deviation of arrivals in the public signal case. (a) Average autocorrelations of returns across 100 simulations for $\sqrt{\omega} = 10$. (b) Average autocorrelations of variance of returns for $\sqrt{\omega} = 10$. (c) Average autocorrelations of returns for $\sqrt{\omega} = 20$. (d) Average autocorrelations of variance of returns for $\sqrt{\omega} = 20$. (e) Average autocorrelations of returns for $\sqrt{\omega} = 30$. (f) Average autocorrelations of variance of returns for $\sqrt{\omega} = 30$. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



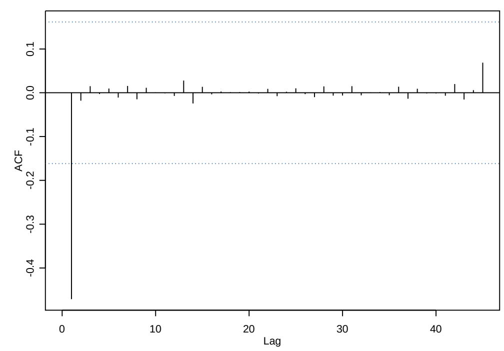
(a) Simulated prices



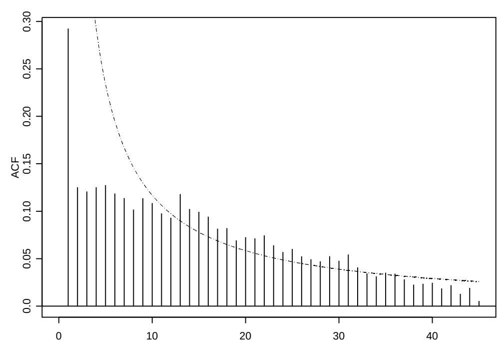
(b) Simulated returns



(c) Simulated variance of returns

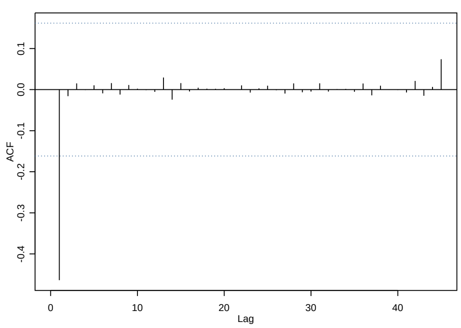


(d) ACF of returns

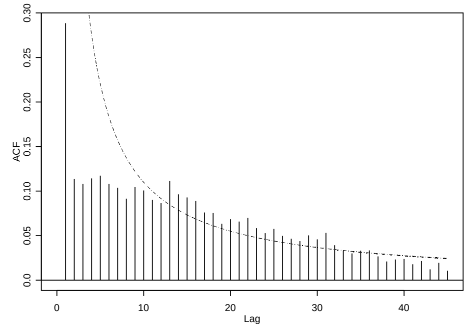


(e) ACF of variance returns

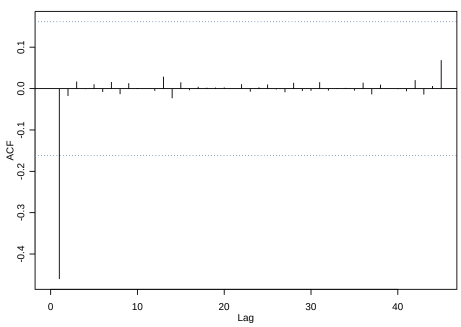
Figure 8: SIMULATION RESULTS IN THE PRIVATE SIGNAL CASE. (a) Time series of simulated prices. (b) Time series of simulated returns. (c) Time series of simulated variance of returns. (d) Average autocorrelations of returns across 100 simulations (e) Average autocorrelations of variance of returns. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



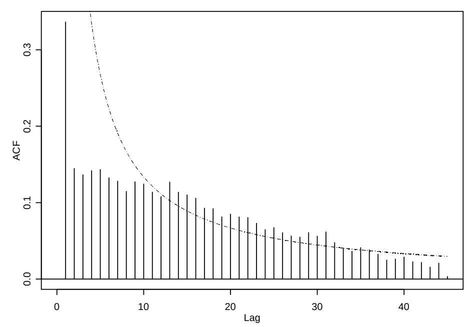
(a) ACF of returns for $\alpha = 80$



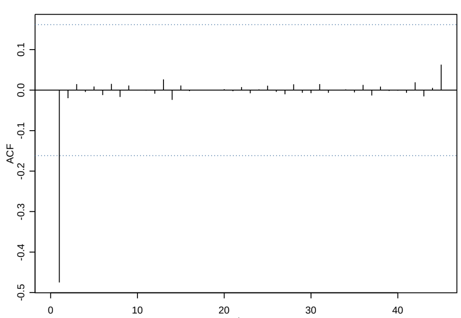
(b) ACF of variance returns for $\alpha = 80$



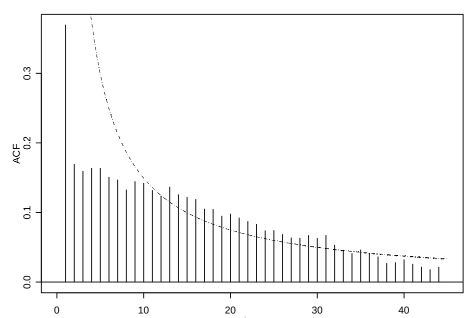
(c) ACF of returns for $\alpha = 100$



(d) ACF of variance of returns for $\alpha = 100$

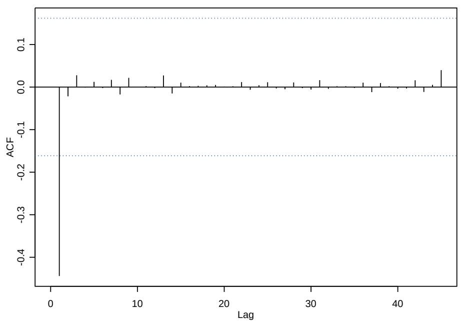


(e) ACF of returns for $\alpha = 120$

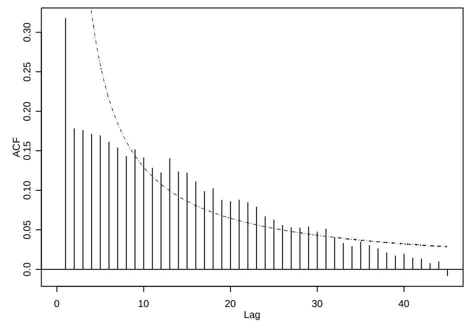


(f) ACF of variance of returns for $\alpha = 120$

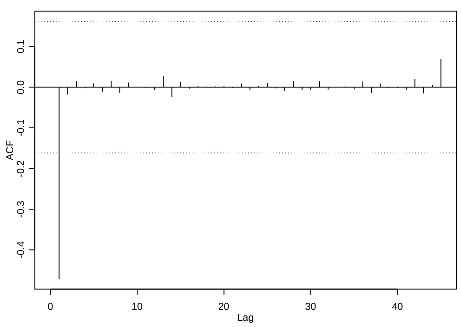
Figure 9: Dependence structure (autocorrelation function (ACF) of r_t and Var(r_t)) and carrying cost in the private signal case. (a) Average autocorrelations of returns for $\alpha = 80$ across 100 simulations. (b) Average autocorrelations of variance of returns for $\alpha = 80$. (c) Average autocorrelations of returns for $\alpha = 100$. (d) Average autocorrelations of variance of returns for $\alpha = 100$. (e) Average autocorrelations of returns for $\alpha = 120$. (f) Average autocorrelations of variance of returns for $\alpha = 120$. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



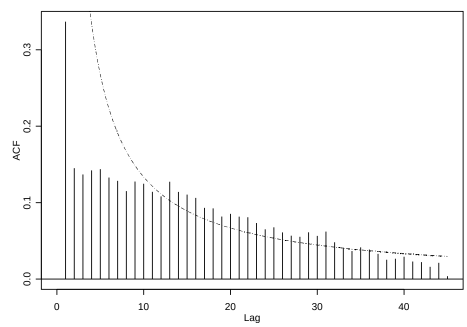
(a) ACF of returns for $n = 30$



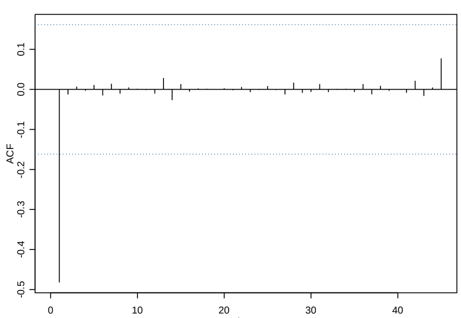
(b) ACF of variance of returns for $n = 30$



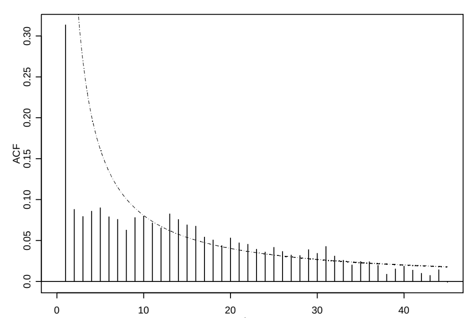
(c) ACF of returns for $n = 50$



(d) ACF of variance of returns for $n = 50$

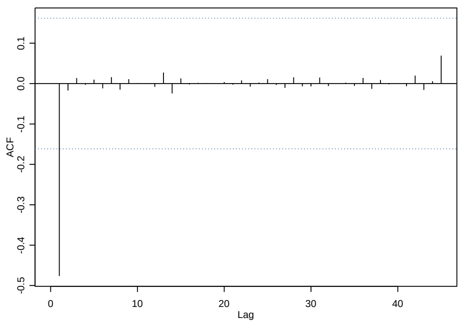


(e) ACF of returns for $n = 80$

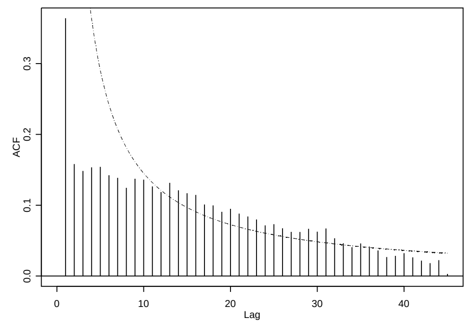


(f) ACF of variance of returns for $n = 80$

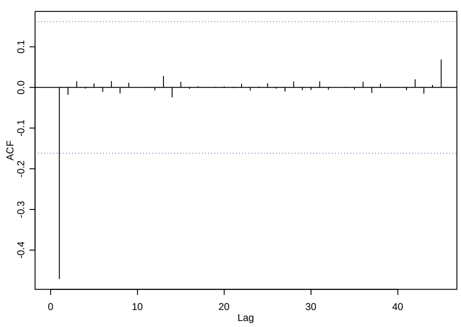
Figure 10: Dependence structure (autocorrelation function (ACF) of r_t and $\mathrm{Var}(r_t)$) and mean arrivals in the private signal case. (a) Average autocorrelations of returns across 100 simulations for $n = 30$. (b) Average autocorrelations of variance of returns for $= 50$. (c) Average autocorrelations of returns for $n = 50$. (d) Average autocorrelations of variance of returns for $n = 80$. (e) Average autocorrelations of returns for $n = 80$. (f) Average autocorrelations of variance of returns for $n = 80$. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



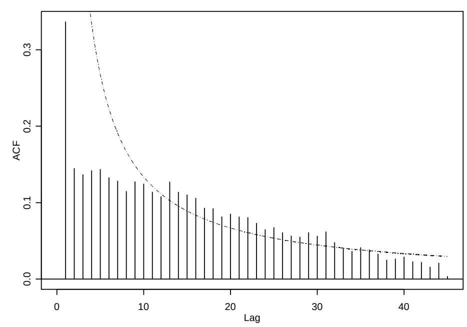
(a) ACF of returns for $\sqrt{\omega} = 10$



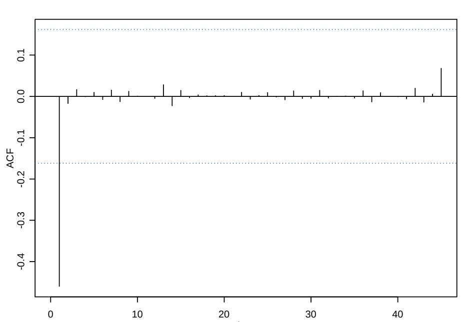
(b) ACF of variance of returns for $\sqrt{\omega} = 10$



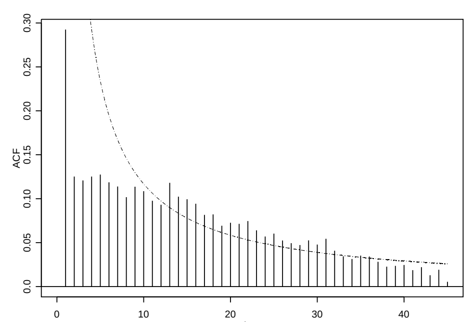
(c) ACF of returns for $\sqrt{\omega} = 20$



(d) ACF of variance of returns for $\sqrt{\omega} = 20$

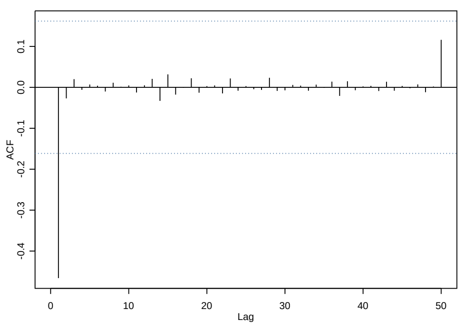


(e) ACF of returns for $\sqrt{\omega} = 30$

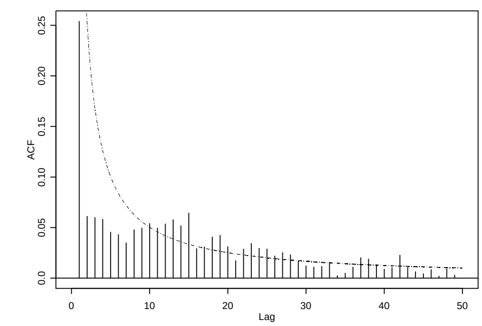


(f) ACF of variance of returns for $\sqrt{\omega} = 30$

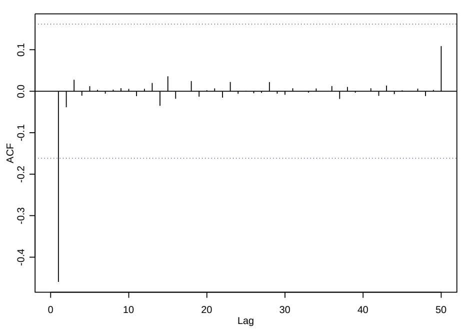
Figure 11: Dependence structure (autocorrelation function (ACF) of r_t and $\mathrm{Var}(r_t)$) and variance of arrivals in the private signal case. (a) Average autocorrelations of returns across 100 simulations for $\sqrt{\omega} = 10$. (b) Average autocorrelations of variance of returns for $\sqrt{\omega} = 10$. (c) Average autocorrelations of returns for $\sqrt{\omega} = 20$. (d) Average autocorrelations of variance of returns for $\sqrt{\omega} = 20$. (e) Average autocorrelations of returns for $\sqrt{\omega} = 30$. (f) Average autocorrelations of variance of returns for $\sqrt{\omega} = 30$. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



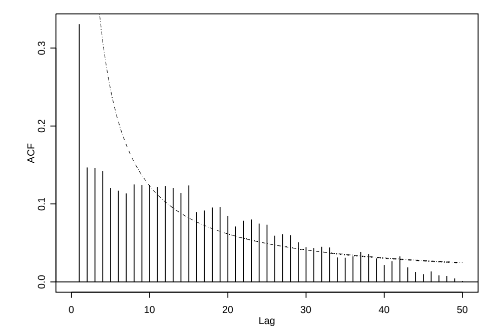
(a) ACF of r_t with public signal



(b) ACF of Var(r_t) with public signal

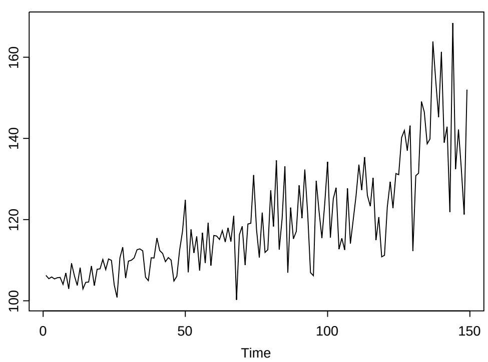


(c) ACF of r_t with private signal

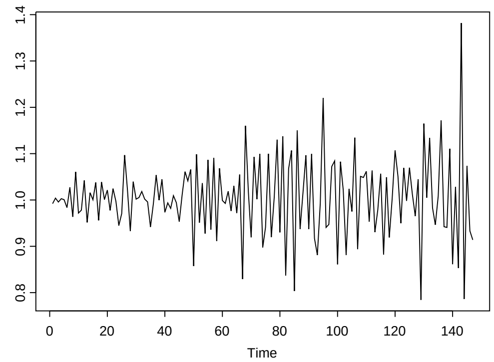


(d) ACF of Var(r_t) with private signal

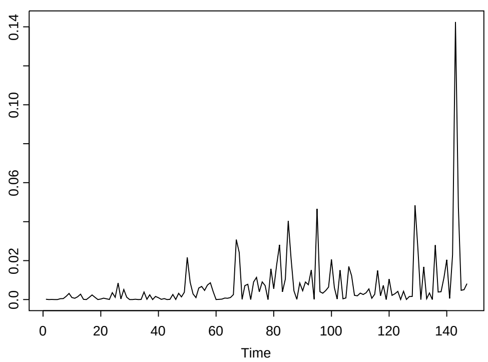
Figure 12: The effects of heterogeneous priors on autocorrelation function (ACF) of r_t and Var(r_t). (a) Average autocorrelations of r_t across 100 simulations with public signal. (b) Average autocorrelations of Var(r_t) with public signal. (c) Average autocorrelations of r_t with private signal. (d) Average autocorrelations of Var(r_t) with private signal. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.



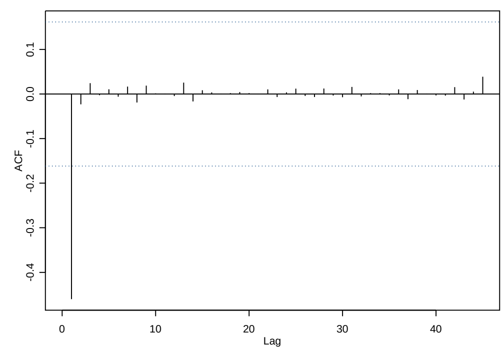
(a) Simulated prices



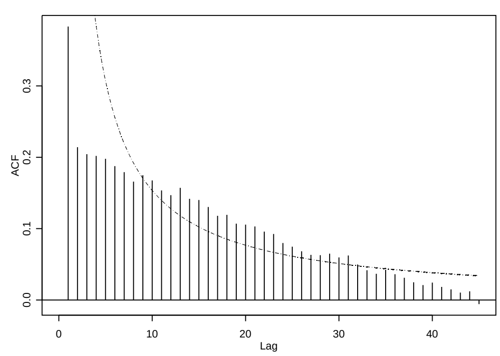
(b) Simulated returns



(c) Simulated variance of returns



(d) ACF of returns



(e) ACF of variance of returns

Figure 13: SIMULATION RESULTS WHEN TRADERS USE SOPHISTICATED GUESS. (a) Time series of simulated prices. (b) Time series of simulated returns. (c) Time series of simulated variance of returns. (d) Average autocorrelations of returns (e) Average autocorrelations of variance of returns. For ACF plots, a hyperbolic decay function of autocorrelations is imposed.

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